Modules of solutions of the Helmholtz equation arising from eigenfunctions of the Dirac operator

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Abstract

We study modules of solutions of the equation $DF = F$, where F is a function in the plane with values in the quaternions and D is the Dirac operator. The functions F will belong to the Sobolev-type space of all functions in $L^p(\Omega, |x|^{-3}dx)$ jointly with their angular and radial derivatives, and where Ω is the complement of the unit disk in $\mathbb{R}^2$. The resulting spaces are right Banach modules over the quaternions. When $p = 2$ we calculate the reproducing kernel of this space and explain its reproducing properties when $p \neq 2$.

1 Introduction.

Much work has been done in the study of spaces of eigenfunctions of the Dirac operator D in the context of the Clifford analysis in $\mathbb{R}^n$. Some authors have studied Hilbert modules of these functions in various domains and norms, considering the representation of its elements and the construction of the corresponding reproducing kernels. For example it has been considered spaces of these functions belonging to $L^2(B)$ or $L^2(\mathbb{R}^n, e^{-|x|^2}dx)$ (see [3, 4, 20]). In [1] it was studied the space of all complex solutions of the Helmholtz equation

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$$\Delta u + u = 0 \tag{1}$$

in $\mathbb{R}^2$ belonging to the Sobolev-type space of all functions in $L^p(\Omega, |x|^{-3}dx)$ jointly with their angular and radial derivatives, and where Ω here and in the rest of the paper is the complement of the unit disk in $\mathbb{R}^2$. When $p = 2$, this space consists precisely of all Herglotz wave functions in the plane, namely, the image of the Fourier transform of L^2 densities in the unit circle, which coincides with the space of solutions of (1) with far-field-pattern in L^2 of the unit circle (see [8, 9, 11]).
Motivated by the decomposition of the Helmholtz operator

$$\Delta + 1 = (1 + D)(1 - D),$$

(see below the definition of D) we consider in this paper the quaternionic versions of the spaces in [1]. Then we define and study the submodules of functions satisfying

$$DF = F, \tag{2}$$

(as well as the modules of functions such that $DF = -F$). In the case $p = 2$, we will have a right submodule of quaternionic Herglotz wave functions in the plane. We find a quaternionic orthonormal basis of this module and we calculate its reproducing Bergman kernel as a submodule of the quaternionic version of Sobolev spaces described above. Finally, we consider the case $p \neq 2$. The resulting spaces are Banach right modules. We describe their basic features and study the reproducing properties of the Bergman projection on them.

We start by recalling some notation from [1]. For $1 < p < \infty$, $\mathcal{H}^p$ stands for the Banach space of all measurable complex functions u in Ω such that $u, \frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta} \in L^1_{loc}(\Omega)$ and

$$\|u\|_{\mathcal{H}^p} = \left\{ \int_\Omega \left(|u|^p + |\frac{\partial u}{\partial r}|^p + |\frac{\partial u}{\partial \theta}|^p \right) |x|^{-3} dx \right\}^{1/p} < \infty,$$

where $x = (s, t)$ has polar coordinates (r, θ),

$$\frac{\partial u}{\partial r} = \frac{s}{r}\frac{\partial u}{\partial s} + \frac{t}{r}\frac{\partial u}{\partial t},$$
$$\frac{\partial u}{\partial \theta} = -t\frac{\partial u}{\partial s} + s\frac{\partial u}{\partial t},$$

(the derivatives are taken in $\mathcal{D}'(\Omega)$) and $dx = ds\,dt$.

The Banach space of all functions $u \in \mathcal{H}^p$ which satisfy the Helmholtz equation in $\mathbb{R}^2$ is denoted by $\mathcal{W}^p$.

2 Hyperholomorphic functions and solutions of the Helmholtz equation in $\mathbb{R}^2$.

Let $\mathbb{H}$ stand for the usual real quaternions. An element $q \in \mathbb{H}$ will be written as

$$q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$$

with $q_i \in \mathbb{R}$, $i = 0, \ldots, 3$, and the quaternionic imaginary units $\{e_i\}$ satisfy the relations:

$$e_1e_2 = e_3, \;\; e_2e_3 = e_1, \; e_3e_1 = e_2, \; e_ie_j = -e_je_i.$$

We will provide $\mathbb{H}$ with the Euclidean norm $|\cdot|$ of $\mathbb{R}^4$. The conjugate $\bar{q}$ of q, is given by

$$\bar{q} = q_0 - q_1e_1 - q_2e_2 - q_3e_3$$

and $q\bar{q} = |q|^2$. We denote $\operatorname{Re} q = q_0$. $\mathbb{H}$ has the structure of a real non-commutative, associative algebra without zero divisors.

Consider now $\mathbb{H}$-valued functions defined in $\mathbb{R}^2$. On the bi-$\mathbb{H}$-module $C^2(\mathbb{R}^2, \mathbb{H})$, the two-dimensional Helmholtz operator (1) with the wave number 1, $\Delta + 1$, acts componentwise, namely, if $u \in C^2(\mathbb{R}^2, \mathbb{H})$, $u = \sum_{k=0}^{3} u_ke_k$, $(e_0 = 1)$ then

$$(\Delta + 1)[u] = \Delta[u] + u = \sum_{k=0}^{3} (\Delta[u_k] + u_k)\, e_k,$$

where $\Delta = \partial_s^2 + \partial_t^2$, $\partial_s = \frac{\partial}{\partial s}$, $\partial_t = \frac{\partial}{\partial t}$ and $(s, t) \in \mathbb{R}^2$.

Consider the following partial differential operators with quaternionic coefficients:

$$D = e_1\partial_s + e_2\partial_t \text{ and } D_r = \partial_s \circ M^{e_1} + \partial_t \circ M^{e_2}$$

called the left and the right Dirac operators; where for $a \in \mathbb{H}$ we denote M^a the operator of the multiplication by a on the right-hand side. Both operators D and D_r are square roots of the negative Laplace operator:

$$-\Delta = D^2 = D_r^2. \tag{3}$$

They are defined on the bi-$\mathbb{H}$-module $C^1(\mathbb{R}^2, \mathbb{H})$, there equality (3) holds. The operator D is right-linear while D_r is left-linear. The equality (3) implies that the following decompositions of the Helmholtz operator hold:

$$\Delta + 1 = (1 + D)(1 - D) = (1 - D)(1 + D) =$$

$$(1 + D_r)(1 - D_r) = (1 - D_r)(1 + D_r).$$

Since a similar decomposition of the Laplace operator is crucial in the definition of the usual holomorphic functions, let us consider, by analogy, the following.

Definition 2.1. *A function $u \in C^1(\mathbb{R}^2, \mathbb{H})$ is called 1-hyperholomorphic (or simply hyperholomorphic in this paper) if $(1 - D)[u] = u - D[u] = 0$ in $\mathbb{R}^2$.*

Remark 2.2. *More precisely hyperholomorphic functions should be called, left-hyperholomorphic since there is a "symmetric" definition for* $1 - D_r$*, as well as for* $1 + D$ *and* $1 + D_r$*.*

We denote then

$$\mathfrak{M} = \ker(1 - D);\ \widetilde{\mathfrak{M}} = \ker(1 + D),$$

We can also define

$$\mathfrak{M}_r = \ker(1 - D_r);\ \widetilde{\mathfrak{M}}_r = \ker(1 + D_r).$$

Note that both $\mathfrak{M}$ and $\widetilde{\mathfrak{M}}$ have a natural structure of a right $\mathbb{H}$-module, while $\mathfrak{M}_r$ and $\widetilde{\mathfrak{M}}_r$ are left $\mathbb{H}$-modules.

The above definitions are particular cases of the more general situation considered in [15] where the operator D acting on functions defined in a domain $U \subset \mathbb{R}^2$ is allowed to be perturbed not only by a real constant 1, but by an arbitrary complex number α: a quaternion-valued function u is called (left)-α-hyperholomorphic if $\alpha u + D[u] = 0$. This idea can be extended to quaternionic values of α as well, but one has to choose if the multiplication is on the left or on the right. In [15] some essential properties of such functions were established. Main integral formulas for α-hyperholomorphic functions were constructed in [16]. All the proofs and details can be found in those papers, see also [10, Appendix 4]. Some developments of the topic are presented in [14] and [7]. The exact relation between α-hyperholomorphic function and solutions to the corresponding Helmholtz equation can be found in [15, 16], see also [10, Chap. 2].

On $C^1(\mathbb{R}^2, \mathbb{H})$ set $\pi = \frac{1}{2}(1 + D)$, $\tilde{\pi} = \frac{1}{2}(1 - D)$, and let Π and $\widetilde{\Pi}$ be their restrictions onto $\ker(1 + \Delta)$ respectively: $\Pi = \pi|_{\ker(1+\Delta)}$, $\widetilde{\Pi} = \tilde{\pi}|_{\ker(1+\Delta)}$. These operators have the following properties that can be easily verified:

i) $\Pi^2 = \Pi; \widetilde{\Pi}^2 = \widetilde{\Pi}$,

ii) $\Pi \circ \widetilde{\Pi} = \widetilde{\Pi} \circ \Pi = \mathbb{O}$, the zero operator,

iii) $\Pi + \widetilde{\Pi} = \mathbf{I}$, the identity operator,

iv) $Range(\Pi) = \ker(\tilde{\pi}|_{C^2(\mathbb{R}^2,\mathbb{H})}), Range(\widetilde{\Pi}) = \ker(\pi|_{C^2(\mathbb{R}^2,\mathbb{H})})$.

Since $\ker \tilde{\pi} = \mathfrak{M}$ and $\ker \pi = \widetilde{\mathfrak{M}}_r$ we may conclude that

$$\ker(1 + \Delta) = \mathfrak{M} \oplus \widetilde{\mathfrak{M}},$$

in other words, given $u \in \ker(1 + \Delta)$, there exist two hyperholomorphic functions $f \in \mathfrak{M}$ and $g \in \widetilde{\mathfrak{M}}$ such that

$$u = f + g,$$

moreover, $f = \frac{1}{2}(1+D)[u]$, $g = \frac{1}{2}(1-D)[u]$, and the decomposition of u is unique. This is an analogue of the fact that each harmonic function of two real variables is a (non-unique!) sum of a holomorphic and an anti-holomorphic function.

Let (r, θ) be the polar coordinates of a $x \in \mathbb{R}^2$, $x \neq 0$. Then we can write the Dirac operator D as (see [3]) :

$$D = w\left(\frac{\partial}{\partial r} + \frac{1}{r}\Gamma\right) \tag{4}$$

where $\Gamma = -e_3 \frac{\partial}{\partial \theta}$ and $w = \cos(\theta)e_1 + \sin(\theta)e_2$. The following functions called spherical monogenics (see [3]) are of a special importance to us:

$$\begin{aligned} p_n(\theta) &= cos(n\theta)e_1 - sin(n\theta)e_2, \\ q_n(\theta) &= cos(n\theta)e_1 + sin(n\theta)e_2. \end{aligned}$$

Also define

$$E_n(\theta) = \cos(n\theta) + \sin(n\theta)e_3.$$

Then we have the following identities:

1. $p_n(\theta) = E_{-n}(\theta)e_1$,
2. $q_n(\theta) = E_n(\theta)e_1$,
3. $wp_n = q_{n+1}e_1$,
4. $wq_{n+1} = p_n e_1$.

Moreover,

$$\begin{aligned} \Gamma p_n &= -np_n \\ \Gamma q_n &= nq_n. \end{aligned} \tag{5}$$

We will be using the following properties of the Bessel functions. For every integer n, the Bessel function $J_n(r)$ can be defined (see [18, p. 20]) by

$$J_n(r) = \int_0^{2\pi} e^{i(r\sin\theta - n\theta)} \frac{d\theta}{2\pi}.$$

For every $n \geq 1$ (see [1, 18]),

$$\int_1^\infty J_n^2(r)\,\frac{dr}{r^2} = \frac{1}{\pi}\frac{1}{n^2 - 1/4} + o\left(\frac{1}{n!2^n}\right). \tag{6}$$

The Bessel functions J_n satisfy the recurrence relations

$$2J_n'(r) = J_{n-1}(r) - J_{n+1}(r), \tag{7}$$

$$-rJ_{n+1}(r) = rJ_n'(r) - nJ_n(r). \tag{8}$$

We have the estimate

$$|J_n(r)| \leq \frac{r^n}{n!2^n} e^{r^2/4}. \tag{9}$$

Throughout this paper C will denote a positive generic constant that might change on each occurrence. Points x and y in $\mathbb{R}^2$ will always have polar coordinates (r, θ) and (ρ, φ) respectively. T will denote the unit circle in $\mathbb{R}^2$.

3 Function modules of $\mathbb{H}$-valued functions.

Let $\mathcal{H}^p_{\mathbb{R}}$ and $\mathcal{W}^p_{\mathbb{R}}$ be the spaces consisting of all real functions in $\mathcal{H}^p$ and $\mathcal{W}^p$ respectively. Then we define the quaternionic modules

$$\mathcal{H}^p_{\mathbb{H}} = \mathcal{H}^p_{\mathbb{R}} \otimes \mathbb{H},$$
$$\mathcal{W}^p_{\mathbb{H}} = \mathcal{W}^p_{\mathbb{R}} \otimes \mathbb{H}.$$

As expected,

$$\mathcal{H}^p_{\mathbb{H}} = \{F : \Omega \subset \mathbb{R}^2 \to \mathbb{H} \ : \ \|F\|_{\mathcal{H}^p_{\mathbb{H}}} < \infty\},$$

where

$$\|F\|_{\mathcal{H}^p_{\mathbb{H}}} = \left\{ \int_\Omega (|F|^p + |\frac{\partial F}{\partial r}|^p + |\frac{\partial F}{\partial \theta}|^p)|x|^{-3} dx \right\}^{1/p},$$

and

$$\mathcal{W}^p_{\mathbb{H}} = \{F \in \mathcal{H}^p_{\mathbb{H}} \ : \ \Delta F + F = 0 \text{ in } \mathbb{R}^2\}.$$

Formally, such tensor products are real spaces but of course we can endow them with a structure of quaternionic modules, more exactly, both are quaternionic bi-modules although we will consider each of them if necessary, as a right module or as a left module. Since $\mathcal{H}^p$ and $\mathcal{W}^p$ are Banach spaces, it easily follows that $\mathcal{H}^p_{\mathbb{H}}$ and $\mathcal{W}^p_{\mathbb{H}}$ are quaternionic Banach bi-modules.
Next we introduce the spaces of main interest to us. These are modules of hyperholomorphic functions in the plane whose restrictions to Ω belong to $\mathcal{H}^p_{\mathbb{H}}$.

Definition 3.1. *For $1 < p < \infty$, we define*

$$\mathfrak{M}^p = \mathcal{H}^p_{\mathbb{H}} \cap \mathfrak{M},$$

$$\widetilde{\mathfrak{M}}^p = \mathcal{H}^p_{\mathbb{H}} \cap \widetilde{\mathfrak{M}}.$$

Clearly $\mathfrak{M}^p$ and $\widetilde{\mathfrak{M}}^p$ are right $\mathbb{H}$-modules.

Remark 3.2. *a) With obvious modifications we can define the left modules $\mathfrak{M}^p_r$ and $\widetilde{\mathfrak{M}}^p_r$.*

b) The rest of the paper will be focused on $\mathfrak{M}^p$ but all the presented results are valid for each of the modules defined above with the ad-hoc modifications.

In the proof of [1, Th.2] it is shown that the convergence of $\{u_k\}$ to u in $\mathcal{W}^p$ implies convergence in $C^\infty(\mathbb{R}^2)$ (every partial derivative of u_k converges to the corresponding derivative of u uniformly on compact subsets of $\mathbb{R}^2$). The same will happen if u_k and u belong to $\mathcal{W}^p_{\mathbb{H}}$. In particular $\mathfrak{M}^p$ is closed in $\mathcal{W}^p_{\mathbb{H}}$, hence $\mathfrak{M}^p$ is a Banach right module for every $1 < p < \infty$.

As mentioned before, $\mathcal{H}^p_{\mathbb{H}}$ is a bi-module. We shall study now the case $p = 2$. Let first $\mathcal{H}^2_{\mathbb{H}}$ be considered as a *right* $\mathbb{H}$-module, then the formula

$$\langle F, G\rangle = \int_\Omega (\overline{F}G + \overline{\frac{\partial F}{\partial r}}\frac{\partial G}{\partial r} + \overline{\frac{\partial F}{\partial \theta}}\frac{\partial G}{\partial \theta})\frac{dx}{|x|^3}$$

define a quaternionic inner product converting $\mathcal{H}^2_{\mathbb{H}}$ into a quaternionic right Hilbert module. Analogously for $\mathcal{H}^2_{\mathbb{H}}$ as a *left* $\mathbb{H}$-module the formula

$$\langle F, G\rangle_l := \int_\Omega (F\overline{G} + \frac{\partial F}{\partial r}\overline{\frac{\partial G}{\partial r}} + \frac{\partial F}{\partial \theta}\overline{\frac{\partial G}{\partial \theta}})\frac{dx}{|x|^3}$$

plays the same role, and $\mathcal{H}^2_{\mathbb{H}}$ becomes a left Hilbert module.
The following decomposition is similar to those introduced in [4, 20].

Theorem 3.3. *Let F be a function from the plane to the quaternions. Then $F \in \mathfrak{M}^2$ if and only if F can be written as*

$$F(r,\theta) = \sum_{n=0}^\infty G_n(r,\theta)\lambda_n, \tag{10}$$

where $G_n(r,\theta) = (J_n(r) - wJ_{n+1}(r))p_n(\theta)$, $\lambda_n \in \mathbb{H}$ and $\sum_{n=o}^\infty |\lambda_n|^2 < \infty$. Furthermore,

$$\sum_{n=o}^\infty |\lambda_n|^2 \sim \|F\|^2_{\mathcal{H}^2_{\mathbb{H}}}.$$

Proof. Let $F \in \mathfrak{M}^2$. Since $\mathfrak{M} \subset C^\infty(\mathbb{R}^2, \mathbb{H})$, then for each $r > 0$ we can express F as its Fourier series

$$F(r,\theta) = \sum_{n=0}^\infty \cos(n\theta)a_n(r) + \sin(n\theta)b_n(r).$$

We can write this series in terms of the spherical monogenics $p_n(\theta)$ and $q_n(\theta)$ to obtain

$$F(r,\theta) = \sum_{n=0}^\infty p_n(\theta)A_n(r) + q_n(\theta)B_n(r)$$

The smoothness of F implies that the series converges absolutely for each (r,θ) and uniformly in each compact subset and it can be differentiated term by term. From (4) and (5) we get that

$$D(p_n(\theta)A_n(r)) = q_{n+1}(\theta)e_1(A'_n(r) - \frac{n}{r}A_n(r)),$$

and

$$D(q_n(\theta)B_n(r)) = p_{n-1}(\theta)e_1(B'_n(r) + \frac{n}{r}B_n(r)).$$

Since $DF = F$, the uniqueness of the coefficients in the Fourier series expansion implies that

$$B_n(r) = e_1(A'_{n-1}(r) - \frac{n-1}{r}A_{n-1}(r)), \tag{11}$$

and

$$A_n(r) = e_1(B'_{n+1}(r) + \frac{n+1}{r}B_{n+1}(r)). \tag{12}$$

It follows that $A_n(r)$ satisfies Bessel's equation of order n,

$$A_n(r)'' + \frac{1}{r}A_n(r)' + \frac{r^2 - n^2}{r^2}A_n(r) = 0.$$

Therefore $A_n(r) = J_n(r)\lambda_n + N_n(r)\beta_n$, where $N_n(r)$ is the Neumann function of order n ([18]). Since N_n has a singularity at $r = 0$ and $A_n(r)$ is regular at that point, it follows that $\beta_n = 0$ for all n. From (8) and (11) we obtain that

$$B_n(r) = -e_1 J_n(r)\lambda_{n-1}, \quad n > 1.$$

Then we have

$$F(r,\theta) = \sum_{n=0}^{\infty}(J_n(r) - wJ_{n+1}(r))p_n(\theta)\lambda_n.$$

Now we will see that $\sum_{n=0}^{\infty}|\lambda_n|^2 < \infty$. From the last decomposition we find that

$$\frac{\partial}{\partial\theta}F(r,\theta) = \sum_{n=0}^{\infty}p_n(\theta)nJ_n(r)e_3\lambda_n + \sum_{n=1}^{\infty}q_n(\theta)nJ_n(r)e_2\lambda_{n-1}.$$

Then by Bessel's inequality we have that

$$\int_0^{2\pi}|\frac{\partial}{\partial\theta}F(r,\theta)|^2 d\theta \geq \sum_{n=0}^{\infty}n^2 J_n(r)^2|\lambda_n|^2.$$

From (6), there is a constant $C > 0$ such that for every N

$$\begin{aligned}\sum_{n=0}^{N}|\lambda_n|^2 &\leq C\sum_{n=0}^{N}|\lambda_n|^2\int_1^{\infty}n^2 J_n^2(r)\frac{dr}{r^2} = \\ &= C\int_1^{\infty}\sum_{n=0}^{N}|\lambda_n|^2 n^2 J_n^2(r)\frac{dr}{r^2} \\ &\leq C\int_1^{\infty}\int_0^{2\pi}|\frac{\partial}{\partial\theta}\,F(r,\theta)|^2 d\theta\frac{dr}{r^2} \leq C\|F\|^2.\end{aligned}$$

To prove the converse, let F satisfy (10) with $\sum_{n=o}^{\infty}|\lambda_n|^2 < \infty$. Then it is easy to see that the series converges in $\mathcal{W}^2_{\mathbb{H}}$ which is continuously embedded in $C^{\infty}(\mathbb{R}^2, \mathbb{H})$. Hence $DF = F$, since every term of the series satisfies this equation. ∎

Remark 3.4. *a) Series of the form*

$$\sum a_n J_n(r),$$

with complex coefficients, called Neumann series have properties of convergence similar to power series (see [18]). In particular they have radius of uniform convergence in compact sets. Then we see that the expansion (10) is also valid for $F \in \mathfrak{M}$ with uniform and absolute convergence in compact subsets of $\mathbb{R}^2$, as well as the series of every partial derivative of F.

b) We can write any quaternionic solution of the Helmholtz equation (1) as

$$\begin{aligned} F(r,\theta) &= \sum_{n=0}^{\infty} \cos(n\theta) J_n(r) a_n + \sum_{n=1}^{\infty} \sin(n\theta) J_n(r) b_n \\ &= \sum_{n=-\infty}^{\infty} J_n(r) E_n(\theta) \lambda_n, \end{aligned}$$

where $a_n, b_n, \lambda_n \in \mathbb{H}$. Noticing that the function G_n in (10) can be written as $G_n(r,\theta) = E_{-n}(\theta) J_n(r) e_1 + E_{n+1}(\theta) J_{n+1}(r)$, we conclude that F belongs to $\mathfrak{M}$ if and only if

$$e_1 \lambda_{n+1} - \lambda_{-n} = 0, \; n \geq 1.$$

This distinguishes the elements of $\mathfrak{M}$ in the module of solutions of the Helmholtz equation in $\mathbb{R}^2$, resembling the fact that holomorphic functions in the disk centered at zero are those harmonic functions with zero Fourier coefficients in θ for $n < 0$.

Proposition 3.5. *The family $\{G_n\}_{n \in \mathbb{N}}$ is orthogonal in $\mathfrak{M}^2$. Moreover there exists a constant $\kappa > 0$ such that*

$$\gamma_n = \|G_n\|_{\mathcal{H}^2_{\mathbb{H}}} = \kappa + O(1/n). \tag{13}$$

Hence $\{G_n/\gamma_n\}_{n \in \mathbb{N}}$ is a basis for $\mathfrak{M}^2$.

Proof. For any n and m we have that

$$\langle G_n, G_m \rangle = \int_\Omega \overline{G_n}(x) G_m(x) |x|^{-3} dx +$$

$$\int_\Omega \overline{\frac{\partial G_n(x)}{\partial r}} \frac{\partial G_m(x)}{\partial r} |x|^{-3} dx + \int_\Omega \overline{\frac{\partial G_n(x)}{\partial \theta}} \frac{\partial G_m(x)}{\partial \theta} |x|^{-3} dx$$

$$= I_1 + I_2 + I_3.$$

Now, from

$$\overline{G_n} G_m = -p_n p_m (J_n J_m + J_{n+1} J_{m+1}) + p_n q_{m+1} e_1 (J_n J_{m+1} - J_{n+1} J_m)$$

and from the orthogonality of the spherical monogenics, it follows that

$$I_1 = \delta_{nm}\beta_n,$$

with $\beta_n = \int_1^\infty (J_n^2(r) + J_{n+1}^2(r))\frac{dr}{r^2}$, and where δ_{nm} is the Kronecker delta. Similarly we obtain that

$$I_2 = \delta_{nm}\beta_n',$$

with $\beta_n' = \int_1^\infty ({J_n'}^2(r) + {J_{n+1}'}^2(r))\frac{dr}{r^2}$.

For I_3 we integrate by parts to obtain

$$I_3 = -\int_\Omega \overline{G_n}\frac{\partial^2 G_m}{\partial\theta^2}|x|^{-3}dx.$$

A computation shows that

$$\frac{\partial^2 G_m}{\partial^2\theta} = -n^2 G_n + (2n+1)J_{n+1}(r)q_{n+1}(\theta)e_1,$$

then

$$\overline{G_n}\frac{\partial^2 G_m}{\partial^2\theta} = -n^2\overline{G_n}G_m - \\ (p_n q_{m+1}e_1 J_n J_{m+1} + p_n p_m J_{n+1}J_{m+1})\,(2n+1).$$

Thus we have as before that

$$I_3 = \delta_{nm}(n^2\beta_n + \alpha_n),$$

with $\alpha_n = (2n+1)\int_1^\infty J_{n+1}^2(r)\frac{dr}{r^2}$.

Hence

$$\gamma_n^2 = \beta_n(n^2+1) + \beta_n' + \alpha_n$$

and from (6), it follows that there is a $\kappa > 0$ such that $\gamma_n = \kappa + O(1/n)$. ∎

Now we proceed to the basic properties of $\mathfrak{M}^p$ for $1 < p < \infty$.

Theorem 3.6. *Let $1 < p < \infty$ and $F \in \mathfrak{M}^p$ such that*

$$F(x) = \sum_{n=0}^\infty G_n(x)\lambda_n,$$

with $\{\lambda_n\} \in \mathbb{H}$. Then the series converges in $\mathfrak{M}^p$.

Proof. Consider the Dirichlet kernel

$$D_n(\varphi) = \frac{\sin(n+1/2)\varphi}{\sin(\varphi/2)} = \sum_{j=-n}^n e^{ij\varphi},$$

where $\varphi \in [0, 2\pi]$, i is the imaginary unit of the complex numbers, and the partial sum operators

$$S_n g(\theta) = D_n * g(\theta) = \sum_{j=-n}^{n} e^{ij\varphi} a_n$$

are defined for any complex function $g \in L^1(T)$ with Fourier series $Sg = \sum_n e^{ij\varphi} a_n$. Recall that if $g \in L^p(T)$, with $p > 1$, then $S_n g$ converges to g in $L^p(T)$ and there exists a constant $C > 0$ such that

$$\|S_n g\|_{L^p(T)} \le C \|g\|_{L^p(T)}, \text{ for } g \in L^p(T). \tag{14}$$

We can obviously replace the complex numbers in the statements above by quaternions of the form $q_0 + q_3 e_3$ and obtain the same results for $g \in L^p_{\mathbb{H}}(T)$ with Fourier series $\sum_n E_n(\varphi) a_n$, $a_n \in \mathbb{H}$. Let $F = \sum G_n \lambda_n \in \mathfrak{M}^p$ and let $S_N F(r, \theta) = S_N F(r, \cdot)(\theta)$. Then $S_N F$ converges to F in $\mathcal{H}^p_{\mathbb{H}}$. In fact, let

$$\psi_N(r) = \|S_N F(r, \cdot) - F(r, \cdot)\|^p_{L^p(T)}.$$

Then $\psi_N(r)$ converges to zero pointwise, and (14) implies that $\psi_N(r) \le C \|F(r, \cdot)\|^p_{L^p(T)}$. Then by the dominated convergence theorem

$$\lim_{N\to\infty} \|S_N F - F\|^p_{L^p(\Omega, |x|^{-3}dx)} = \lim_{N\to\infty} \int_\Omega \psi_N(r) \frac{dr}{r^2} = 0.$$

Also

$$\lim_{N\to\infty} \left\| \frac{\partial}{\partial r}(S_N F - F) \right\|^p_{L^p(\Omega, |x|^{-3}dx)} = \lim_{N\to\infty} \left\| \frac{\partial}{\partial \theta}(S_N F - F) \right\|^p_{L^p(\Omega, |x|^{-3}dx)} = 0$$

follows with the same argument, since

$$\frac{\partial}{\partial r} S_N F = S_N \left(\frac{\partial}{\partial r} F \right) \text{ and } \frac{\partial}{\partial \theta} S_N F = S_N \left(\frac{\partial}{\partial \theta} F \right).$$

If we write

$$G_n(r, \theta) = E_{-n}(\theta) J_n(r) e_1 + E_{n+1}(\theta) J_{n+1}(r),$$

then

$$\begin{aligned} S_N F(r, \theta) &= \sum_{n=0}^{N} G_n(r, \theta) \lambda_n - E_{(N+1)}(\theta) J_{(N+1)}(r) \lambda_N. \\ &= \sum_{n=0}^{N} G_n(r, \theta) \lambda_n - R_{N+1}(r, \theta), \end{aligned}$$

where $R_n(r, \theta) = E_n(\theta) J_n(r) \lambda_{n-1}$.

Since

$$\left\| F - \sum_{n=0}^{N} G_n(x) \lambda_n \right\|_{\mathcal{H}^p_{\mathbb{H}}} \le \|F - S_N F\|_{\mathcal{H}^p_{\mathbb{H}}} + \|R_{N+1}\|_{\mathcal{H}^p_{\mathbb{H}}},$$

then the proof will be complete if we can prove that $\|R_N\|_{\mathcal{H}^p_{\mathbb{H}}}$ tends to zero as N tends to ∞. To this end, write

$$R_N(r,\theta) = E_N * F(r,\cdot)(\theta). \tag{15}$$

By the Riemann-Lebesgue theorem, R_N tends to zero pointwise. Young's inequality implies that

$$\|R_N(r,\cdot)\|_{L^p(T)} \le \|F(r,\cdot)\|_{L^p(T)}.$$

Then a dominated convergence argument can be used to show that R_N converges to zero in $L^p(\Omega, |x|^{-3}\,dx)$. Once again, taking $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial r}$ inside the convolution (15) and repeating the previous argument we conclude that $\|R_N\|_{\mathcal{H}^p_{\mathbb{H}}}$ converges to zero. ∎

4 The Reproducing kernel.

We proceed now to construct the reproducing kernel for $\mathfrak{M}^2$ as a submodule of $\mathcal{H}^2_{\mathbb{H}}$. Let

$$K(x,y) = \sum_{n=0}^{\infty} \frac{G_n(x)\overline{G_n(y)}}{\gamma_n^2} \tag{16}$$

From (13) and (see [5]) $\sum_{n\in\mathbb{Z}} J_n(r)^2 = 1$, it follows that this series converges absolutely and uniformly on compact subsets of $\mathbb{R}^2 \times \mathbb{R}^2$ as well as the series of any partial derivative of $K(x,y)$. Also by Theorem 3.3, $K(\cdot,y) \in \mathfrak{M}^2$ for each $y \in \mathbb{R}^2$ and

$$K(\cdot,y) = \sum_{n=0}^{\infty} \frac{G_n(\cdot)\overline{G_n(y)}}{\gamma_n^2} \tag{17}$$

with convergence in $\mathfrak{M}^2$. We define for every $F \in \mathcal{H}^2_{\mathbb{H}}$,

$$P[F](y) = \langle K(\cdot,y), F\rangle. \tag{18}$$

Theorem 4.1. *The operator P is a continuous right $\mathbb{H}$- linear projection of $\mathcal{H}^2_{\mathbb{H}}$ onto $\mathfrak{M}^2$. For any $F \in \mathcal{H}^2_{\mathbb{H}}$ and $y \in \Omega$ we have that*

$$PF(y) = \sum_{n=0}^{\infty} \frac{G_n(y)}{\gamma_n^2} < G_n, F >,$$

with uniform and absolute convergence on compact subsets of $\mathbb{R}^2$.

Proof. The proposition follows from (17). That the convergence is uniform on compact sets is a consequence of the estimate (9) and the orthonormality of $\gamma_n^{-1}G_n$ and (13). ∎

Now we study the action of the projection P on $\mathcal{H}^p_{\mathbb{H}}$. The uniform estimate ([6, Lemma 3.4])

$$|J_n(r)| \le C r^{-1/3},$$

valid for $r \ge 1$, together with the recurrence formula (7) and the estimate (9) imply that

$$PF(y) = \langle K(\cdot, y), F\rangle =$$

$$\int_\Omega \overline{K(x,y)} F(x) \frac{dx}{r^3} + \int_\Omega \overline{\frac{\partial}{\partial r} K(x,y)} \frac{\partial}{\partial r} F(x) \frac{dx}{r^3} + \int_\Omega \overline{\frac{\partial}{\partial \theta} K(x,y)} \frac{\partial}{\partial \theta} F(x) \frac{dx}{r^3}$$

is well defined for $F \in \mathcal{H}^p_{\mathbb{H}}$, that is, the integrals exist for every $y \in \mathbb{R}^2$. Moreover, we can differentiate $\frac{\partial}{\partial \rho} PF$ and $\frac{\partial}{\partial \theta} PF$ inside the integral to conclude that $PF \in \mathfrak{M}$ for every $F \in \mathcal{H}^p_{\mathbb{H}}$. Also, the estimates for the Bessel functions above imply that the expansion

$$PF(y) = \sum_{n=0}^{\infty} \frac{G_n(y)}{\gamma_n^2} \langle G_n, F\rangle \tag{19}$$

is valid for such F.

Proposition 4.2. *For every $1 < p < \infty$, the operator P can be extended as bounded operator from $\mathcal{H}^p_{\mathbb{H}}$ into $L^p_{\mathbb{H}}(\Omega, |x|^{-3}\, dx)$, both seen as right modules or left modules.*

Proof. Let $F \in \mathcal{H}^p_{\mathbb{H}}$, then

$$\begin{aligned} PF(y) &= \sum_{n=0}^{\infty} \frac{G_n(y)}{\gamma_n^2} \langle G_n, F\rangle \\ &= \sum_{n=0}^{\infty} E_{-n}(\varphi) \frac{J_n(r) e_1}{\gamma_n^2} \langle G_n, F\rangle \\ &+ \sum_{n=0}^{\infty} E_{n+1}(\varphi) \frac{J_{n+1}(r)}{\gamma_n^2} \langle G_n, F\rangle . \end{aligned}$$

Let $c_n = \gamma_{n-1}^2$ if $n > 0$ and $c_n = \gamma_{-n}^2$ for $n \le 0$, and consider the Fourier multiplier operator

$$\mathcal{M}\left(\sum_{n=-\infty}^{\infty} e^{in\theta} a_n\right) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} a_n$$

acting on complex functions with Fourier series $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$.

Then by (13) $\mathcal{M}$ and $\mathcal{M}^{-1}$ are continuous in $L^p(T)$ (see [17, Prop. 4.1, Ch 5]). If we let $\mathcal{M}$ act on quaternionic expansions

$$\sum_{n=-\infty}^{\infty} E_n(\varphi) a_n,\ a_n \in \mathbb{H},$$

we see that $\mathcal{M}$ and $\mathcal{M}^{-1}$ are continuous in $L^p_{\mathbb{H}}(T)$ also. Define $\widetilde{P} = \mathcal{M} \circ P$. Then from (19),

$$\widetilde{P}F(y) = \left\langle \widetilde{K}(\cdot, y), F \right\rangle,$$

with

$$\widetilde{K}(x,y) = \sum_{n=0}^{\infty} G_n(x)\overline{G_n(y)}$$
$$= \sum_{n=0}^{\infty}(J_n(r) - wJ_{n+1}(r))E_n(\varphi-\theta)(J_n(\rho)+w'J_{n+1}(\rho)), \tag{20}$$

where $w' = \cos(\varphi)e_1 + \sin(\varphi)e_2$. Then by the continuity of $\mathcal{M}$ and $\mathcal{M}^{-1}$, the proposition is equivalent to proving the same statement for $\widetilde{P}$. From the expression (20) for $\widetilde{K}$, it is clear that the proof will be complete if we can prove that for every $k, l \in \mathbb{Z}$, the integral operators acting in Ω with respect to $|x|^{-3}\,dx$ and with kernels $\overline{N(x,y)}$, $\overline{\frac{\partial N}{\partial r}(x,y)}$,$\overline{\frac{\partial N}{\partial \theta}(x,y)}$ (multiplying in the left) define bounded operators in $L^p_{\mathbb{H}}(\Omega, |x|^{-3}\,dx)$, where

$$N(x,y) = \sum_{n=l}^{\infty} J_n(\rho)J_{n+k}(r)E_n(\varphi-\theta).$$

This is proved in Proposition 5.2 of the appendix. ■

Now we are ready to prove that the projection P is reproducing for $\mathfrak{M}^p$.

Theorem 4.3. *Let $1 < p < \infty$ and $F \in \mathcal{H}^p_{\mathbb{H}}$. Then $F \in \mathfrak{M}^p$ if and only of $PF = F$.*

Proof. Let $F \in \mathcal{H}^p_{\mathbb{H}}$. We already know that $PF \in \mathfrak{M}$, then clearly $PF = F$ implies that $F \in \mathfrak{M}^p$. To prove the converse, let $F \in \mathfrak{M}^p$. By Theorem 3.6 there exists a sequence of functions $F_n \in \mathfrak{M}^p \cap \mathfrak{M}^2$ converging to F in $\mathfrak{M}^p$. Since P is a projection in $\mathfrak{M}^2$ we have by Proposition 4.2 that in $L^p_{\mathbb{H}}(\Omega, |x|^{-3}\,dx)$,

$$PF = \lim_{n\to\infty} PF_n = \lim_{n\to\infty} F_n = F.$$

■

5 Appendix.

In this section we turn to the notation of the complex numbers , namely $x = re^{i\theta}$, $y = \rho e^{i\varphi}$, $x \cdot y$ will denote the dot product of x and y.

Let $k \in \mathbb{Z}$ and

$$N_1(x,y) = \sum_{n=-\infty}^{\infty} J_n(r)J_{n+k}(\rho)e^{in(\varphi-\theta)},$$

$$N_2(x,y) = \sum_{n=l}^{\infty} J_n(r)J_{n+k}(\rho)e^{in(\varphi-\theta)}.$$

By the summation theorems for the Bessel functions [5, 8.53], we have that $N_1(x,y) = e^{ik\psi}J_k(|x-y|)$, provided that x and y form a triangle with sides r, ρ and $R = |x-y|$, where $0 < \psi < \pi/2$ is the angle opposite to ρ if $\rho < r$ and is the angle opposite to r if $r < \rho$.

Lemma 5.1. *Let ψ be defined as above, with $x, y \in \Omega$. Then*

$$\left|\frac{\partial \psi}{\partial r}\right|, \left|\frac{\partial \psi}{\partial \theta}\right| \le 4\frac{r\rho}{R^2}.$$

Proof. Assume that $\rho < r$ and $\theta \neq \varphi$. Then

$$\sin\psi = \frac{\rho}{R}\sin(\theta - \varphi)$$

so that

$$\frac{\partial \psi}{\partial r}\cos\psi = \rho\sin(\theta-\varphi)\frac{\partial}{\partial r}\frac{1}{R},$$

$$\frac{\partial \psi}{\partial \theta}\cos\psi = \rho\sin(\theta-\varphi)\frac{\partial}{\partial \theta}\frac{1}{R} + \frac{\rho}{R}\cos(\theta-\varphi).$$

From the cosine law

$$\cos\psi = \frac{r^2 + R^2 - \rho^2}{2rR},$$

also

$$\frac{\partial}{\partial r}\frac{1}{R} = -\frac{(x-y)}{R^3}\cdot\frac{x}{r},$$
$$\frac{\partial}{\partial \theta}\frac{1}{R} = -\frac{(x-y)}{R^3}\cdot ix.$$

Then we have

$$\frac{\partial \psi}{\partial r} = -\frac{2\rho(x-y)\cdot x}{(r^2+R^2-\rho^2)\,R^2}\sin(\theta-\varphi),$$

and

$$\frac{\partial \psi}{\partial \theta} = -\frac{2\rho r(x-y)\cdot ix}{(r^2+R^2-\rho^2)\,R^2}\sin(\theta-\varphi) + \frac{2\rho r}{(r^2+R^2-\rho^2)}\cos(\theta-\varphi).$$

Since $r > 1$ and $|r\sin(\theta-\varphi)| \le R$ we obtain

$$\left|\frac{\partial \psi}{\partial r}\right| \le \frac{2\rho}{R^2} \le \frac{2\rho r}{R^2}$$
$$\left|\frac{\partial \psi}{\partial \theta}\right| \le \frac{4\rho r}{R^2}.$$

The proof in the case $r < \rho$ is analogous. ∎

Proposition 5.2. *The integral operators acting in Ω with respect to $|x|^{-3}\,dx$ and with kernels $N_i(x,y)$, $\frac{\partial N_i}{\partial r}(x,y)$, $\frac{\partial N_i}{\partial \theta}(x,y)$ define bounded operators in $L^p(\Omega, |x|^{-3}\,dx)$, for $i = 1, 2$, and $1 < p < \infty$.*

Proof. If M_l is the projection defined in trigonometric polynomials

$$M_l(\sum a_n e^{in\theta}) = \sum_{n\geq l} a_n e^{in\theta},$$

then M_l can be extended as a continuous operator in $L^p(S^1)$ for every $1 < p < \infty$ (this follows from the continuity of the conjugate function, see [17]). Then it is enough to prove the proposition for $i = 1$. In fact,

$$M_l \left(\int_\Omega N_1(x,y) f(x) \, |x|^{-3} \, dx \right) = \int_\Omega N_2(x,y) f(x) \, |x|^{-3} \, dx,$$

and we deal with the other kernels in the same way (M_l is acting in θ). Now we prove that the operator

$$Tf(y) = \int_\Omega \frac{\partial}{\partial \theta} N_1(x,y) f(x) \, |x|^{-3} \, dx$$

is bounded in $L^p(\Omega, |x|^{-3} \, dx)$. The kernels $N_1(x,y)$, $\frac{\partial}{\partial r} N_1(x,y)$ are treated in a similar way (easier since they are less singular).

The case $k = 0$ was proved in [1]. So we assume $k \neq 0$.

By (7) we write

$$\begin{aligned}
\partial_\theta N_1(x,y) = {} & ik\psi e^{ik\psi} \frac{\partial \psi}{\partial \theta} J_k(|x-y|) + \\
& \frac{e^{ik\psi}}{2} \left(J_{k-1}(|x-y|) - J_{k+1}(|x-y|) \right) \frac{x-y}{|x-y|} \cdot ix \\
= {} & I_1 + I_2.
\end{aligned}$$

Since $J_k(0) = 0$ and from the fact that $|J_n(t)| \leq C_n t^{-1/2}$ for t large , it follows that $|I_1|$ is bounded by $Cr\rho/R$ if $R < 1/2$ and by $Cr\rho/R^{3/2}$ for $R \geq 1/2$. Since $x \cdot ix = 0$ we have

$$(x-y) \cdot ix = -y \cdot ix = -(x-y) \cdot iy.$$

Hence $|I_2|$ is bounded by $\min(r,\rho)$ if $R < 1/2$ and by $r\rho/R^{3/2}$ for $R \geq 1/2$. We conclude that

$$\begin{aligned}
& \left| \frac{\partial}{\partial \theta} N_1(x,y) \right| \leq \\
& \chi_{R<1/2}(x,y) \left(\min(r,\rho) + r\rho/R \right) + \frac{r\rho}{(1+R)^{3/2}} \\
& = K_1(x,y) + K_2(x,y).
\end{aligned}$$

An easy calculation shows that there exists $C > 0$ such $\int_\Omega K_1(x,y) \, |x|^{-3} \, dx \leq C$ for every $y \in \Omega$. Then by the Shur's test (see [19]) K_1 defines a bounded operator in $L^p(\Omega, |x|^{-3} \, dx)$. The kernel K_2 also defines a bounded operator on $L^p(\Omega, |x|^{-3} \, dx)$, this is included in the proof of [1, Prop. 1]. ∎

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