Fixed points theorems and quasi-variational inequalities in G-convex spaces

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Abstract

We obtain a generalized continuous selection theorem and a coincidence theorem for generalized convex spaces. Some new Himmelberg type theorems and Eilenberg-Montgomery and Gorniéwicz type fixed point theorems for mappings with KKM property are established in noncompact LG-spaces. Moreover, applications to these fixed point theorems for existence of equilibria are given.

1   Introduction

Let X be a nonempty set, we denote by 2^X the family of all subsets of X, by $\mathcal{F}(X)$ family of all nonempty finite subsets of X and $|A|$ the cardinality of $A \in \mathcal{F}(X)$. Suppose that Y is a nonempty set and $F : X \to 2^Y$ is a multivalued mapping, fibers $F^-(y)$ for $y \in Y$ defined by $F^-(y) = \{x \in X : y \in F(x)\}$. For topological spaces X and Y, a multivalued mapping $F : X \to 2^Y$ is said to be compact if the closure $clF(X)$ of its range $F(X)$ is compact in Y. A multivalued mapping F is said to be upper semicontinuous (u.s.c.) if for each closed set $B \subseteq Y$, the set $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ is closed subset of X; lower semicontinuous (l.s.c.) if for each open set $B \subseteq Y$, the set $F^-(B)$ is open.

Let f be a real bifunction on $X \times Y$, then f is called λ-transfer lower semicontinuous (l.s.c.) on Y if for each $(x, y) \in X \times Y$ with $f(x, y) > \lambda$ there exist $x' \in X$ and a

neighborhood $U(y)$ of y in Y such that $f(x', z) > \lambda$ for all $z \in U(y)$. The bifunction f is said to be λ-transfer upper semicontinuous (u.s.c.) on Y if $-f$ is λ-transfer l.s.c. on Y. It is easily seen that a lower (upper) semicontinuous bifunction is λ-transfer lower (upper) semicontinuous for each λ. A nonempty topological space is acyclic if all of its reduce homology groups over rational vanishes.

The following class $\mathbf{A}(X, Y)$ of approachable multivalued mappings was introduced by Ben-El-Mechaiekh *et al.* [1]. Let $(X, \mathcal{U})$, $(Y, \mathcal{V})$ be uniform topological spaces with bases $\mathcal{U}$ and $\mathcal{V}$ of symmetric entourages for the uniformities on X and Y respectively. For each $U \in \mathcal{U}$ and $V \in \mathcal{V}$, let

$$W = \{((x, y), (x', y')) \in (X \times Y) \times (X \times Y) : (x, x') \in U, (y, y') \in V\}.$$

Then the family $\mathcal{W} = (W)_{U \in \mathcal{U}, V \in \mathcal{V}}$ is a base of symmetric entourages for the product uniformity, and the associated uniform topology on $X \times Y$ is the product of the uniform topologies on X and Y. Let $F : X \to 2^Y$ be a multivalued mapping. For given element $W \in \mathcal{W}$, a function $f : X \to Y$ is said to be a W-approximative selection of F if and only if $Gr(f) \subseteq W[Gr(F)]$, where $Gr(f)$ and $Gr(F)$ denote the graphs of f and F, respectively. A multivalued mapping F is said to be approachable if F admits a continuous W-approximative selection for each $W \in \mathcal{W}$. The class $\mathbf{A}(X, Y)$ of multivalued mappings is defined by

$$\mathbf{A}(X, Y) := \{F : X \to 2^Y : F \text{ is approachable}\}.$$

A generalized convex space or G-convex space was first introduced by Park and Kim [24]. A G-convex space $(X, D; \Gamma)$ consist of a topological space X and a nonempty set D such that for each $A = \{a_0, a_1, ..., a_n\} \in \mathcal{F}(D)$ there exist a subset $\Gamma(A)$ of X and a continuous function $\Phi_A : \Delta_n \to \Gamma(A)$ such that for each $L = \{a_{i_0}, a_{i_1}, ..., a_{i_l}\} \subseteq A$ implies $\Phi_A(\Delta_l) \subset \Gamma(L)$, where Δ_n is an n-simplex with vertices $v_0, v_1, ..., v_n$, $\Delta_l = co\{v_{i_0}, v_{i_1}, ..., v_{i_l}\}$ the face of Δ_n corresponding to L. When $D = X$, we shall write $(X; \Gamma)$ in place of $(X, X; \Gamma)$. If $D \subseteq X$, $(X, D; \Gamma)$ is a G-convex space and $K \subset X$, then K is called G-convex if for each $A \in \mathcal{F}(D), A \subset K$ implies $\Gamma(A) \subset K$. The G-convex hull of K denoted by *G-coK*, is the set $\bigcap\{B \subset X : B$ is a G-convex subset of X containing K$\}$. A function $f : X \to \mathbf{R}$ is called *G*-quasiconvex (resp.*G*-quasiconcave) if for each $\lambda \in \mathbf{R}$, the set $\{x \in X : f(x) < \lambda\}$ (resp. $\{x \in X : f(x) > \lambda\}$) is *G*-convex.

Notice that G-convex spaces contain most of the well known spaces such as convex spaces, H-spaces, L-spaces, C-spaces and hyperconvex metric spaces. For details see Park [18-22] and references therein.

A G-convex space $(X, D; \Gamma)$ is called an LG-space if $(X, \mathcal{U})$ is a uniform space such that D is dense in X and if there exists a symmetric basis $\{V_\lambda\}_{\lambda \in I}$ for uniformity $\mathcal{U}$ such that for each $\lambda \in I$, $\{x \in X : C \bigcap V_\lambda[x] \neq \emptyset\}$ is G-convex whenever $C \subseteq X$ is G-convex, where $V_\lambda[x] = \{x' \in X : (x, x') \in V_\lambda\}$.

Let $(X, D; \Gamma)$ be a G-convex space and Y be a topological space. The following class $\mathfrak{B}(X, Y)$ of better admissible multivalued mappings was introduced by Park [4]. This class consists of multivalued maps $F : X \to 2^Y$ such that for any $A \in \mathcal{F}(D)$ with $|A| = n + 1$ and for any continuous function $f : F(\Gamma(A))) \to \Delta_n$, the composition map $f o F o \Phi_A : \Delta_n \to \Delta_n$ has a fixed point. It is well known that the class of better admissible contains $\mathfrak{A}_c^\kappa(X, Y)$, the admissible class, and many other

important classes of multivalued mappings [19]. A multivalued map $F : D \rightarrow 2^X$ is called a KKM map if for each $A \in \mathcal{F}(D)$, $\Gamma(A) \subset \bigcup_{x \in A} F(x)$. More generally if $G : D \rightarrow 2^Y$, $F : X \rightarrow 2^Y$ are two multivalued maps such that for any $A \in \mathcal{F}(D)$, $F(\Gamma(A)) \subseteq G(A)$, then G is said to be a generalized KKM mapping with respect to F. Let $F : X \rightarrow 2^Y$ be a multivalued mapping such that if $G : D \rightarrow 2^Y$ is a generalized KKM mapping with respect to F, then the family $\{clG(x) : x \in D\}$ has the finite intersection property. In this case we say that F has the KKM property. We define

$$\mathfrak{K}(X,Y) := \{F : X \rightarrow 2^Y : F \text{ has the generalized KKM property}\}.$$

When X is a convex subset of a topological vector space, the class $\mathfrak{K}(X,Y)$ was introduced and studied by Chang and Yen [2]. This concept is further extended for G-convex spaces by Lin *et al.* [17].

Motivated by the concept of c measure of noncompactness of Hahn [10] for topological vector spaces, we define this notion in a similar way for G-convex spaces. Let $(X, D; \Gamma)$ be a G-convex space, $D \subset X$, and C a cone in a vector space with partial ordering $\leq$. Let c be a real number with $c \geq 1$. A function $\Psi : 2^X \rightarrow C$ is called a c-measure of noncompactness on X provided that the that the following conditions hold for any $Z \in 2^X$:

(1) $\Psi(clZ) = \Psi(Z)$;

(2) $\Psi(\text{G-co } Z) \leq c\Psi(Z)$;

(3) if $x \in X$, then $\Psi(Z \cup \{x\}) = \Psi(Z)$;

(4) if $Z_1 \subset Z$, then $\Psi(Z_1) \leq \Psi(Z)$.

If $T : X \rightarrow 2^X$, then T is called Ψ-pseudocondensing map if, whenever $\Psi(Z) \leq c\Psi(T(Z))$ for $Z \in 2^X$, then Z is relatively compact.

In particular, if $c = 1$, then T is called Ψ-condensing. Note that if T is a compact mapping, then T is of course, Ψ-condensing.

Throughout this paper, all topological spaces are assumed to be Hausdorff.

2   Continuous selection theorem and Fixed point theorems

In this section we shall obtain a new version of existence of a continuous selection for multivalued mappings on noncompact subsets of a G-convex space and we apply this result for obtaining a coincidence theorem and fixed point theorems in G-convex spaces.

In order to obtain our continuous selection result we used the following notation. Assume J is a well ordered indexing set and let $\{e_\alpha : \alpha \in J\}$ be a given abstract set. Define

$$E := \{x = \sum_{\alpha \in J} \lambda_\alpha e_\alpha : \lambda_\alpha \in \mathbf{R} \text{ and at most finitely many } \lambda_\alpha \neq 0\}.$$

We provide E by the induced topology of $l_1(J)$. For each nonempty subset I of J the convex hull of $\{e_\alpha : \alpha \in I\}$ is denoted by Δ_I.

The following proposition improves proposition 3.8 of Ben-El-Mechaiekh *et al.* [1], theorem 3.2 of Horvath [11], lemma 2 of Kim and Tan [13] and theorem 1 of Wu and Shen [29].

Proposition 2.1. *Let* $(X, D; \Gamma)$ *be a G-convex space and* Y *be a normal space. Suppose that* $S : Y \to 2^D$ *and* $T : Y \to 2^X$ *are two multivalued mappings such that:*

(1) *for each* $y \in Y$ *and for every* $L \in \mathcal{F}(S(y))$ *one has* $\Gamma(L) \subseteq T(y)$,

(2) *there exist a nonempty paracompact subset* K *of* Y *and finite subset* M *of* D *such that* $Y \setminus K \subseteq \{IntS^-(x) : x \in M\}$,

(3) $K = \bigcup\{IntS^-(x) : x \in S(K)\}$,

(4) *for each* $A \in \mathcal{F}(D)$ *we have* $\Phi_A(\Delta_l) = \Phi_L(\Delta_l)$ *for any* $L \subseteq A$ *where* $l + 1 = |L|$.

Then T *has a continuous selection.*

Proof. Since K is paracompact, there is a locally finite open refinement $\mathcal{V} := \{V_\alpha : \alpha \in I\}$ of the open cover $\{IntS^-(x) : x \in S(K)\}$, where I is an index set. Therefore for each $\alpha \in I$, there exists $x_\alpha \in S(K)$ such that $V_\alpha \subseteq IntS^-(x_\alpha) \subseteq S^-(x_\alpha)$. Let $\{x_\alpha : \alpha \in I\} \bigcup M = \{x_\beta : \beta \in J\} = C$, we can suppose that J is a well ordered set. Moreover $C \subseteq S(Y) \subseteq D$. Assume that $\{h_\beta : \beta \in J\}$ is a partition of unity subordinated to $\mathcal{V} \bigcup \{IntS^-(x) : x \in M\} = \{V_\beta : \beta \in J\}$ and $h : Y \to \Delta_J$ is defined by

$$h(y) = \sum_{\beta \in J} h_\beta(y) e_\beta.$$

If $h_\beta(y) \neq 0$, then $y \in V_\beta$ and so $x_\beta \in S(y)$. Hence if $J_y = \{\beta \in J : h_\beta(y) \neq 0\}$ and $L = \{x_\beta : \beta \in J_y\}$, then L is finite and $L \subseteq S(y)$, therefore by condition (1) $\Phi_L(\Delta_l) \subseteq T(y)$, where $J_y = l + 1$. Thus by the virtue of condition (4) $f(y) := \Phi_L o h(y)$ is a continuous selection of T.

Remarks. (a) Condition (4) of the above proposition is satisfied by a wide classes of spaces. Namely H-spaces, B'-simplicial convexity of Ben-El-Mechaiekh *et al.* [1] and ω-connected spaces considered by Park [20].
(b) In proposition 2.1 if K is compact, then without condition (4) we can conclude that there exist continuous functions $h : Y \to \Delta_n$ and $g : \Delta_n \to X$ for some $n \in \mathbf{N}$, such that $g o h = f$ is a continuous selection of T. Therefore, we obtain theorem 2.1 of Ding and Park [5] and theorem 1 of Yu and Lin [33].

The following coincidence theorem is an improvement of theorem 3.1 of Ding and Tarafdar [6] and theorem 3.1 of Park [20].

Theorem 2.2. *Let* $(X, D; \Gamma)$ *be G-convex space and* Y *be a normal space. Suppose that* $S : Y \to 2^D$ *and* $T : Y \to 2^X$ *are two multivalued mappings satisfying the conditions (1) and (3) of proposition 2.1, condition (2) of proposition 2.1 for a nonempty compact subset* K *and* $F \in \mathfrak{B}(X, Y)$. *Then there exist a point* $x_0 \in X$

and a point $y_0 \in Y$ such that $x_0 \in T(y_0)$ and $y_0 \in F(x_0)$.

Proof. By part (b) of the above remark, there are continuous functions $g : \Delta_n \to X$ and $h : Y \to \Delta_n$ such that $goh = f$ is a continuous selection of T. By our assumptions on $\mathfrak{B}(X, Y)$, the map $hoFog : \Delta_n \to 2^{\Delta_n}$ has a fixed point $t_0 \in hoFog(t_0)$. Hence if $x_0 = g(t_0)$, then there exists $y_0 \in F(x_0)$ where $t_0 = h(y_0)$. But $x_0 = g(t_0) = goh(y_0) \in T(y_0)$ and so the proof is complete.

As a consequence of the above theorem we obtain the following result which improves corollary 3.1 of Ding and Tarafdar [6] and corollary 2.2 of Tarafdar [26].

Corollary 2.3. *In theorem 2.2 if we replace F by any continuous mapping $f : X \to Y$, then there is a point $x_0 \in X$ such that $x_0 \in T(f(x_0))$.*

In the case when $X = Y$ and $f = I$ is the identity mapping on X, corollary 2.3 reduces to the following corollary which contains corollary 2.3 of Tarafdar [26], lemma 2 of Wu [28] and particular case of theorem 6.4 of Park [20] for paracompact C-spaces.

Corollary 2.4. *Let $(X, D; \Gamma)$ be a normal G-convex space. Suppose that $S : X \to 2^D$ and $T : X \to 2^X$ are satisfied the conditions of theorem 2.2, then T has a fixed point.*

In the following lemmas we show the richness of the space $\mathfrak{K}(X, Y)$ of all multivalued mappings with the KKM property. The first lemma is an analogous result of lemma 3.1 of Ding [4] and part (2) of lemma 2 of Park [22].

Lemma 2.5. *Let $(X, D; \Gamma)$ be a G-convex space and Y be a topological space. Then those elements $F \in \mathfrak{B}(X, Y)$ such that for any $A \in \mathcal{F}(D)$ with $|A| = n + 1$, the set $F(\Phi_A(\Delta_N))$ in its induced topology is a normal space, belong to $\mathfrak{K}(X, Y)$.*

In particular case, any element $F \in \mathfrak{B}(X, Y)$ for which F is u.s.c. and compact values, belongs to $\mathfrak{K}(X, Y)$.

Proof. Assume that $F \in \mathfrak{B}(X, Y)$ and $G : D \to 2^Y$ is a generalized KKM mapping with respect to F such that $G(x)$ is closed for each $x \in D$. If the family $\{G(x) : x \in D\}$ does not have the finite intersection property, then there exists a finite subset $A = \{x_0, x_1, ..., x_n\}$ of D such that $\bigcap_{i=0}^n G(x_i) = \emptyset$. Thus $F(\Phi_A(\Delta_n)) \subseteq \bigcup_{i=0}^n V_i$, where $V_i = F(\Phi_A(\Delta_n)) \setminus G(x_i)$. Since $F(\Phi_A(\Delta_n))$ is normal space, then there exists a partition of unity $\{h_i : i = 0, ..., n\}$ subordinated to the open cover $\{V_i : i = 0, ..., n\}$. Define $h : F(\Phi_A(\Delta_n)) \to \Delta_n$ as $h(y) = \sum_{i=0}^n h_i(y)e_i$, then by our assumptions on $\mathfrak{B}(X, Y)$, the map $hoFo\Phi_A : \Delta_n \to 2^{\Delta_n}$ has a fixed point $t_0 \in hoFo\Phi_A(t_0)$. So $h^{-1}(t_0) \bigcap Fo\Phi_A(t_0) \neq \emptyset$. If $y \in h^{-1}(t_0) \bigcap Fo\Phi_A(t_0)$ and $J_y = \{i : h_i(y) \neq 0\}$, then $i \in J_y$ if only if $y \in V_i$ and so $y \in \bigcap_{i \in J_y} V_i$. But $y \in Fo\Phi_A(\Delta_{J_y}) \subseteq \bigcup_{i \in J_y} G(x_i)$, which is a contradiction.

As a consequence of the above lemma we obtain the following result.

Lemma 2.6. *Let $(X, D; \Gamma)$ be a G-convex space, Y be a topological space and $F : X \to 2^Y$ be an u.s.c. with acyclic compact values, then $F \in \mathfrak{K}(X, Y)$.*

Lemma 2.7. *Let $(X, D; \Gamma)$ be an LG-space and $F : X \to 2^X$ be u.s.c. G-convex compact values, then $F \in \mathfrak{K}(X, X)$.*

Proof. Suppose that $A = \{x_0, x_1, ... x_n\}$ is a finite subset of D. Then $Fo\Phi_A : \Delta_n \to 2^X$ is u.s.c. and G-convex values hence by proposition 3.9 of [1] $Fo\Phi_A \in \mathbf{A}(\Delta_n, X)$. Now, since Δ_n is compact by lemma 2.4 of [1] the composition $hoFo\Phi_A : \Delta_n \to 2^{\Delta_n}$ is approachable for any continuous map $h : F(\Phi_A(\Delta_n)) \to \Delta_n$. Lemma 4.1 of [1] implies that this composition map has a fixed point, therefore $F \in \mathfrak{B}(X, X)$. Thus by lemma 2.5, we have $F \in \mathfrak{K}(X, X)$.

Remark. The proof of the above lemma also shows that if $F \in \mathbf{A}(X, X)$ is u.s.c. and compact, then $F \in \mathfrak{B}(X, Y)$, and therefore it belongs to $\mathfrak{K}(X, X)$.

In the following theorem we establish an almost fixed point theorem for a multivalued map which is a similar result to theorem 3.1 of Ding [4], theorem 2 of Park [22] and the first part of theorem 5.1 of Park [23] in our context.

Theorem 2.8. *Let $(X, D; \Gamma)$ be an LG-space, Y be a compact subset of X and $F \in \mathfrak{K}(X, Y)$ with G-convex values. Then for each $U \in \mathcal{U}$ there exists $x_U \in X$ such that $F(x_U) \cap U[x_U] \neq \emptyset$.*

Proof. Since Y is compact and D is dense in X, there exists $A = \{x_0, x_1, ..., x_n\} \in \mathcal{F}(D)$ such that $Y \subseteq \bigcup_{x \in A} U[x]$. Now define a mapping $G : D \to 2^Y$ by $G(x) = Y \setminus U[x]$, for all $x \in X$, then $G(x)$ is closed for each $x \in X$ and $\cap_{x \in A} G(x) = \emptyset$. Hence, G is not a generalized KKM mapping with respect to F. Therefore there exists $B = \{x_0, x_1, ..., x_n\} \subseteq A$ such that $F(\Gamma(B)) \not\subseteq \bigcup_{i=0}^n G(x_i)$. So there exists $y' \in F(\Gamma(B))$ such that $y' \notin \bigcup_{i=0}^n G(x_i)$. From the definition of G it follows that $y' \in U[x_i]$ for all $i \in \{0, 1, ..., n\}$. If $y' \in F(x')$ for some $x' \in \Gamma(B))$, then $B \subseteq \{x \in X : F(x') \cap U[x] \neq \emptyset\}$. Therefore, $F(x') \cap U[x'] \neq \emptyset$.

Theorem 2.9. *Let $(X, D; \Gamma)$ be an LG-space and $F \in \mathfrak{K}(X, X)$ with G-convex values. Suppose that $T : X \to 2^X$ is closed values, u.s.c. and compact and $F(x) \subseteq T(x)$ for each $x \in X$, then T has a fixed point.*

Proof. By the above theorem for $Y = clT(X)$ and for each $U \in \mathcal{U}$ there exists $x_U \in X$ such that $F(x_U) \cap U[x_U] \neq \emptyset$ and so $T(x_U) \cap U[x_U] \neq \emptyset$. If $y_U \in T(x_U) \cap U[x_U]$, then since T is compact and closed we can assume there is a point $\hat{x} \in X$ such that x_U and y_U are convergent to $\hat{x}$ and $\hat{x} \in T(\hat{x})$.

Remark. In the case when for each $x \in X$, $\{x\}$ is G-convex, or equivalently, since X is Hausdorff, for each $x \in X$, and each $U \in \mathcal{U}$, $U[x]$ is G-convex, the condition of G-convexity values of F in theorems 2.8 and 2.9 is not necessary. Hence theorem 2.9 slightly improves theorems 5.2 and 5.4 of Park [18]. Moreover by the remark which follows lemma 2.7, theorem 2.9 implies also corollary 4.4 of Ben-El-Mechaiekh *et al.* [1]. Therefore, we have the following corollary which improves also theorem 2 of Chang and Yen for locally convex spaces [2].

Corollary 2.10. *Let $(X, D; \Gamma)$ be an LG-space and for each $x \in X$, $\{x\}$ be G-convex. Then any $T \in \mathfrak{K}(X, X)$ which is compact, closed values and u.s.c. has a fixed point.*

From lemma 2.7 and theorem 2.9 we can conclude the main result of Park [21] and therefore, the main result of Watson [27].

Corollary 2.11. *Let $(X, D; \Gamma)$ be an LG-space and $T : X \to 2^X$ be an u.s.c. closed values, G-convex values and compact, then T has a fixed point.*

Remark. In the above corollary instead of LG-space and G-convexity values of T, we can assume that $(X; \Gamma)$ is a G-convex space provided with a uniform structure $\mathcal{U}$ such that for each $U \in \mathcal{U}$ and $x \in X$ the set $\{y \in X : T(x) \cap U[y] \neq \emptyset\}$ is G-convex. In this case we obtain a refinement of theorem 2.3 of Hou [12].

From lemma 2.6, theorem 2.9 and corollary 2.10 we obtain the main result of Yuan [32].

Corollary 2.12. *Let $(X, D; \Gamma)$ be an LG-space such that for each $x \in X$, $\{x\}$ is G-convex. Suppose that $T : X \to 2^X$ is an u.s.c., compact, closed and acyclic values, then T has a fixed point.*

By proposition 2.1 and corollary 2.10 we conclude the following result which improves corollary 4.7 of Ben-El-Mechaiekh *et al.* [1], theorem 3.1 of Kirk, Sims and Yuan [15] and theorem 6 of Park [21].

Corollary 2.13. *Let $(X, D; \Gamma)$ be a normal LG-space such that for each $x \in X$, $\{x\}$ is G-convex. Suppose that $S : X \to 2^D$ and $T : X \to 2^X$ are two multivalued mappings such that T is compact and all of the conditions of the proposition 2.1 hold. Then T has a fixed point.*
Proof. By proposition 2.1, T has a continuous selection $f : X \to X$. But $clf(X) \subseteq clT(X)$ and T is compact, therefore f is also compact. Hence corollary 2.10 implies that f has a fixed point $x_0 = f(x_0) \in T(x_0)$.

When our multivalued mapping is l.s.c., we have the following fixed point theorem in LG-spaces which contains the second part of theorem 5.1 of Park [23].

Theorem 2.14. *Let $(X, D; \Gamma)$ be an LG-space and Y be a compact space. Suppose that $S : Y \to 2^X$ is l.s.c. and G-convex values, $F \in \mathfrak{B}(X, Y)$ and $T : X \to 2^X$ is u.s.c., compact with closed values such that $SoF \subseteq T$, then T has a fixed point.*
Proof. Suppose that $U \in \mathcal{U}$, since Y is compact and S is l.s.c., then there is a finite subset $A = \{x_0, x_1, ..., x_n\}$ of D such that $Y \subseteq \bigcup_{i=0}^n S^-(U[x_i])$. Assume that $\{h_i : i = 0, 1, ..., n\}$ is a partition of unity subordinated to $\{S^-(U[x_i]) : i = 0, 1, ..., n\}$, $h : Y \to \Delta_n$ is defined by $h(y) = \sum_{i=0}^n h_i(y)e_i$ and $\Phi_A : \Delta_n \to \Gamma(A)$. Therefore, the map $hoFo\Phi_A : \Delta_n \to 2^{\Delta_n}$ has a fixed point $t_0 \in \Delta_n$. Suppose that $\Phi_A(t_0) = x_U$, then since $t_0 \in hoF(x_U)$ there exists $y \in F(x_U)$ such that $h(y) = t_0$. If $J_y = \{i : h_i(y) \neq 0\}$ and $i \in J_y$, then $y \in S^-(U[x_i])$. Hence $\{x_i : i \in J_y\} \subseteq \{x : S(y) \cap U[x] \neq \emptyset\}$. Since $S(y)$ is G-convex, then $S(y) \cap U[\Phi_A oh(y)] \neq \emptyset$. But $\Phi_A oh(y) = \Phi_A(t_0) = x_U$ and $S(y) \subseteq SoF(x_U)$, therefore $SoF(x_U) \cap U[x_U] \neq \emptyset$. So, for each element U, there exist $x_U, y_U \in X$ such that $y_U \in S(x_U)$ and $y_U \in U[x_U]$. Since $clT(X)$ is compact

and T is u.s.c. with closed values, then we can conclude x_U, y_U have subnet converge to some $\hat{x} \in clT(X)$ and $\hat{x} \in T(\hat{x})$. This completes our proof.

As a consequence of the above theorem we obtain an analogous result to coincidence theorem 7 of Wu [28] and Eilenberg-Montgomery and Gorniéwicz 's theorem [25, lemma 1].

Corollary 2.15. *Let $(X, D; \Gamma)$ be an LG-space, $\{x\}$ be G-convex for each $x \in X$ and Y be a compact space. Suppose that $f : Y \to X$ is continuous map and $F \in \mathfrak{B}(X, Y)$ is u.s.c. and closed values, then the multivalued maps foF and Fof have fixed points.*
Proof. It is enough in theorem 2.14, to set $S = f$ and $T = foF$.

By a similar proof as that it was given by Chen and Yen[2 , Proposition 3(ii)], we can obtain the following lemma.

Lemma 2.16. Let $(X, D; \Gamma)$ be a G-convex space and Y, Z topological spaces. If $T \in \mathfrak{K}(X, Y)$ and if $f : Y \to Z$ is continuous, then $fT \in \mathfrak{K}(X, Z)$.

Now by using lemma 2.16 and corollary 2.10 we obtain a refinement of corollary 2.15.

Corollary 2.17. Let $(X, D; \Gamma)$ be an LG-space such that for each $x \in X$, $\{x\}$ is G-convex, and Y be a compact space. Suppose that $f : Y \to X$ is continuous map and $T \in \mathfrak{K}(X, Y)$ is u.s.c. and closed values, then the multivalued maps foT and Tof have fixed points.

As another application of our fixed point theorems we have the following coincidence point theorem which is similar to theorems 2.3 and 2.4 of Wu et al. [30] and refines our theorem 2.2 in LG-spaces.

Theorem 2.18. Let Y be a topological space and let $Z \subset Y$ be a nonempty compact subset. Suppose that $(X, D; \Gamma)$ is an LG-space such that for each $x \in X$, $\{x\}$ is G-convex and $T : Y \to 2^X$ has a continuous selection. If $F \in \mathfrak{K}(X, Z)$ is u.s.c., then there is a point $z_0 \in Z$ and a point $x_0 \in X$ such that $z_0 \in F(x_0)$ and $x_0 \in T(z_0)$.
Proof. Suppose that f is a continuous selection of T. Then by corollary 2.17 foF has a fixed point $x_0 \in X$. Hence, there exists $z_0 \in F(x_0)$ such that $x_0 = f(z_0) \in T(z_0)$.

Remark. If T satisfies the conditions of theorem 2.2, we have a continuous selection for T.

In order to obtain our result for Ψ-pseudocondensing map, we need the following lemma. The proof of this lemma is similar to that for topological vector spaces, see for example lemma 3.1 of I. S. Kim et al. [14] and therefore, it is omitted.

Lemma 2.19. Let $(X, D; \Gamma)$ be a G-convex space such that $D \subset X$, and Z be a closed G-convex subset of X. If $T : Z \to 2^Z$ is Ψ-pseudocondensing map, then there exists a nonempty compact G-convex subset $K \subset Z$ such that $T(K) \subset K$.

The following theorem is a analogous result to theorem 3.2 of I. S. Kim et al. [14] and theorem 3 of Lin and Yu [16] for LG-spaces.

Theorem 2.20. Let $(X, D; \Gamma)$ be an LG-space such that for each $x \in X$, $\{x\}$ is G-convex. Then any Ψ-psedocondensing maps $T \in \mathfrak{K}(X, X)$ which is u.s.c. and closed valued has a fixed point.
Proof. By lemma 2.19, there is a nonempty compact G-convex subset K of X such that $T(K) \subset K$. Since $T \in \mathfrak{K}(X, X)$, it is easy to see that $T|_K \in \mathfrak{K}(K, K)$ is closed and compact. Hence an appeal to corollary 2.10 completes the proof.

3 Generalized quasi-variational inequalities

In this section some applications of our results in the previous section for obtaining a solution of quasi-variational inequalities are given. As a consequence of the theorem 2.2, we obtain the following result that is similar to theorem 7.1 of Granas and Liu [8], theorem 10 of Ha [9] and theorem 3.3 of Yuan [31] for G-convex spaces. Those theorems improve the well known Fan's minimax inequality [7].

Theorem 3.1. *Let $(X; \Gamma)$ be a G-convex space, Y be a normal topological space and $F \in \mathfrak{B}(X, Y)$. Suppose that $\varphi, \psi : X \times Y \to \mathbf{R}$ are two real valued bifunctions such that:*

(1) $\varphi(x, y)$ is G-quasiconvex in x,

(2) $\psi(x, y)$ is λ-transfer u.s.c. in y for every λ,

(3) $\varphi(x, y) \leq \psi(x, y)$ for each $(x, y) \in X \times Y$,

(4) there is a nonempty compact subset K of Y and a finite subset $M \in \mathcal{F}(X)$ such that for every $y \in Y \backslash K$ there exists $x \in M$ such that $\psi(x, y) < \inf_{x \in X, y \in F(x)} \varphi(x, y)$.

Then there is an $y_0 \in Y$ such that

$$\inf_{x \in X, y \in F(x)} \varphi(x, y) \leq \psi(x, y_0)$$

for all $x \in X$.
Proof. Let $\inf_{x \in X, y \in F(x)} \varphi(x, y) = \alpha$. Assume the conclusion is false, then for each $y \in Y$ there exists $x \in X$ such that $\psi(x, y) < \alpha$. Define $S, T : Y \to 2^X$ by

$$S(y) = \{x \in X : \psi(x, y) < \alpha\} \quad \text{and} \quad T(y) = \{x \in X : \varphi(x, y) < \alpha\}$$

for each $y \in Y$. By our assumption $S(y) \neq \emptyset$, condition (3) implies that $S(y) \subset T(y)$ and by condition (1), $T(y)$ is G-convex. Moreover, condition (2) implies the condition (3) of proposition 2.1 holds. Therefore, by theorem 2.2 there exists $x_0 \in X$ and $y_0 \in Y$ such that $y_0 \in F(x_0)$ and $x_0 \in T(y_0)$, hence $\varphi(x_0, y_0) < \alpha$, which is a contradiction.

As a consequence of proposition 2.1 and corollary 2.10, we obtain a solution of quasi-equilibrium problem which is similar to theorem 5 of Chen, Lin and Park [3]. In this way, we deduce a solution of quasi-variational inequalities in G-convex spaces, see; [3, 15, and 18].

Theorem 3.2. *Let $(X, D; \Gamma)$ be a paracompact and perfectly normal LG-space such that for each $x \in X$, $\{x\}$ is G-convex and the condition (4) of proposition 2.1 holds. Assume that $T : X \to 2^X$ is a multivalued mapping with G-convex values, open fibers and compact such that $\bar{T} : X \to 2^X$ is u.s.c., where $\bar{T}(x) = clT(x)$ for $x \in X$. Suppose that φ is a real valued bifunction on $X \times X$ such that*

(i) $\varphi(x, y)$ *is l.s.c. in first variable,*

(ii) $\varphi(x, y)$ *is G-quasiconcave in second variable and $\varphi(x, x) \leq 0$ for all $x \in X$.*

Then there exists $\hat{x} \in \bar{T}(\hat{x})$ such that $\varphi(\hat{x}, x) \leq 0$ for all $x \in T(\hat{x})$.

Proof. Suppose that Y is fixed points of $\bar{T}$, then by corollary 2.10 Y is nonempty. Assume $S(x) = \{y \in X : \varphi(x, y) > 0\}$ for all $x \in X$ and $U = \{x \in X : S(x) \cap T(x) \neq \emptyset\}$. If there exists $\hat{x} \in Y \setminus U$, then the theorem is proved. Now let $Y \subseteq U$ and $F(x) = T(x) \cap S(x)$, then $Y \subseteq U = \bigcup_{x \in X} F^-(x)$. Since T has open fibers and $x \mapsto \varphi(x, y)$ is l.s.c., then F has open fibers. Moreover since T is G-convex values and $y \mapsto \varphi(x, y)$ is G-quasiconcave for each $x \in X$, F has G-convex values and since X is perfectly normal and paracompact, U is paracompact. Hence, we can apply proposition 2.1 to assure that there exists a continuous selection $f : U \to X$ such that $f(x) \in F(x)$ for all $x \in U$. Define the multivalued mapping $G : X \to 2^X$ by

$$G(x) = \begin{cases} f(x) & \text{if } x \in U, \\ \bar{T}(x) & \text{if } x \in X \setminus U. \end{cases}$$

Then G is u.s.c. and $G(x)$ is G-convex for each $x \in X$. Moreover since $G(x) \subseteq T(x)$ for all $x \in X$, then G is compact. Therefore by corollary 2.10, G has a fixed point $\hat{x}$. Note that, if $\hat{x} \in U$, then $\hat{x} = f(\hat{x}) \in F(\hat{x})$. Hence $\varphi(\hat{x}, \hat{x}) > 0$ which contradicts condition (ii). If $\hat{x} \in X \setminus U$, then $\hat{x} \in \bar{T}(\hat{x})$. Therefore $\hat{x} \in Y$, which is a contradiction. So the proof is complete.

Corollary 3.3. *Let $(X, D; \Gamma)$ be a G-convex space and T be the same as in theorem 3.2. Suppose that Y is a topological space, $H : X \to 2^Y$ a multivalued mapping having a continuous selection $h : X \to Y$ and $\psi : X \times Y \times X :\to \mathbf{R}$. If the following conditions are fulfilled:*

(i) $\psi(x, y, z)$ *is l.s.c. in (x, y) and is G-quasiconcave in z,*

(ii) *for each $x \in X$ and each $y \in T(x)$, $\psi(x, y, x) \leq 0$,*

then there exist $\hat{x} \in \bar{T}(\hat{x})$ and $\hat{y} \in H(\hat{x})$ such that

$$\psi(\hat{x}, \hat{y}, x) \leq 0 \qquad\qquad\qquad \forall x \in T(\hat{x}).$$

Proof. Put $\varphi(x, y) = \psi(x, h(x), y)$ for all $(x, y) \in X \times X$ and apply the above theorem.

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