Bubble motion and evolution described with quaternions

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Abstract

Based on observations by one or several cameras of the trajectory of a bubble it is interpolated. We show that this can be done in a very efficient way using dual quaternions. It is shown how quaternionic splines are a solution of this interpolation problem.

1 Motion of Bubbles and its Characteristics

There are several forms of making bubbles in a fluid, injecting gas and cavitation are the main ones.

In the first case long duration bubbles of several shapes and behaviours are obtained, in the second one small and short duration bubbles are obtained which coalesce very fast.

In the first case the formation of the bubble is determined by the size of the orifice from where air is injected. The speed of the air through the orifice and the shape of it determine the ulterior behaviour of the bubble.

In the second case it depends in the procedure used to produce the cavitation, two procedures, with pumps or with ultrasound are the most commonly used.

According to Keung[1] we will refer to several empirical formulas that determine the volume of the bubble:

$$V_p = \frac{2\pi R\sigma}{g\Delta\rho}$$

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for the volume and:

$$a = \left(\frac{z}{2}\frac{R\sigma}{\Delta\rho g}\right)^{1/2}$$

for the radius of the bubble.

Here:
$R =$ orifice radius in cm
$\sigma =$ surface tension, dynes/cm
$g =$ gravitation in cm/s^2
$\Delta\rho =$ difference between the density of the liquid and the density of the bubble in $gram/cm^3$

The flow rate of the gas also has influence in the size with the formula:

$$a = G_s^n$$

Where G_s is the air flow rate and $n = 0.1..0.44$

2 Bubble Shape

Diameter $<$ 0.1cm solid spherical
Diameter $>$ 0.01cm deviational from spherical
Diameter $>$ 0.1cm ellipsoidal

3 Movement

The movement depends strongly on Reynolds number:

$$R_e = \frac{U_a}{\gamma}$$

here:
$U =$ bubble velocity
$a =$ bubble radius
$\gamma =$ kinematic viscosity of the fluid

then for:

$R_e < 1$
$a < 0.01cm$ Stokes law regime:

$$v = \frac{1}{3}\left(\frac{ga^2}{\gamma}\right)$$

in this case bubbles move vertically without oscillating

for:

$1 < R_e < 800$
$0.01 < a < 0.1cm$

$$v = 2\sqrt{\frac{ga}{0.9}}$$

bubbles move in a helicoidal trajectory

for:

$R_e \geq 800$

$a > 0.1cm$ Bubbles are ellipsoidal and its movement is irregular they tend to subdivide

4 The Data set

A device consisting on a cubic tank of one cubic meter with transparent faces and a device producer of the bubbles inside is used to film ascending bubbles with two perpendicular cameras.

To obtain a good contrast the bubbles must be of around 1 mm diameter or greater.

Helicoidal or irregular movement is obtained and the shape of the bubble changes during its ascension.

5 Hypothesis on the composition of the motion

In the first case we have only a vertical movement, the helicoidal movement or the irregular one are more interesting.

We will decompose the movement in 4:

1) The vertical ascending movement
2) The rotation around the axis of ascension.
3) The translation in the radius of rotation
4) The nutation around the axis of the bubble of the upper pole of it.

6 Quaternionic Spline Reconstruction of the Movement

The idea is to decompose the movement in translations (the ascending movement and the radial movement around the axis of the ascension) and rotation (around the axis of ascension). For that we use dual quaternions as follows:

The adjunction of a dual unit ϵ with $\epsilon^2 = 0$ to the quaternions yields the non-commutative ring $\mathcal{H}[\epsilon]$ of the dual quaternions:

According to Juettler [2] a dual quaternion:

$$Q = Q^0 + \epsilon Q^\epsilon = (q^0 + \vec{q}^0) + \epsilon(q^\epsilon + \vec{q}^\epsilon) \tag{1}$$

Consists in a real part $Q^0 = Re\, Q \in \mathcal{H}$ and the dual part $Q^\epsilon = DuQ \in \mathcal{H}$

The dual quaternion:

$$2v_0 + \epsilon\vec{v} \quad (v_0 \in R, \vec{v} \in R^3) \tag{2}$$

corresponds to the translation with the displacement vector :

$$\frac{1}{v_0}\vec{v} \;\; v_0 \neq 0 \tag{3}$$

The quaternion:

$$D = d_0 + \vec{d} \in \mathcal{H} \quad D \neq 0 \tag{4}$$

describes a rotation around the origin

A spatial displacement is a composition of the translation and the rotation, it corresponds to the dual quaternion:

$$Q = Q^0 + Q^\epsilon = (2v_0 + \epsilon\vec{v}) * (d_0 + \vec{d}) \tag{5}$$

This dual quaternion satisfies Plücker condition:

$$Du(Q * \overline{Q}) = q^0 q^\epsilon + \vec{q}^0 \cdot \vec{q}^\epsilon = 0 \tag{6}$$

Resulting from this, a motion may be considered a curve on the quadric hypersurface of the real projective 7-space.

7 Definition

Consider the polynomial:

$$Q(t) = \sum_{i=0}^{n} b_i^n(t) B_i \quad t \in R \tag{7}$$

where

$$b_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} \tag{8}$$

are the Bernstein polynomials and the coefficients $B_i \in H$

This polynomial satisfies also Plücker relation.

Let $m+1$ spatial displacements:

$$P_i = (2 + e\vec{s_i}) * (r_{i,0} + \vec{r_i}) \tag{9}$$

Where the first parentheses is the Trans(P_i) and the second the Rot(P_i) and $\vec{s_i}$, $\vec{r_i} \in R^3$; $r_{i,0} \in R$ with parameters $t_i \in R$

The displacement P_i is assumed to describe the position of the moving space $\hat{E}^3$ with respect to the fixed space E^3 at the point on time t

This interpolation fulfills the following requirements:

1)The interpolating motion is found by solving a set of linear equations.

2) The result of the interpolation doesn't depend on the choice of the orientations of the

fixed and the moving space.

3) The interpolation problem may be considered in a mechanical or in a geometrical point of view.

8 The interpolation of the rotational part

The interpolation of the Q-motion has the form:

$$Q_{rot} = \sum_{j=0}^{k} b_j^k(t) C_j \tag{10}$$

The coefficients C_j are the unknowns and it has to satisfy the interpolation conditions:

$$Q_{rot}(t_i) = \lambda_i(r_{i,0} + \vec{r_i}) \quad (i = 0, ..., m) \tag{11}$$

9 Algorithm

1) Estimate the velocities $\vec{v_i}$ of the origin and the angular velocities $\vec{w_i}$ of the moving space at the given positions P_i
2) Construct the rotational cubic Q-motion with the algorithm 1
3) Construct the translational cubic Q-motion
4) Define piecewise the motion as the composition of both
The translational spline Q-motion is given by:

$$Q_{tras}^{(i)} = 2 + \epsilon \sum_{j=0}^{3} d_j^3 \left(\frac{t - t_i}{t_{i+1} - t_i} \right) \vec{p_j^{(i)}} \tag{12}$$

The rotational:

$$Q_{rot}^{(i)} = 2 + \epsilon \sum_{j=0}^{3} d_j^3 \left(\frac{t - t_i}{t_{i+1} - t_i} \right) C_j^{(i)} \tag{13}$$

10 Our proposal

Basically our proposal consists on using the physical data we may have about the movement as initial velocity, accelerations or so; in order to define the spline we are going to use. This use also admits to do extrapolations and not only interpolations with some sense.

The advantages of this procedure is that we are not constrained for example to consider translations with constant velocities or rotations with constant angular velocity, this makes the resulting movement more "real".

Let now be $P = \{p_1, p_2, ..., p_n\}$ a set of given positions of a moving object and suppose we have also the set $V = \{v_1, v_2, ..., v_n\}$ of the velocities in such positions, we can think also in having some $A = \{a_k, .., a_m\}$ set of accelerations in some of such positions.

Velocities and accelerations may be calculated in a previous development using for example only successive positions or they may be directly given data.

Of course our hypothesis is that $P \in \mathcal{H}$ as well as V and A

If we have a function $X(t)$ that interpolates P then it is desirable that $\dot{X}$ interpolate V and $\ddot{X}$ interpolate A.

Now the steps of the interpolation processes proposed by Jüttler may be applied to each of the sets but combining them by its physical sense.

As each P is a quaternion $\{p_0, p_1, p_2, p_3\}$ where $p_0 = \cos\theta/2$, $p_1 = x_p(t)\sin\theta/2$, $p_2 = y_p(t)\sin\theta/2$ and $p_3 = z_p(t)\sin\theta/2$ where $\theta = \omega t$ is the angle of rotation around the axis x_p, y_p, z_p we have that all our components are depending upon the time in a translational and in a rotational sense.

11 Some consequences

There are some very important consequences and advantages of the use of quaternions in a B-spline environment

One of them is that when we use B-splines we are really minimizing the energy by putting the second derivative to zero, that is:

$$\vec{X} = \vec{X}(t) \tag{14}$$

implies

$$\frac{d}{dt}\left(\frac{\vec{X}}{X}\right) = \frac{\vec{X}(X \wedge \dot{X})}{X^3} = \frac{(X \times \dot{X}) \times X}{X^3} \tag{15}$$

In order to obtain C^2 continuity (needed to describe a movement) we will resort to a well known Boehm[3] parametric cubic B-spline.

As we are dealing with parametric quaternion functions to obtain our continuity requirements we will do piecewise interpolation "connecting" the movement between the consecutive positions.

12 An example

To obtain the movement between the positions we may do as Jüttler:

$$X_{tras}^{(i)} = 2 + \epsilon \sum_{j=0}^{3} d_j^3 \left(\frac{t - t_i}{t_{i+1} - t_i}\right) \vec{p_j^{(i)}} \tag{16}$$

The rotational:

$$X_{rot}^{(i)} = 2 + \epsilon \sum_{j=0}^{3} d_j^3 \left(\frac{t - t_i}{t_{i+1} - t_i}\right) C_j^{(i)} \tag{17}$$

$X(t)$ will then be the quaternion multiplication of both

To do this we have to compute the $m+1$ coefficients $C_j \in H$ of the rotational motion and compute the $m+1$ coefficients $\vec{p}_j \in R^3$ of the translational motion. This means as seen above to solve:

$$\sum_{j=0}^{k} b_j^k(t_i) C_j = R_i^{(0)} \quad i = 0, ..., m \tag{18}$$

and

$$\sum_{j=0}^{l} b_j^l(t_i)\vec{p_j} = \vec{s_i} \quad i = 0, ..., m \tag{19}$$

Then we don't have to precalculate $R_i^{(0)}$ as Jüttler does because we have them (the V_i) the same with the $\vec{s_i}$

But we also have that $\dot{X}(t) = V(t)$ and that we can find $V(t)$ as:

$$V_{tras}^{(i)} = 2 + \epsilon \sum_{j=0}^{3} d_j^3 \left(\frac{t - t_i}{t_{i+1} - t_i} \right) \vec{v_j^{(i)}} \tag{20}$$

and

$$V_{rot}^{(i)} = 2 + \epsilon \sum_{j=0}^{3} d_j^3 \left(\frac{t - t_i}{t_{i+1} - t_i} \right) V_j^{(i)} \tag{21}$$

So we have also to compute the $V \in H$ and the $v \in R^3$ which now means to solve:

$$\sum_{j=0}^{k} v_j^k(t_i)V_j = A_i^{(0)} \quad i = 0, ..., m \tag{22}$$

and

$$\sum_{j=0}^{l} v_j^l(t_i)\vec{v_j} = \vec{a_i} \quad i = 0, ..., m \tag{23}$$

where we can also know from the data $A_i^{(0)}$ and $\vec{a_i}$ but having the advantage of knowing also that:

$$\dot{X}(t) = V_{tras}^{(i)} \cdot V_{rot}^{(i)} \tag{24}$$

and that something equal happens with $\ddot{X}(t)$ so there are some ordinary differential equations linking all.

Note that it is different than just finding a parametrical 3D curve but it uses the same tools, basically B-splines and other CAGD tools.

As can also be seen there are a lot of different variations from the above depending upon the application and the specific data of our problem, our idea in this case was just to focus the attention on the possibility of using all this tools at the same time and the advantages it may represent.

13 A more complicated problem

Suppose now that we have a set of reference systems $\mathcal{S} = \{S_1, S_2, ..., S_n\}$ and in each one a set of observational data taken all not necessarily simultaneously. That may be the case of several cameras pointing at the bubble that may not be completely synchronized. Let them be $\mathcal{D} = \{D_{S_1}^{t_1}, D_{S_2}^{t_2}........D_{S_n}^{t_n}\}$

Now if we take t as a parameter then there is a quaternion transformation for each set

$$T_i = \overline{q} * D_{S_i}^{t_i} * q \tag{25}$$

that puts all the data in the same reference system. Then we proceed as in the first algorithm to determine the attitude and trajectory of the object with respect to any given reference system.

As they will be of course gaps between one and other set of data in time and space we can use the above algorithms to interpolate and obtain the whole movement.

The most important is that data from one and other source may overlap in time but there can't be contradicting between them because they are all images of the same body so the procedure will merge all the data sets to obtain a continuous movement.

14 Example

A vector r_{G_0} of the position of a reference system from where part of the data comes on the system G_0 can be presented as:

$$r_{G_0}(t) = \Delta r_{G_0}(t) + r_{0G_0} + v_{0G_0}(t - t_0) \tag{26}$$

G_0 is to note that we are taking that system of reference.

We must note that the coordinates of the vector $\Delta r_{S_0}(t_q)$ are known but only at the system S_0, we have to determine the coordinates of this vector with respect to G_0

Let $\Delta R_{G_0}(t)$ and $\Delta R_{S_0}(t)$ be the quaternions such that their vectorial parts are equal to the values of the vector $\Delta r(t)$ in the systems of reference G_0 and S_0 respectively.

As G_0 and S_0 are connected by an equation already seen the relation between $\Delta R_{G_0}(t)$ and $\Delta R_{S_0}(t)$ is the following:

$$\Delta R_{S_0}(t) = \overline{P} \Delta R_{G_0}(t) P \tag{27}$$

So that:

$$\Delta R_{G_0}(t) = P \Delta R_{S_0}(t) \overline{P} \tag{28}$$

Let then $R_{G_0}(t)$, R_{0G_0} and V_{0G_0} be the quaternions corresponding to the vectors $r_{G_0}(t)$, r_{0G_0} and v_{0G_0} respectively. The quaternions $R_{G_0}(t)$ and V_{0G_0} define the coordinates and initial velocity unknown of the system at the time $t = t_0$ with the quaternionic notation we have that:

$$R_{G_0}(t) = P \Delta R_{S_0}(t) \overline{P} + R_{0G_0} + V_{0G_0}(t - t_0) \tag{29}$$

From another point of view, the localization of one system referred to the other G_0 is the localization of the perspective of view.

The quaternion $C(t_h)$ corresponds to the position with respect to the system $G(t_h)$.

We need to interpolate $\Delta R_{S_0}(t_q)$ to find their values at the observation time t_h. According to the mean square method the function:

$$F(R_{0G_0}, V_{0G_0}) = \sum_{h=1}^{n} ||R_{G_0}(t_h) - U(t_h) C(t_h) \overline{U(t_h)}||^2 \tag{30}$$

must be minimized in relation to R_{0G_0} and V_{0G_0}

Consequently, deriving partially and imposing that such derivatives are cero we find the following system of equations:

$$nR_{0G_0} + m_1 V_{0G_0} = \sum_{h=1}^{n} T(t_h) \tag{31}$$

$$m_1 R_{0G_0} + m_2 V_{0G_0} = \sum_{h=1}^{n} T(t_h)(t_h - t_0) \tag{32}$$

or

$$m_b = \sum_{h=1}^{n} (t_h - t_0), \quad b = 1, 2 \tag{33}$$

and

$$T(t_h) = U(t_h)C(t_h)\overline{U(t_h)} - P\Delta R_{S_0}(t_h)\overline{P} \tag{34}$$

The existence of the solution means that:

$$\det \begin{pmatrix} n & m_1 \\ m_1 & m_2 \end{pmatrix} \neq 0 \tag{35}$$

The solution are two quaternions such that their vectorial parts are the localization and velocity we are looking for:

$$R_{0G_0} = \frac{(m_2 \sum_{h=1}^{n} T(t_h) - m_1 \sum_{h=1}^{n} T(t_h)(t_h - t_0))}{(nm_2 - m_1^2)} \tag{36}$$

$$V_{0G_0} = \frac{(n \sum_{h=1}^{n} T(t_h)(t_h - t_0) - m_1 \sum_{h=1}^{n} T(t_h))}{(nm_2 - m_1^2)} \tag{37}$$

15 Vortex explanation of the Movement

Now we will again focus on the physical problem in order to try an analytical approach of it.

An ascending bubble with some initial velocity creates a source at the top of it and a sink at the bottom which at is turn create each one a vortex.

Such vortices may create a rotation of the bubble over itself and being it a deformable object put its center of mass excentric creating the helicoidal movement around an axis.

The drag forces over a strong helicoidal trajectory may decrease the radius of the helix.

Each one of this movements and vortices have a quaternionic representation.

We will show in the Navier Stokes environment how vorticity may be introduced.

16 Quaternionic Representation of the Vortices

The Navier Stokes equations for incompressible viscous flow are:

$$\frac{Du}{Dt} = -\bigtriangledown P + \frac{1}{R}\bigtriangledown^2 u \ \ in\, D \tag{38}$$
$$\bigtriangledown \cdot u = 0 \ \ in\, D \tag{39}$$
$$u = 0 \ \ on\, \partial D \tag{40}$$

Taking ξ as the vorticity:

$$\xi = \bigtriangledown \times v \tag{41}$$

we have the vorticity transport equation:

$$\frac{D\xi}{Dt} = (\xi \cdot \bigtriangledown)u + \frac{1}{R}\bigtriangledown^2 \xi \tag{42}$$

Here u is the velocity, P the pressure, R the Reynolds number.
As $\bigtriangledown \cdot u = 0$ and $\xi = \bigtriangledown \times u$
there exists a vector function $\psi(x)$ such that $u = \bigtriangledown \times \psi$ then

$$\bigtriangledown^2 \psi = -\xi \tag{43}$$

In 3D ψ is the velocity potential in 2D is the stream function. The solution to the above differential equation is:

$$\psi(x,t) = \int L(x-z)\xi(z)dz \tag{44}$$

where:

$$L(x) = \begin{Bmatrix} \frac{-1}{2\pi} log|x| & x \in R^2 \\ \frac{1}{4\pi}\frac{1}{|x|} & x \in R^3 \end{Bmatrix} \tag{45}$$

since

$$u = \bigtriangledown \times \psi$$

we have that:

$$u(x,t) = \int K(x-z)\xi(z)dz \tag{46}$$

where K is:

$$K(x) = \frac{1}{2\pi}\frac{(-x_2, x_1)}{|x|^2} \ \ x \in R^2$$
$$K(x) = \frac{1}{4\pi|x|^3}\begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix} \ \ x \in R^3$$

The kernel K is singular for both dimensions. This idea leads to the so called vorticity equations. In fact the transformation produces a non-linear system of differential equations for the two unknown vector functions u and ξ

Another idea is, applying ideas of quaternionic and Clifford analysis to find a transformation into one non-linear equation only for the vorticity ξ. For this reason we use the higher-dimensional version of the Borel-Pompeiu formula:

$$TDu(x) = u(x) - Fu(x) \tag{47}$$

where T is the T-operator (Teodorescu transform), D the Dirac-operator and F the Cauchy integral. The Cauchy integral depends only on the boundary values of u.

That means that if $u = 0$ on the boundary then this part can be deleted of the formula.

Moreover, Du means for a quaternion valued function $(0, u)$ (u is the vector of velocity)

$$Du = (-div\, u,\ rot\, u) \tag{48}$$

As we are working with divergence free vectors and consequently

$$Du = (0,\ rotu)$$

Remembering that

$$rot\ u = \nabla \times u$$

we have that

$u = TDu$ and with $Du = rot\, u = \xi$ it follows

$$u = T\xi \tag{49}$$

This is an expression to describe the velocity u explicitly by the vorticity ξ. If the boundary values of u are not zero but some known quantity then we have this additional known summand F(boundary values of u). The operators T and F are defined following Guerlebeck [3] as:

$$(T_G u)(x) = -\int_G e(x-y)u(y)dG_y$$

$$(F_\gamma u)(x) = \int_\gamma e(x-y)\alpha(y)u(y)d\gamma_y$$

α is the outer normal to γ at the point y and $e(x)$ the fundamental solution (generalized Cauchy kernel) of the Dirac-operator.

In this way substituting in the above equations we obtain a nonlinear equation in ξ instead a system in u and ξ. To find representation formulas and numerical methods for ξ is one of the goals of the project. Because we have to evaluate only the vorticity (and not in addition the velocity, too) a better efficiency of this approach is expected.

17 Vortex and rotations

Now, a vortex is basically rotation and that rotation is expressed as a quaternion in many ways. We have seen an analytic expression where vorticity is the only variable left in Navier Stokes but that is a quaternionic variable an therefore represents a rotation.

The question is what is that rotation?

Remember that:

$$\xi = \bigtriangledown \times v$$

So if both are quaternions this is a rotation or a displacement and can be expressed as double quaternion where:

$$\xi = \mathbf{GH} \tag{50}$$

that is formed by two unit quaternions **G** and **H** such that:

$$\xi = G = DQ \quad v = H = \overline{D}Q \tag{51}$$

So ξ may be decomposed, or can either represent, in a "real" vortex and a stream, equivalent to the rotation and the displacement.

This can be used, having as data a field of vortices, to interpolate, using global interpolation, between this kind of quaternions and the result in a given point will be the same than using Navier Stokes.

Particularly this is useful to represent the movement of the bubble because all its movements may be represented as vortices.

18 Conclusions and Future Work

Our goal is to continue the experiments using moving cameras to follow the bubble in order to have an accurate description of the changes in shape.

A model based in the quaternionic expression of the vorticity is needed to substitute the interpolated movement obtained with observations.

The changes in the movement and shape and coalescence due to the increase of the initial velocity must be studied as well as the direction of the orifice.

References

[1] Jin Keung Choi Non spherical bubble behavior in vortex flow fields Proceedings IABEM2002

[2] Juettler B, Visualization of moving objects using dual quaternion curves, Comput and graphics vol 18 No 3 pp 315-326, 1994

[3] Bohem W Cubic B-spline curves and surfaces in computer aided geometric design. Computing 19:29-34,1977.

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