Projective limits of paratopological vector spaces

Carmen Alegre*

Abstract

In this paper we describe various methods of constructing new paratopological vector spaces from given ones. First, we introduce the notion of a right dual of a paratopological vector space with the aim to define a right weak topology. We prove that, in a certain class of normed spaces, the classical weak topology is determined by a right weak topology. Next, the quotient topology in the context of paratopological vector spaces is discussed. Finally, we consider the projective limit of paratopological vector spaces and prove that every pseudoconvex space is a projective limit of quasi-normed spaces.

1 Introduction and preliminaries

A *paratopological group* [5] is a triple $(X, +, \tau)$ such that $(X, +)$ is a group and τ is a topology on X for which the operation $+$ is continuous.

If $(X, +, \tau)$ is a paratopological group, then so is $(X, +, -\tau)$, where $-\tau = \{A \subseteq X : -A \in \tau\}$ is called *the conjugate topology* of τ. Clearly, the map $x \to -x$ is a homeomorphism from (X, τ) onto $(X, -\tau)$.

It is well known that if $(X, +, \tau)$ is a paratopological group, then $(X, +, \tau^*)$ is a Hausdorff topological group, where $\tau^* = \tau \vee (-\tau)$.

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Paratopological groups are called quasi-topological groups in [8] and [17] (see also [18] and [15]), where these structures are studied by means of the properties of three canonical quasi-uniformities that one can introduce in a natural way on a paratopological group.

If $(X,+,\cdot)$ is a real vector space and τ is a topology on X, $(X,+,\cdot,\tau)$ is said to be a *paratopological vector space* if $(X,+,\tau)$ is a paratopological group such that for each neighborhood U of rx, with $x \in X$ and $r \in \mathbb{R}^+$ (the set of nonnegative real numbers), there exist a neighborhood V of x and an $\varepsilon > 0$ such that $[r, r+\varepsilon[\cdot V \subseteq U$.

These spaces were introduced in [1], calling them pseudotopological vector spaces.

If $(X,+,\cdot,\tau)$ is a paratopological vector space, $(X,+,\cdot,-\tau)$ is also a paratopological vector space. Furthermore $(X,+,\cdot,\tau^*)$ is a Hausdorff topological vector space [1].

In the following both the paratopological vector space $(X,+,\cdot,\tau)$ and the paratopological group $(X,+,\tau)$ will be simply denoted by (X,τ), or by X, if no confusion arises.

By the continuity of the sum, if $\mathcal{B}$ is a base of neighborhoods of the origin of (X,τ), then $x+\mathcal{B}$ is a base of neighborhoods of x. So, the whole topological structure of X is determined by a base of neighborhoods of the origin.

A subset A of a real vector space X is called *semibalanced* (quasi-balanced in [1]) if for each $x \in A$, $rx \in A$ whenever $0 \leq r \leq 1$. It is *absorbent* if for each $x \in X$ there is some $t > 0$ such that $x \in rA$ for all $r \geq t$. The set A is called *convex* if for all $x, y \in A$, $rx + (1-r)y \in A$ whenever $0 \leq r \leq 1$. Clearly, every convex set containing the origin is semibalanced.

Every neighborhood of the origin in a paratopological vector space is absorbent and contains a semibalanced neighborhood of the origin [1].

If there is a base of neighborhoods of the origin consisting of convex sets, the paratopological vector space is called a *pseudoconvex vector space* ([3]),or simply a *pseudoconvex* space if no confusion arises. Such spaces are an extension of the locally convex vector spaces.

A useful tool for the analytical description of certain convex sets is the concept of a subnorm. A nonnegative real valued function p defined on a real vector space X is said to be a *subnorm* if it is subadditive and positively homogeneous.

If U is a convex and semibalanced neighborhood of the origin in the paratopological vector space (X,τ), then the Minkowski functional of U is a subnorm on X [1].

If $\mathcal{Q}$ is a family of subnorms on a real vector space X, there is a coarsest topology on X which turns X into a pseudoconvex vector space in which every subnorm of $\mathcal{Q}$ is upper semicontinuous [1].

By a quasi-metric on a set X (compare [9]) we mean a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$: (i) $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$, and (ii) $d(x,y) \leq d(x,z) + d(z,y)$.

According to [7], a *quasi-norm* on a real vector space X is a subnorm q on X that satisfies the following condition: $q(x) = q(-x) = 0 \Leftrightarrow x = 0$.

Quasi-norms are called *nonsymmetric norms* in [4] and *asymmetric norms* in [11] and [12].

A *quasi-normed space* is a pair (X,q) such that X is a real vector space and q is a quasi-norm on X.

If q is a quasi-norm on a real vector space, then the function q^{-1} defined on X by $q^{-1}(x) = q(-x)$ is also a quasi-norm on X, called the *conjugate* of q. The function q^* defined on X by $q^*(x, y) = \max\{q(x), q^{-1}(x)\}$ is a norm on X.

Each quasi-norm q on a real vector space X induces a quasi-metric d_q on X defined by

$$d_q(x, y) = q(y - x),$$

for all $x, y \in X$. We refer to the topology $\mathcal{T}(d_q)$ as the topology *induced* by q.

Furthermore, and by Theorem 4.6 of [7], $(X, \mathcal{T}(d_q))$ is clearly a pseudoconvex vector space where the sets of the form

$$V(0, \varepsilon) = \{x \in X : q(x) < \varepsilon\}, \quad \varepsilon > 0,$$

constitute a fundamental system of neighborhoods of 0 for the topology $\mathcal{T}(d_q)$.

It seems interesting to point out that in the last years it has been shown that several nonsymmetric structures from topological algebra and functional analysis, as locally convex cones and quasi-normed (semi)linear spaces, constitute efficient tools in the study of some questions in Theoretical Computer Science ([10],[21], [23], [24]) and Approximation Theory ([4], [6], [14], [20]), respectively.

2 The right weak topology

We define the *right dual* of a paratopological vector space (X, τ) as the space of functions

$$X_r^{'} = \{f : (X, \tau) \to (\mathbb{R}, u) : f \text{ is linear and continuous}\}$$

where $(\mathbb{R}, u)$ is the paratopological vector space induced by the quasi-norm u defined on $\mathbb{R}$ by $u(x) = max\{x, 0\}$.

Note that $f : X \to \mathbb{R}$ is in $X_r^{'}$ if and only if it is a linear and upper semicontinuous function on (X, τ).

A recent study of duality in the realm of quasi-normed spaces is given in [12].

Obviously, $X_r^{'}$ is not a vector space in general for the pointwise usual operations. Indeed, denote by id the identity function on $(\mathbb{R}, u)$. Then id belongs to the right dual of $(\mathbb{R}, u)$. However, it is clear that $-id$ is not continuous from $(\mathbb{R}, u)$ into itself.

It is easy to check that $X_r^{'}$ admits the structure of a cone in the sense of [14].

We define the *left dual* of (X, τ) as the right dual of the paratopological vector space $(X, -\tau)$ and denote it by $X_l^{'}$. It is clear that $X_r^{'} = -X_l^{'}$.

If X is a topological vector space, then $X_r^{'} = X_l^{'}$ since in this case a linear function f is upper semicontinuous if and only if it is lower semicontinuous. Therefore $X_r^{'}$ is the topological dual of X, that is, the real vector space of all the linear and continuous functions on X, denoted by $X^{'}$ in the usual terminology of topological vector spaces.

In the introduction the concept of a quasi-normed space was recalled. In [7] we exhibited a natural class of examples of quasi-normed spaces, the normed linear lattices. In fact, we proved that whenever X is a normed lattice, the quasi-norm $q(x) = \|x^+\|$ with $x^+ = sup\{x, 0\}$ determines the topology and order of X in the sense of [9]. Moreover, if $X^{'}$ is the topological dual of the normed lattice $(X, \| \, \|)$ and $X_r^{'}$ is the right dual of the pseudoconvex space determined by the quasi-normed space (X, q), it follows, by Proposition 4.3 of [7], that X_r' is the cone of positive elements of X'.

If (X, τ) is a paratopological vector space, the sets

$$V^{\epsilon}_{f_1, f_2, \ldots, f_n} = \{x \in X : f_1(x) < \epsilon, f_2(x) < \epsilon, \ldots, f_n(x) < \epsilon\}$$

with $f_1, f_2, \ldots, f_n \in X_r'$, $\epsilon > 0$, $n \in \mathbb{N}$, form, by Theorem 4.2 of [1] a base $\mathcal{B}$ of neighborhoods of the origin for a topology turning X into a pseudoconvex vector space. We define this topology as *the right weak topology* and denote it by $\sigma(X, X_r')$.

By definition, it is obvious that $\sigma(X, X_r')$ is the coarsest topology on X under which all the elements of X_r' are upper semicontinuous functions.

Proposition 1. *Let (X, τ) be a paratopological vector space. The right dual of the paratopological vector space $(X, \sigma(X, X_r'))$ coincides with X_r'.*

Proof. Since $\sigma(X, X_r')$ is coarser than τ, X_r' contains the right dual of $(X, \sigma(X, X_r'))$. On the other hand, if $f \in X_r'$ then $f^{-1}(]-\infty, \epsilon[)$ is a $\sigma(X, X_r')$ -neighborhood of the origin, so that f is in the right dual of $(X, \sigma(X, X_r'))$. ■

Next, we shall prove that if (X, τ) is a T_0 pseudoconvex space, its dual X_r' separates points in X. To this end, we begin by establishing a lemma which is a consequence of an algebraic version of the Hahn-Banach theorem.

Lemma 1. *Let X be a real vector space and $a \in X$. If p is a subnorm on X, then there is a linear function $f : X \to \mathbb{R}$ such that $-p(-x) \leq f(x) \leq p(x)$ for all $x \in X$ and $f(a) = p(a)$.*

Proof. Let M be the vector subspace spanned by a. Define $f_1(\lambda a) = \lambda p(a)$ for all $\lambda \in \mathbb{R}$. If $\lambda \geq 0$, $f_1(\lambda a) = p(\lambda a)$ and if $\lambda < 0$, $f_1(\lambda a) = \lambda p(a) \leq 0 \leq p(\lambda a)$, so $f_1 \leq p$. Therefore, by Theorem 3.2 of [22], there exists a linear function $f : X \to \mathbb{R}$ such that f extends f_1 and $-p(-x) \leq f(x) \leq p(x)$. Clearly $f(a) = p(a)$. ■

Proposition 2. *Let (X, τ) be a T_0 pseudoconvex space. If $f(a) = 0$ for every $f \in X_r'$, then $a = 0$.*

Proof. Suppose that $a \neq 0$. Then, since (X, τ) a T_0 pseudoconvex space, there is an upper semicontinuous subnorm p on X such that $p(a) \neq 0$ or $p(-a) \neq 0$. If $p(a) \neq 0$, by Lemma 1, there is $f : X \to \mathbb{R}$ linear such that $-p(-x) \leq f(x) \leq p(x)$ for all $x \in X$ and $f(a) = p(a)$. Therefore, since $p^{-1}(]-\infty, \epsilon[) \subset f^{-1}(]-\infty, \epsilon[)$, $f \in X_r'$ and $f(a) \neq 0$ which is a contradiction. If $p(-a) \neq 0$ the proof is similar. ■

Proposition 3. *If (X, τ) is a T_0 pseudoconvex space then $(X, \sigma(X, X_r'))$ is a T_0 (pseudoconvex) space.*

Proof. By the previous proposition, we have that

$$\bigcap_{V \in \mathcal{B}} V \cap (-V) = \bigcap_{f \in X_r'} \bigcap_{\epsilon > 0} f^{-1}(]-\epsilon, \epsilon[) = \bigcap_{f \in X_r'} f^{-1}(0) = 0.$$

Thus, $(X, \sigma(X, X_r'))$ is obviously a T_0 space. ■

Let us recall that if (X,τ) is a T_0 paratopological vector space, then (X,τ^*) is a Hausdorff topological vector space, where $\tau^* = \tau \vee (-\tau)$. If X' is the topological dual of (X,τ^*) then $\sigma(X,X')$ is the usual weak topology of X. We shall study the relationship between $\sigma(X,X')$ and the right weak topology of X.

Proposition 4. *If (X,τ) is a T_0 paratopological vector space, then the topology $\sigma(X,X_r') \vee (-\sigma(X,X_r'))$ is coarser than $\sigma(X,X')$.*

Proof. Since $X_r' \subset X'$, then $\sigma(X,X_r')$ is coarser than $\sigma(X,X')$.
Now, since $(X,\sigma(X,X'))$ is a topological vector space, the topology $-\sigma(X,X_r')$ is also coarser than $\sigma(X,X')$. Hence, $\sigma(X,X_r') \vee (-\sigma(X,X_r'))$ is coarser than $\sigma(X,X')$. ∎

The effort to find examples in which the topology $\sigma(X,X_r') \vee (-\sigma(X,X_r'))$ is not equal to $\sigma(X,X')$ was in vain. In the sequel of this section we present two results that might support a conjecture on the equality.

Proposition 5. *If (X,τ) is a finite dimensional T_0 paratopological vector space, then $\sigma(X,X_r') \vee (-\sigma(X,X_r')) = \sigma(X,X')$.*

Proof. The statement follows immediately from the well-known fact that a finite-dimensional vector space has only one topology under which it is a separated convex space. ∎

Theorem 1. *Let $(X,\| \ \|)$ be a real normed lattice, and let $q(x) = \|x^+\|$. If (X,τ) is the pseudoconvex space determined by the quasi-norm q and X_r' is its right dual, then $\sigma(X,X_r') \vee (-\sigma(X,X_r')) = \sigma(X,X')$.*

Proof. First, it is convenient to notice that the sets

$$B(u_1,u_2,\ldots,u_n\,\epsilon) = \bigcap_{i=1}^{n} u_i^{-1}(]-\epsilon,\epsilon[)$$

with $u_1,u_2,\ldots,u_n \in X_r'$ and $\epsilon > 0$ are a base of neighborhoods of the origin for the topology $\sigma(X,X_r') \vee (-\sigma(X,X_r'))$.

Let V be a $\sigma(X,X')$-neighborhood of the origin. Then, there exist $\epsilon > 0$ and $v_1,v_2,\ldots,v_n \in X'$ such that

$$\bigcap_{i=1}^{n} v_i^{-1}(]-\epsilon,\epsilon[) \subset V.$$

By the Riesz decomposition theorem [16], $v_i = (v_i)^+ - (-v_i)^+$, where $(v_i)^+$ and $(-v_i)^+$ are linear continuous and positive functions on $(X,\| \ \|)$. So, by Corollary 1 of [2], $(v_i)^+$ and $(-v_i)^+ \in X_r'$ for each i. Therefore, the set

$$W = B(v_1^+,v_2^+,\ldots,v_n^+,\epsilon/2) \cap B((-v_1)^+,(-v_2)^+,\ldots,(-v_n)^+,\epsilon/2)$$

is a $\sigma(X,X_r') \vee (-\sigma(X,X_r'))$-neighborhood of the origin and it holds that

$$W \subset \bigcap_{i=1}^{n} v_i^{-1}(]-\epsilon,\epsilon[) \subset V.$$

∎

3 Quotient paratopological vector spaces

Let X be a real vector space and let M be a vector subspace of X and let ϕ be the canonical mapping of X onto X/M, that is, the mapping which assigns to each $x \in X$ its equivalence class $\hat{x} = x + M$.

If (X,τ) is a paratopological vector space and $\mathcal{U}$ is a base of semibalanced neighborhoods of the origin, by Theorem 4.2 of [1], the sets $\phi(U)$ with $U \in \mathcal{U}$ form a base of neighborhoods of the origin in a topology $\hat{\tau}$ on X/M, called *quotient topology*, under it X/M is a paratopological vector space. Besides, if (X,τ) is a pseudoconvex space, the quotient paratopological vector space is a pseudoconvex space.

On the other hand, if p is the Minkowski functional of the convex and semibalanced neighborhood U of the origin, the Minkowski functional of $\phi(U)$ is given by

$$\hat{p}(\hat{x}) = inf\{\lambda : \lambda \geq 0, \hat{x} \in \lambda\phi(U)\}.$$

Now, $\hat{x} \in \lambda\phi(U)$ if and only if $\hat{x} = \hat{y}$ with $y \in \lambda U$, then

$$\hat{p}(\hat{x}) = inf\{inf\{\lambda : \lambda \geq 0, y \in \lambda U\}, y \in \hat{x}\}.$$

Hence

$$\hat{p}(\hat{x}) = inf\{p(y) : y \in \hat{x}\}.$$

Next we study some properties of quotients of paratopological spaces.

Proposition 6. *If (X,τ) is a paratopological vector space and F is a vector subspace of X, then F is closed in (X,τ) if and only if F is closed in $(X,-\tau)$.*

Proof. Suppose F is closed in (X,τ) and $x \notin F$. Since $-x \notin F$, there is a neighborhood U of the origin in (X,τ) such that $-x+U \subset X-M$. Therefore, $x-U \subset X-M$ and so $X-M$ is closed in $(X,-\tau)$. ■

Proposition 7. *X/M is a T_1 space if and only if M is a closed subset of (X,τ).*

Proof. Since $M = \hat{0}$ in the vector space X/M, by Proposition 2.3.3 of [13], we conclude the result. ■

Proposition 8. *If X/M is a T_0 space then M is closed in (X,τ^*)*

Proof. Suppose M is not closed in (X,τ^*). Then, if $x \notin M$, we have

$$(x + U \cap (-U)) \cap M \neq \emptyset$$

for all neighborhoods U of the origin in (X,τ). Thus,

$$x \in U \cap (-U) + M \subset (U + M) \cap (-U + M) = \phi(U) \cap \phi(-U),$$

and therefore $x \in \bigcap_{U\in\mathcal{B}} \phi(U) \cap \phi(-U)$. Now, since X/M is a T_0 space, we have that

$$\bigcap_{U\in\mathcal{B}} \phi(U) \cap \phi(-U) = M,$$

thus $x \in M$, which is a contradiction. ■

In the following example we show that the converse of this property is not true.

Example 1. Consider the paratopological vector space $(\mathbb{R}^2, u)$, where $\mathbb{R}$ is the usual vector space and u is the topology induced by the quasi-norm defined on $\mathbb{R}^2$ by $u(x, y) = max\{x^+, y^+\}$. Let $M = \{(x, y) \in \mathbb{R}^2 : y = x\}$. M is a closed vector subspace in $(\mathbb{R}^2, u^*)$, but $\mathbb{R}^2/M$ is not a T_0 space since $\hat{u}$ is the trivial topology on $\mathbb{R}^2/M$.

This example shows also that if (X, τ) is a paratopological vector space, then the topology $\hat{\tau} \vee (-\hat{\tau})$ is coarser than the quotient topology of $\tau \vee -\tau$. Furthermore, we have shown that the quotient of a T_0 paratopological space is not a T_0 space.

4 Projective limits of paratopological vector spaces

In this section we define the projective limit of a family of paratopological vector spaces. We shall follow a process that is analogous to the one which appears in [19] for the definition of the projective limit of locally convex spaces.

Theorem 2. *Let X be a vector space, and for each $i \in I$ let f_i be a linear mapping from X into the paratopological vector space (X_i, τ_i), such that $\bigcap_{i \in I} f_i^{-1}(0) = \{0\}$. Then there is a coarsest topology on X for which all the f_i are continuous and which makes X into a paratopological vector space. If $\mathcal{B}_i$ is a base of neighborhoods in X_i, the finite intersections of the sets $f_i^{-1}(U)$ $(U \in \mathcal{B}_i,\ i \in I)$, form a base $\mathcal{B}$ of neighborhoods of the origin for X with respect to this topology.*

Proof. By Theorem 4.2 of [1], the sets of $\mathcal{B}$ form a base of neighborhoods of the origin in a topology on X under which X is a paratopological vector space and, obviously, it is the coarsest topology making the functions f_i continuous. ∎

The vector space X endowed with this topology is called the *projective limit* of the paratopological vector spaces X_i with the mappings f_i. Obviously, if X_i is a pseudoconvex space for all $i \in I$, the projective limit is also a pseudoconvex space.

Proposition 9. *If for each $i \in I$, X_i is a T_0 space, then its projective limit X is a T_0 space.*

Proof. Since X_i is a T_0 space for all $i \in I$, we have that

$$\bigcap_{V \in \mathcal{B}} V \cap (-V) = \bigcap_{i \in I} f_i^{-1}\Big(\bigcap_{U \in \mathcal{B}_i} U \cap (-U)\Big) = \bigcap_{i \in I} f_i^{-1}(0) = \{0\}.$$

∎

In [3] the concepts of right-bounded and right-precompact sets in a paratopological vector space were defined. A subset A of a paratopological vector space (X, τ) is *right-bounded* if for each neighborhood V of the origin there is $s > 0$ such that $A \subseteq tV$ whenever $t > s$. A is *right-precompact* if for each neighborhood V of the origin there are points $a_1, a_2, \ldots, a_n$ of A such that $A \subset \bigcup_{i=1}^{n}(a_i + V)$. If A is right-bounded or right-precompact in $(X, -\tau)$, A is called left-bounded or left precompact in (X, τ), respectively. We can give a characterization of these sets in a projective limite in a way that is analogous to the characterization of bounded or precompact sets in a projective limit of topological vector spaces.

Proposition 10. *Let X be the projective limit of the paratopological spaces X_i with the mappings f_i. Then a subset A of X is right-bounded, or right- precompact, if and only if each $f_i(A)$ has the same property.*

One of the most immediate examples of a projective limit is the right weak topology on any paratopological vector space X, obtained by taking for (f_i) the set of all upper semicontinuous linear forms of X (so that each X_i is $\mathbb{R}$).

It is well known that every separated locally convex space is a projective limit of normed spaces. We next show the analogous property in the realm of pseudoconvex spaces.

Theorem 3. *Let X be a real vector space and τ a topology on X. Then (X, τ) is a T_0 pseudoconvex space if and only if (X, τ) is a projective limit of quasi-normed spaces.*

Proof. Suppose (X, τ) is a T_0 pseudoconvex space and let $\mathcal{B}$ be a base of convex and semibalanced neighborhoods of the origin. Let $\mathcal{P}$ be the family of Minkowski functionals of the elements of $\mathcal{B}$. If $p \in \mathcal{P}$, p is a subnorm and the set $M_p = p^{-1}(0) \cap (-p)^{-1}(0)$ is a vector subspace of X. For each $p \in \mathcal{P}$, we consider the quotient space X/M_p. In this space the function $\hat{p}(\hat{x}) = \inf\{p(x+m) : m \in M_p\}$ is a subnorm. Now, we shall show that $\hat{p}$ is a quasi-norm.

Indeed, if $\hat{p}(\hat{x}) = \hat{p}(-\hat{x}) = 0$, then given $n \in \mathbb{N}$ there is $m \in M_p$ such that $p(x+m) < 1/n$. Thus, $p(x) \leq p(x+m) + p(-m) < 1/n$ and therefore $p(x) = 0$. In an analogous way we prove that $p(-x) = 0$. Hence $x \in M_p$ and then $\hat{x} = \hat{0}$.

Next, we shall prove that (X, τ) is the projective limit of the quasi-normed spaces $(X/M_p, \hat{p})$ with the canonical mappings ϕ_p.

Since (X, τ) is a T_0 space, we have that

$$\bigcap_{p \in \mathcal{P}} p^{-1}(0) \cap (-p)^{-1}(0) = \{0\},$$

and then $\bigcap_{p \in \mathcal{P}} \phi_p^{-1}(\hat{0}) = \{0\}$.

On the other hand, since ϕ_p are continuous, the projective limit is coarser than τ. Now, if U is a neighborhood of the origin in (X, τ), there is $p \in \mathcal{P}$ such that $V = \{x \in X : p(x) < 1\} \subset U$. We prove that U is a neighborhood of the origin in the projective limit, by showing that $\phi_p^{-1}(\{\hat{x} : \hat{p}(\hat{x}) < 1\}) \subset V$.

If $\hat{p}(\hat{y}) < 1$, there is $x \in \hat{y}$ such that $p(x) < 1$. Since $x - y \in M_p$ we have that $p(y) \leq p(y-x) + p(x) = p(x) < 1$ and so $y \in V$.

The converse is obvious, since a quasi-normed space is a pseudoconvex space and the projective limit of T_0 pseudoconvex spaces is a T_0 pseudoconvex space as we have observed above. ■

In the next example we construct the quasi-normed spaces whose projective limit defines the right weak topology, according to the proof of the above theorem.

Example 2. Let (X, τ) be a paratopological vector space and $\sigma(X, X_r')$ its right weak topology. First, we prove that if $f \in X_r'$ and $U = \{x \in X : f(x) < \epsilon\}$, then the Minkowski functional of U coincides with $\frac{1}{\epsilon} f^+$, where $f^+(x) = (f(x))^+$.

In fact, if q is the Minkowski functional of U, since $\frac{1}{\epsilon}f(x) \leq \frac{1}{\epsilon}f^+(x)$, we have that $q \leq \frac{1}{\epsilon}f^+$. Now, if $f(x) \geq 0$, then $\frac{1}{\epsilon}f^+(x) = \frac{1}{\epsilon}f(x)$ and so $\frac{1}{\epsilon}f^+(x) \leq q(x)$. On the other hand, if $f(x) \leq 0$, then $\frac{1}{\epsilon}f^+(x) = 0$, hence $\frac{1}{\epsilon}f^+(x) \leq q(x)$. Thus, $q = \frac{1}{\epsilon}f^+$.

Let V be the $\sigma(X, X'_r)$-neighborhood of the origin

$$V = \{x \in X : f_1(x) < \epsilon, f_2(x) < \epsilon, \ldots, f_n(x) < \epsilon\}$$

with $f_1, f_2, \ldots, f_n \in X'_r$ and $\epsilon > 0$, and let p be the Minkowski functional of V. If $V_i = \{x \in X : f_i(x) < \epsilon\}$ and p_i its Minkowski functional, since

$$V = \bigcap_{1 \leq i \leq n} V_i,$$

it follows that $p = sup\{p_i : 1 \leq i \leq n\} = sup\{\frac{1}{\epsilon}f_i^+ : 1 \leq i \leq n\}$.

We shall now prove that

$$p^{-1}(0) \cap (-p)^{-1}(0) = \bigcap_{1 \leq i \leq n} Ker f_i.$$

In fact, if $p(x) = p(-x) = 0$, then $nx \in V$ and $-nx \in V$ for each $n \in \mathbb{N}$, therefore $f_i(x) \leq 0$ and $f_i(-x) \leq 0$. Thus, $f_i(x) = 0$, for each i. Now, if $f_i(x) = 0$, then $\lambda x \in V$ and $\lambda(-x) \in V$ for each $\lambda > 0$, and thus $p(x) = p(-x) = 0$.

Furthermore,

$$\hat{p}(\hat{x}) = inf\{p(x+m) : m \in \bigcap_{1 \leq i \leq n} Ker f_i\} =$$

$$inf\{\frac{1}{\epsilon} sup_{1 \leq i \leq n}(f_i(x+m))^+ : m \in \bigcap_{1 \leq i \leq n} Ker f_i\} =$$

$$= \frac{1}{\epsilon} sup_{1 \leq i \leq n}(f_i(x))^+ = p(x)$$

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E.U. Informática, Departamento de Matemática Aplicada,
Universidad Politécnica de Valencia,
46071 Valencia, Spain
E-mail: calegre@mat.upv.es