Rational approximation in Orlicz spaces on Carleson curves

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Abstract

We define some subclasses of Orlicz spaces of functions and establish here a direct theorem of the approximation theory by rational functions.

1 Introduction and main results

Let Γ be a rectifiable Jordan curve in the complex plane $\mathbf{C}$ and let $G := Int\Gamma$, $G^{-} := Ext\Gamma$. Without loss of generality we suppose that $0 \in G$. Further let $T := \{w \in \mathbf{C} : |w| = 1\}$, $U := IntT$, $U^{-} := ExtT$. We denote by φ and φ_1 the conformal mappings of G^{-} and G onto U^{-} normalized by the conditions

$$\varphi(\infty) = \infty, \ \lim_{z\to\infty} \frac{\varphi(z)}{z} > 0$$

and

$$\varphi_1(0) = \infty, \ \lim_{z\to 0} \ z\varphi_1(z) > 0$$

respectively and let ψ and ψ_1 be the inverse mappings of φ and φ_1.

Let also $L_p(\Gamma)$ and $E_p(G)$ $(1 \leq p < \infty)$ be the Lebesgue space of measurable complex valued functions on Γ and the Smirnov class of analytic functions in G respectively. Since Γ is rectifiable, we have $\varphi' \in E_1(G^{-})$, $\varphi_1' \in E_1(G)$ and ψ',

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$\psi_1' \in E_1(U^-)$, which imply that the functions φ' and φ_1' admit the nontangential limits *a. e.* on Γ belonging to $L_1(\Gamma)$, and $\psi'$and ψ_1' have nontangential limits *a. e.* on T belonging to $L_1(T)$ [8, pp. 419-453].

For $z \in \Gamma$ and $\varepsilon > 0$, we denote by $\Gamma(z,\varepsilon)$ the portion of Γ in the open disk of radius ε centered at z, i. e. $\Gamma(z,\varepsilon) := \{t \in \Gamma\colon |t-z| < \varepsilon\}$. Further let $|\Gamma(z,\varepsilon)|$ denotes the length of $\Gamma(z,\varepsilon)$.

Definition 1. Γ *is called a Carleson curve if the condition*

$$\sup_{z\in\Gamma} \sup_{\varepsilon>0} \frac{1}{\varepsilon} |\Gamma(z,\varepsilon)| < \infty$$

holds.

A convex and continuous function $M : [0,\infty) \to [0,\infty)$ for which $M(0) = 0$, $M(x) > 0$ for $x > 0$ and

$$\lim_{x\to 0} \frac{M(x)}{x} = 0, \ \lim_{x\to\infty} \frac{M(x)}{x} = \infty$$

is called an $N-$function. The complementary $N-$function of M is defined by

$$N(y) := \max_{x\geq 0} \{xy - M(x)\}$$

for $y \geq 0$.

Let M be an $N-$function and N be its complementary function. We denote by $L_M(\Gamma)$ the linear space of Lebesgue measurable functions $f : \Gamma \to \mathbf{C}$ satisfying the condition

$$\int_\Gamma M(\alpha |f(z)|) |dz| < \infty$$

for some $\alpha > 0$. $L_M(\Gamma)$ becomes a Banach space with respect to the norm

$$\|f\|_{L_M(\Gamma)} := \sup \left\{ \int_\Gamma |f(z) g(z)| |dz| : g \in L_N(\Gamma), \rho(g,N) \leq 1 \right\} \tag{1}$$

where $\rho(g,N) = \int_\Gamma N(|g(z)|) |dz|$ [17, pp. 52-68].

The norm $\|.\|_{L_M(\Gamma)}$ is called the Orlicz norm and the Banach space $L_M(\Gamma)$ is called an Orlicz space.

It is known that every function in $L_M(\Gamma)$ is integrable on Γ, i.e. $L_M(\Gamma) \subset L_1(\Gamma)$ [17, p. 50].

The $N-$function M is said to satisfy the Δ_2-condition if

$$\limsup_{x\to\infty} \frac{M(2x)}{M(x)} < \infty$$

holds.

The Orlicz space $L_M(\Gamma)$ is reflexive if and only if the $N-$function M and its complementary function N are both satisfy the Δ_2-condition [17, p. 113].

The more general information about Orlicz spaces can be found in [16] and [17].

Let Γ_r be the image of the circle $\{w \in \mathbf{C} : |w| = r, \ 0 < r < 1\}$ under some conformal mapping of U onto G and M be an $N-$function. We denote by $E_M(G)$ the class of functions f analytic in G and satisfying the condition

$$\int_{\Gamma_r} M(|f(z)|) |dz| < \infty$$

uniformly in r.

Definition 2. *[13] The class $E_M(G)$ is called the Smirnov−Orlicz class.*

If $M(x) = M(x,p) := x^p$, $1 < p < \infty$, then the Smirnov−Orlicz class $E_M(G)$ coincides with the usual Smirnov class $E_p(G)$. As was noted in [13], every function of class $E_M(G)$ has *a. e.* nontangential boundary values and the boundary function belongs to $L_M(\Gamma)$.

The class $E_M(G^-)$ can be defined similarly.

For $\varsigma \in \Gamma$ we define the points $\varsigma_h \in \Gamma$ and $\varsigma_{1h} \in \Gamma$ as

$$\varsigma_h := \psi\left[\varphi(\varsigma)e^{ih}\right], \varsigma_{1h} := \psi_1\left[\varphi_1(\varsigma)e^{ih}\right], h \in [0, 2\pi]$$

and the shifts $T_h f$ and $T_{1h} f$ for $f \in L_M(\Gamma)$ by

$$T_h f(\varsigma) := \frac{f(\varsigma_h)}{\varphi'(\varsigma_h)}\varphi'(\varsigma), \varsigma \in \Gamma \tag{2}$$

and

$$T_{1h} f(\varsigma) := \frac{f(\varsigma_{1h})}{[\varphi_1(\varsigma_{1h})]^{-2}\varphi_1'(\varsigma_{1h})}[\varphi_1(\varsigma)]^{-2}\varphi_1'(\varsigma), \varsigma \in \Gamma. \tag{3}$$

For example, if $\Gamma \equiv T$, then $T_h f(w) = f\left(we^{ih}\right)$, $T_{1h} f(w) = f\left(we^{-ih}\right)$ and hence $T_h f(w) \in L_M(\Gamma)$, $T_{1h} f(w) \in L_M(\Gamma)$ as soon as $f \in L_M(\Gamma)$. Moreover, if

$$0 < c_1 \le \left|\varphi'(z)\right| \le c_2 < \infty$$

or

$$0 < c_3 \le \left|\varphi_1'(z)\right| \le c_4 < \infty$$

for $z \in \Gamma$ and with the constants c_1, c_2, c_3, c_4, which are independent of z, then it is easy to verify that $L_M(\Gamma)$ is invariant with respect to the shifts $T_h f$ and $T_{1h} f$. Starting from this we define the functions $\omega_M^*(.,f)$, $\omega_{1M}^*(.,f)$ and $\Omega_M^*(.,f)$ for $\delta \ge 0$ as

$$\omega_M^*(\delta,f) := \sup_{|h|\le\delta}\|f - T_h f\|_{L_M(\Gamma)},$$

$$\omega_{1M}^*(\delta,f) := \sup_{|h|\le\delta}\|f - T_{1h} f\|_{L_M(\Gamma)},$$

$$\Omega_M^*(\delta,f) := \omega_M^*(\delta,f) + \omega_{1M}^*(\delta,f).$$

Let $\omega(\delta)$ be a nonnegative, continuous, nondecreasing real function such that $\omega(0) = 0$, $\omega(\delta) > 0$ for $\delta > 0$, and $\omega(n\delta) \le c_5\, n\, \omega(\delta)$ for every natural number n and with some constant $c_5 > 0$.

We define the classes of functions $H_\Gamma^\omega L_M(\Gamma)$, $H_\Gamma^\omega E_M(G)$ and $H_\Gamma^\omega E_M(G^-)$ as

$$H_\Gamma^\omega L_M(\Gamma) := \{f \in L_M(\Gamma) : \Omega_M^*(\delta,f) \le c_6\, \omega(\delta)\},$$

$$H_\Gamma^\omega E_M(G) := \{f \in E_M(G) : \omega_M^*(\delta,f) \le c_7\, \omega(\delta)\},$$

$$H_\Gamma^\omega E_M\left(G^-\right) := \left\{f \in E_M\left(G^-\right) : \omega_{1M}^*(\delta,f) \le c_8\, \omega(\delta)\right\},$$

where the constants c_6, c_7 and c_8 are independent of f and δ.

It is clear that if $f \in H_\Gamma^\omega L_M(\Gamma)$, then $T_h f \in L_M(\Gamma)$ and $T_{1h} f \in L_M(\Gamma)$.

Our main results are the following.

Theorem 1. *Let Γ be a Carleson curve, $L_M(\Gamma)$ be a reflexive Orlicz space on Γ and $f \in H_\Gamma^\omega L_M(\Gamma)$. Then for each natural number n there exists a rational function $R_n(z,f)$ such that*

$$\|f - R_n(\cdot,f)\|_{L_M(\Gamma)} \leq c\,\omega(1/n) \tag{4}$$

holds with a constant c, which is independent of n.

Corollary 1. *If $f \in H_\Gamma^\omega E_M(G)$, then for each natural number n there exists an algebraic polynomial $P_n(z,f)$ of degree $\leq n$ such that*

$$\|f - P_n(\cdot,f)\|_{L_M(\Gamma)} \leq c\,\omega(1/n) \tag{5}$$

holds with a constant c, which is independent of n.

Corollary 2. *If $f \in H_\Gamma^\omega E_M(G^-)$ then for each natural number n there exists a polynomial $B_n(1/z,f)$ of $1/z$ such that*

$$\|\ f - B_n(\cdot,f)\|_{L_M(\Gamma)} \leq c\,\omega(1/n) \tag{6}$$

holds with a constant c, which is independent of n.

Theorem 1 is new also in the spaces $L_p(\Gamma)$, $1 < p < \infty$. To the best of the authors knowledge in the literature there are no results studying the direct theorems of the approximation theory by polynomials and rational functions in the Orlicz spaces and Smirnov-Orlicz classes.

When Γ is a smooth Jordan curve and $\theta(s)$, the angle between the tangent and the positive real axis expressed as a function of arclength s, has modulus of continuity $\Omega(s,\theta)$ satisfying the Dini-smooth condition

$$\int_0^\delta \frac{\Omega(s,\theta)}{s} ds < \infty,\ \delta > 0$$

some inverse problems of the approximation theory in the Smirnov-Orlicz classes were investigated by Kokilashvili [13].

Under different restrictive conditions upon $\Gamma = \partial G$ the similar problems in $L_p(\Gamma)$ and $E_p(G)$,$1 \leq p < \infty$, spaces were studied in [1], [2], [9], [14], [6], [4], [10], [11].

Throughout this paper we shall denote by c, c_1, c_2, . . . constants depending only on numbers that are not important for the questions of interest.

2 Auxiliary results

Let Γ be a rectifiable Jordan curve and $f \in L_1(\Gamma)$. Then the functions f^+ and f^- defined by

$$f^+(z) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{\varsigma - z} d\varsigma,\ z \in G \tag{7}$$

and

$$f^-(z) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{\varsigma - z} d\varsigma,\ z \in G^- \tag{8}$$

are analytic in G and G^- respectively and $f^-(\infty) = 0$.

The Cauchy singular integral of $f \in L_1(\Gamma)$ at $z \in \Gamma$ is defined by

$$S_\Gamma(f)(z) := \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma \setminus \Gamma(z,\varepsilon)} \frac{f(\varsigma)}{\varsigma - z} d\varsigma,$$

if the limit exists.

For $f \in L_1(\Gamma)$, if one of the functions f^+ and f^- has nontangential limits *a. e.* on Γ, then $S_\Gamma(f)(z)$ exists *a. e.* on Γ and also the other one of the functions f^+ and f^- has nontangential limits *a. e.* on Γ. Conversely, if $S_\Gamma(f)(z)$ exists *a. e.* on Γ, then the functions f^+ and f^- have nontangential limits *a. e.* on Γ. In both cases, the formulae

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2} f(z), \; f^-(z) = S_\Gamma(f)(z) - \frac{1}{2} f(z) \tag{9}$$

holds *a. e.* on Γ [8, p. 431] and hence

$$f = f^+ - f^- \tag{10}$$

a. e. on Γ.

For $f \in L_1(\Gamma)$, if $S_\Gamma(f)(z)$ exists *a. e.* on Γ, we associate the function $S_\Gamma(f)$ taking the value $S_\Gamma(f)(z)$ *a. e.* on Γ. The linear operator S_Γ defined in such way is called the Cauchy singular operator.

The following theorem, proved in [12], characterizes the curves which the singular operator S_Γ is bounded in the reflexive Orlicz space $L_M(\Gamma)$.

Theorem 2 *The singular operator S_Γ is a bounded linear operator in the reflexive Orlicz space $L_M(\Gamma)$, i. e.,*

$$\|S_\Gamma(f)\|_{L_M(\Gamma)} \le c \|f\|_{L_M(\Gamma)}, \; f \in L_M(\Gamma) \tag{11}$$

holds, where c is a constant depending only on M and Γ, if and only if Γ is a Carleson curve.

Let k be a nonnegative integer. Then the function $\varphi'(z)\varphi^k(z)$ has a pole of order k at ∞. Hence there exists a polynomial $B_k(z)$ of degree k and a function $E_k(z)$ analytic in G^- such that $E_k(\infty) = 0$ and

$$\varphi^k(z)\varphi'(z) := B_k(z) + E_k(z)$$

holds for every $z \in G^-$.

The polynomials $B_k(z)$ $(k = 0, 1, 2, ...)$ are called the Faber polynomials of the second kind for G and satisfy the expansion

$$\frac{1}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{B_k(z)}{w^{k+1}} \tag{12}$$

for $z \in G$ and $w \in U^-$ [18, p. 95].

Now let's consider the function $[\varphi_1(z)]^{k-2} \varphi_1'(z)$. This function is analytic in $G \setminus \{0\}$ and has a pole of order k at the point 0. If we denote its principal part at 0 by $\widetilde{B}_k(1/z)$, then there exists an analytic function $\widetilde{E}_k(z)$ in G such that

$$[\varphi_1(z)]^{k-2} \varphi_1'(z) = \widetilde{B}_k(1/z) + \widetilde{E}_k(z)$$

holds for every $z \in G \setminus \{0\}$ and for the principal parts $\widetilde{B}_k(1/z)$ the expansion

$$\frac{w^{-2}}{\psi_1(w) - z} = \sum_{k=0}^{\infty} - \frac{\widetilde{B}_k(1/z)}{w^{k+1}}, z \in G^-, w \in U^- \tag{13}$$

holds [4].

3 Proofs of the new results

Let $f \in L_M(\Gamma)$. Then $f \in L_1(\Gamma)$ and hence the functions

$$f_0(w) := f[\psi(w)]\,\psi'(w)$$

and

$$f_1(w) := f[\psi_1(w)]\,\psi_1'(w)\,w^2$$

are integrable on T. We can associate the series

$$f_0(w) \sim \sum_{k=-\infty}^{\infty} a_k w^k \tag{14}$$

and

$$f_1(w) \sim \sum_{k=-\infty}^{\infty} \widetilde{a}_k w^k \tag{15}$$

for $w \in T$.

Let

$$K_n(\theta) = \sum_{m=-n}^{n} \lambda_m^{(n)} e^{im\theta}$$

be an even, nonnegative trigonometric polynomial satisfying the conditions

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(\theta)\,d\theta = 1 \tag{16}$$

and

$$\int_0^{\pi} \theta K_n(\theta)\,d\theta \le c_9/n \tag{17}$$

for $n = 1, 2, ...$ and with a constant $c_9 > 0$. In special case, the Jackson kernel

$$J_n(\theta) = \frac{3\sin^4(n\theta/2)}{n(2n^2+1)\sin^4(\theta/2)}$$

satisfies these conditions[5, p. 203].

Let's consider the integral

$$I(\theta, z) := \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\varsigma_{-\theta})}{\varphi'(\varsigma_{-\theta})}\frac{\varphi'(\varsigma)}{\varsigma - z}d\varsigma, z \in G.$$

Substituting $\varsigma = \psi(e^{it})$ here, we obtain

$$I(\theta, z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_0\left(e^{i(t-\theta)}\right)\frac{e^{it}}{\psi(e^{it}) - z}dt.$$

Since by (14)

$$f_0\left(e^{it}\right) \sim \sum_{k=-\infty}^{\infty} a_k e^{ikt}$$

and by (12)

$$\frac{e^{it}}{\psi(e^{it}) - z} \sim \sum_{k=0}^{\infty} \frac{B_k(z)}{e^{ikt}}$$

we can associate [3, pp. 74-75] to $I(\theta,z)$ the series expansion, i.e.,

$$I(\theta,z)\sim\sum_{k=0}^{\infty}a_kB_k(z)\,e^{-ik\theta}.$$

Then by the generalized Parseval's identity [3, pp. 225-228]

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(\theta)\,I(\theta,z)\,d\theta=\sum_{k=0}^{n}\lambda_k^{(n)}a_kB_k(z),$$

because the function $K_n(\theta)$ is of bounded variation and $I(\cdot,z)\in L_1([-\pi,\pi])$. Hence we have

$$\frac{1}{4\pi^2 i}\int_{-\pi}^{\pi}K_n(\theta)\,d\theta\int_{\Gamma}\frac{f(\varsigma_{-\theta})}{\varphi'(\varsigma_{-\theta})}\frac{\varphi'(\varsigma)}{\varsigma-z}d\varsigma=\sum_{k=0}^{n}\lambda_k^{(n)}a_kB_k(z)$$

for $z\in G$. Now, we consider the integral

$$I_1(\theta,z):=\frac{1}{2\pi i}\int_{\Gamma}\frac{f\left(\varsigma_{1(-\theta)}\right)}{\varphi_1^{-2}\left(\varsigma_{1(-\theta)}\right)\varphi_1'\left(\varsigma_{1(-\theta)}\right)}\frac{[\varphi_1(\varsigma)]^{-2}\varphi_1'(\varsigma)}{\varsigma-z}d\varsigma,\ z\in G^-.$$

Making the change of variable $\varsigma=\psi_1(e^{it})$ we obtain

$$I_1(\theta,z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f_1\left(e^{i(t-\theta)}\right)\frac{e^{-2it}}{\psi_1(e^{it})-z}dt.$$

Similarly, according to the relations(13) and (15), the function $I_1(\theta,z)$ has the Fourier expansion

$$I_1(\theta,z)\sim-\sum_{k=0}^{\infty}\widetilde{a}_k\widetilde{B}_k(1/z)\,e^{-ik\theta}$$

by [3, pp. 74-75]. Since the kernel $K_n(\theta)$ is of bounded variation and $I_1(.,z)$ is integrable, the generalized Parseval identity [3, pp. 225-228] yields again that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(\theta)\,I_1(\theta,z)\,d\theta=-\sum_{k=0}^{n}\lambda_k^{(n)}\widetilde{a}_k\widetilde{B}_k(1/z),\ z\in G^-,$$

and by definition of $I_1(\theta,z)$ we have

$$\frac{1}{4\pi^2 i}\int_{-\pi}^{\pi}K_n(\theta)\,d\theta\int_{\Gamma}\frac{f\left(\varsigma_{1(-\theta)}\right)}{\varphi_1^{-2}\left(\varsigma_{1(-\theta)}\right)\varphi_1'\left(\varsigma_{1(-\theta)}\right)}\frac{\varphi_1^{-2}(\varsigma)\,\varphi_1'(\varsigma)}{\varsigma-z}d\varsigma$$

$$=-\sum_{k=0}^{n}\lambda_k^{(n)}\widetilde{a}_k\widetilde{B}_k(1/z),z\in G^-.$$

Therefore,

$$P_n(z,f):=\frac{1}{4\pi^2 i}\int_{-\pi}^{\pi}K_n(\theta)\,d\theta\int_{\Gamma}\frac{f(\varsigma_{-\theta})}{\varphi'(\varsigma_{-\theta})}\frac{\varphi'(\varsigma)}{\varsigma-z}d\varsigma,\ z\in G$$

is a polynomial of degree n and

$$Q_n(z,f):=\frac{1}{4\pi^2 i}\int_{-\pi}^{\pi}K_n(\theta)\,d\theta\int_{\Gamma}\frac{f\left(\varsigma_{1(-\theta)}\right)}{\varphi_1^{-2}\left(\varsigma_{1(-\theta)}\right)\varphi_1'\left(\varsigma_{1(-\theta)}\right)}\frac{\varphi_1^{-2}(\varsigma)\,\varphi_1'(\varsigma)}{\varsigma-z}d\varsigma,\ z\in G^-$$

is a polynomial of degree n of $1/z$.

Since the kernel $K_n(\theta)$ is an even function we have

$$P_n(z,f)=\frac{1}{4\pi^2 i}\int_0^{\pi}K_n(\theta)\,d\theta\int_\Gamma\left[\frac{f(\varsigma_\theta)}{\varphi'(\varsigma_\theta)}+\frac{f(\varsigma_{-\theta})}{\varphi'(\varsigma_{-\theta})}\right]\frac{\varphi'(\varsigma)}{\varsigma-z}d\varsigma$$

and

$$Q_n(z,f)=$$

$$\frac{1}{4\pi^2 i}\int_0^{\pi}K_n(\theta)\,d\theta\int_\Gamma\left[\frac{f(\varsigma_{1\theta})}{\varphi_1^{-2}(\varsigma_{1\theta})\,\varphi_1'(\varsigma_{1\theta})}+\frac{f\left(\varsigma_{1(-\theta)}\right)}{\varphi_1^{-2}\left(\varsigma_{1(-\theta)}\right)\varphi_1'\left(\varsigma_{1(-\theta)}\right)}\right]\frac{\varphi_1^{-2}(\varsigma)\,\varphi_1'(\varsigma)}{\varsigma-z}d\varsigma$$

for $z\in G$ and $z\in G^-$ respectively. Then by (2) and (3) we obtain

$$P_n(z,f)=\frac{1}{4\pi^2 i}\int_0^{\pi}K_n(\theta)\,d\theta\int_\Gamma\left[T_\theta f(\varsigma)+T_{(-\theta)}f(\varsigma)\right]\frac{d\varsigma}{\varsigma-z},\ z\in G$$

and

$$Q_n(z,f)=\frac{1}{4\pi^2 i}\int_0^{\pi}K_n(\theta)\,d\theta\int_\Gamma\left[T_{1\theta}f(\varsigma)+T_{1(-\theta)}f(\varsigma)\right]\frac{d\varsigma}{\varsigma-z},\ z\in G^-.$$

Taking the relations (7) and (8) into account we finally get

$$P_n(z,f)=\frac{1}{2\pi}\int_0^{\pi}K_n(\theta)\left[(T_\theta f)^+(z)+\left(T_{(-\theta)}f\right)^+(z)\right]d\theta,\ \ z\in G \tag{18}$$

and

$$Q_n(z,f)=\frac{1}{2\pi}\int_0^{\pi}K_n(\theta)\left[(T_{1\theta}f)^-(z)+\left(T_{1(-\theta)}f\right)^-(z)\right]d\theta,\ \ z\in G^-. \tag{19}$$

Proof of Theorem 1. Let $f\in H_\Gamma^\omega L_M(\Gamma)$. By (16) for $z'\in G$ we have

$$f^+\left(z'\right):=\frac{1}{2\pi}\int_{-\pi}^{\pi}f^+\left(z'\right)K_n(\theta)\,d\theta=\frac{1}{2\pi}\int_0^{\pi}2f^+\left(z'\right)K_n(\theta)\,d\theta,$$

which together with (18) implies that

$$f^+\left(z'\right)-P_n\left(z',f\right)=\frac{1}{2\pi}\int_0^{\pi}K_n(\theta)\left\{2f^+\left(z'\right)-\left[(T_\theta f)^+\left(z'\right)+\left(T_{(-\theta)}f\right)^+\left(z'\right)\right]\right\}d\theta.$$

Limiting $z'\to z\in\Gamma$, along all nontangential paths inside Γ, by (9) we have

$$\begin{aligned}f^+(z)-P_n(z,f)\ &=\ \frac{1}{2\pi}\int_0^{\pi}K_n(\theta)\left[S_\Gamma\left(f-(T_\theta f)\right)(z)+S_\Gamma\left(f-\left(T_{(-\theta)}f\right)\right)(z)\right]d\theta\\&\quad+\frac{1}{4\pi}\int_0^{\pi}K_n(\theta)\left[\left(f-(T_\theta f)\right)(z)+\left(f-\left(T_{(-\theta)}f\right)\right)(z)\right]d\theta\end{aligned}$$

for almost all $z\in\Gamma$. Now using the norm (1) and later applying the Fubini theorem and getting the supremum in the integral sign we obtain

$$\left\|f^+-P_n(.,f)\right\|_{L_M(\Gamma)}=\sup\int_\Gamma\left|f^+(z)-P_n(z,f)\right||g(z)||dz|$$

$$\leq \sup \int_\Gamma \left| \frac{1}{2\pi} \int_0^\pi K_n(\theta) \left[S_\Gamma (f - T_\theta f)(z) + S_\Gamma (f - T_{(-\theta)} f)(z) \right] d\theta \right| | g(z) || dz |$$

$$+ \sup \int_\Gamma \left| \frac{1}{4\pi} \int_0^\pi K_n(\theta) \left[(f - T_\theta f)(z) + (f - T_{(-\theta)} f)(z) \right] d\theta \right| | g(z) || dz |$$

$$\leq \sup \int_\Gamma \left\{ \frac{1}{2\pi} \int_0^\pi K_n(\theta) \left[| S_\Gamma (f - T_\theta f)(z) | + | S_\Gamma (f - T_{(-\theta)} f)(z) | \right] d\theta \right\} | g(z) || dz |$$

$$+ \sup \int_\Gamma \left\{ \frac{1}{4\pi} \int_0^\pi K_n(\theta) \left[| (f - T_\theta f)(z) | + | (f - T_{(-\theta)} f)(z) | \right] d\theta \right\} | g(z) || dz |$$

$$\leq \frac{1}{2\pi} \int_0^\pi K_n(\theta) \left\{ \sup \int_\Gamma \left[| S_\Gamma (f - T_\theta f)(z) | + | S_\Gamma (f - T_{(-\theta)} f)(z) | \right] | g(z) || dz | \right\} d\theta$$

$$+ \frac{1}{4\pi} \int_0^\pi K_n(\theta) \left\{ \sup \int_\Gamma \left[| (f - T_\theta f)(z) | + | (f - T_{(-\theta)} f)(z) | \right] | g(z) || dz | \right\} d\theta$$

$$\leq \frac{1}{2\pi} \int_0^\pi K_n(\theta) \left[\| S_\Gamma (f - T_\theta f) \|_{L_M(\Gamma)} + \left\| S_\Gamma \left(f - T_{(-\theta)} f \right) \right\|_{L_M(\Gamma)} \right] d\theta$$

$$+ \frac{1}{4\pi} \int_0^\pi K_n(\theta) \left[\| f - T_\theta f \|_{L_M(\Gamma)} + \left\| f - T_{(-\theta)} f \right\|_{L_M(\Gamma)} \right] d\theta,$$

where the supremums in the above are taken over all functions $g \in L_N(\Gamma)$, with $\rho(g, N) \leq 1$. By virtue of (11) from this we conclude that

$$\left\| f^+ - P_n(., f) \right\|_{L_M(\Gamma)} \leq c_{10} \int_0^\pi K_n(\theta) \left\{ \| f - T_\theta f \|_{L_M(\Gamma)} + \left\| f - T_{(-\theta)} f \right\|_{L_M(\Gamma)} \right\} d\theta,$$

and then by definition of $\omega_M^*(\cdot, f)$, we have

$$\left\| f^+ - P_n(\cdot, f) \right\|_{L_M(\Gamma)} \leq c_{11} \int_0^\pi K_n(\theta) \, \omega_M^*(\theta, f) \, d\theta. \tag{20}$$

Similarly, for $z' \in G^-$ we obtain

$$f^-\left(z'\right) - Q_n(z', f) = \frac{1}{2\pi} \int_0^\pi K_n(\theta) \left\{ 2 f^-\left(z'\right) - \left[(T_{1\theta} f)^-\left(z'\right) + \left(T_{1(-\theta)} f\right)^-\left(z'\right) \right] \right\} d\theta.$$

Here letting $z' \to z \in \Gamma$ along all nontangential paths outside Γ, by (9) we get

$$\begin{aligned} f^-(z) - Q_n(z, f) &= \frac{1}{2\pi} \int_0^\pi K_n(\theta) \left[S_\Gamma (f - T_{1\theta} f)(z) + S_\Gamma \left(f - T_{1(-\theta)} f \right)(z) \right] d\theta \\ &\quad + \frac{1}{4\pi} \int_0^\pi K_n(\theta) \left[(T_{1\theta} f - f)(z) + \left(T_{1(-\theta)} f - f \right)(z) \right] d\theta \end{aligned}$$

for almost all $z \in \Gamma$. Therefore,

$$\left\| f^- - Q_n(\cdot, f) \right\|_{L_M(\Gamma)} \leq c_{12} \int_0^\pi K_n(\theta) \, d\theta \left\{ \| f - T_{1\theta} f \|_{L_M(\Gamma)} + \left\| f - T_{1(-\theta)} f \right\|_{L_M(\Gamma)} \right\}$$

and by definition of $\omega_{1M}^*(\cdot, f)$ we obtain

$$\left\| f^- - Q_n(\cdot, f) \right\|_{L_M(\Gamma)} \leq c_{13} \int_0^\pi K_n(\theta) \, \omega_{1M}^*(\theta, f) \, d\theta. \tag{21}$$

If we set $R_n(z,f) := P_n(z,f) - Q_n(z,f)$, then by(10), (20), (21) and by definition of $\Omega_M^*(\cdot,f)$ we get

$$\begin{aligned}\|f - R_n(\cdot,f)\|_{L_M(\Gamma)} &\leq \left\|f^+ - P_n(\cdot,f)\right\|_{L_M(\Gamma)} + \left\|f^- - Q_n(\cdot,f)\right\|_{L_M(\Gamma)} \\ &\leq c_{14}\int_0^\pi K_n(\theta)\,\Omega_M^*(\theta,f)\,d\theta \\ &\leq c_{15}\int_0^\pi K_n(\theta)\,\omega(\theta)\,d\theta \\ &= c_{15}\int_0^\pi K_n(\theta)\,\omega(n\theta/n)\,d\theta \\ &\leq c_{16}\omega(1/n)\int_0^\pi K_n(\theta)(n\theta+1)\,d\theta.\end{aligned}$$

This relation and (17) gives (4).

Proof of Corollary 1. Let $f \in H_\Gamma^\omega E_M(G)$. Let's take $z' \in G^-$. Since $f \in E_M(G) \subset E_1(G)$ we have by the Cauchy theorem

$$f^-\left(z'\right) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{\varsigma - z'}d\varsigma = 0.$$

So $f^-(z) = 0$ for almost all $z \in \Gamma$ and hence $f = f^+$ *a. e.* on Γ. By (20) we have

$$\begin{aligned}\|f - P_n(.;f)\|_{L_M(\Gamma)} &\leq c_{17}\int_0^\pi K_n(\theta)\,\omega_M^*(\theta;f)\,d\theta \\ &\leq c_{18}\int_0^\pi K_n(\theta)\,\omega(\theta)\,d\theta \\ &\leq c_{19}\omega(1/n)\end{aligned}$$

and hence (5) is proved.

Proof of Corollary 2. Let $f \in H_\Gamma^\omega E_M(G^-)$ and $z' \in G$. Then by the Cauchy formula we have

$$f^+\left(z'\right) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{\varsigma - z'}d\varsigma = f(\infty).$$

Hence $f^+(z) = f(\infty)$ *a. e.* on Γ and by (9) we have $f = f(\infty) - f^-$ *a. e.* on Γ. Now, setting $B_n(1/z,f) := f(\infty) - Q_n(1/z,f)$ and applying the relation (21) we conclude that

$$\begin{aligned}\|\, f - B_n(\cdot,\ f)\|_{L_M(\Gamma)} &\leq c_{20}\int_0^\pi K_n(\theta)\,\omega_{1M}^*(\theta;f)\,d\theta \\ &\leq c_{21}\int_0^\pi K_n(\theta)\,\omega(\theta)\,d\theta \\ &\leq c\omega(1/n),\end{aligned}$$

and the proof is completed.

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