Taylor Series on the Hyperbolic Unit Ball

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Abstract

Hyperbolic monogenics are introduced as eigenfunctions of the hyperbolic angular Dirac operator Γ. Explicit formulae for the Taylor series of these functions are given, as well as integral formulae related to the hyperbolic angular Dirac operator.

1 Introduction

In this paper we consider a projective model for the m-dimensional hyperbolic unit ball, realized as the manifold of rays inside the future cone in the real orthogonal space $\mathbb{R}^{1,m}$ (see section 5). Using Clifford algebras (see section 2), it is possible to define a Clifford algebra structure on this manifold. This structure enables us to define the Dirac operator on sections of homogeneous line bundles (section 6). In previous papers (see e.g. [7] and [9]) we have calculated the fundamental solution for this operator together with a class of null-solutions. In this paper we will reformulate these results in terms of Gegenbauer functions (see section 3) leading to a function theory for the hyperbolic angular Dirac operator Γ, which arises as a first order differential operator on the hyperbolic unit ball. Indeed, in section 7 we prove some important integral formulae (such as Stokes' theorem and Cauchy's representation formula) and in section 8 we give an explicit formula for the Taylor series of eigenfunctions of this hyperbolic Dirac operator Γ. In the construction of this Taylor series, we make use of the Euclidean Cauchy kernel, which is defined in section 4 where we give an overview of the most important function theoretical results concerning the Dirac operator on flat Euclidean space $\mathbb{R}^m$.

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The main results of this paper were obtained by Ryan and his collaborators and Van Lancker for the case of the m-dimensional *sphere* (i.e. the positively curved space) in references [11], [15] and [16]. As both the hyperbolic unit ball and the sphere are real submanifolds of the complex sphere in $\mathbb{C}^m$, one can argue that these results extend to the hyperbolic situation via holomorphic continuation. However, the authors did not fully exploit the projective nature of the model for Riemannian spaces, in contrast to the approach followed in the present paper. The present approach can also be translated to the spherical situation, offering a way to reinterpret spherical monogenics as modulated versions of monogenic functions on $\mathbb{R}^m$.

One particular interesting case of the operator studied in this paper is the conformal Dirac operator on the hyperbolic unit ball, invariant under the group $Spin(2,m)$ or the conformal group, for which we refer to [11] and [12]. For the more general case of arbitrary manifolds, this has been done by Calderbank and Cnops, see references [2] and [4]. By considering the subgroup $Spin(1,m)$ of the conformal group one obtains a richer class of functions, the so-called hyperbolic monogenics, which are the subject of this paper.

2 Clifford Algebras

Throughout this paper two different Clifford algebras will be used : $\mathbb{R}_{0,m}$ and $\mathbb{R}_{1,m}$. Let us therefore start with the definition of the general universal Clifford algebra $\mathbb{R}_{p,q}$. Let $(e_1,\cdots,e_m)$ be an orthonormal basis for the real orthogonal space $\mathbb{R}^{p,q}$, where $p+q=m$, endowed with the inner product $\vec{x}\cdot\vec{y}=\sum_{k=1}^{p}x_iy_i-\sum_{k=p+1}^{p+q}x_iy_i$. The Clifford algebra $\mathbb{R}_{p,q}$ is the 2^m-dimensional real linear associative algebra defined by the following multiplication rules :

$$\begin{array}{ll} e_i^2=1 & i=1,\ldots,p \\ e_i^2=-1 & i=p+1,\ldots,p+q=m \\ e_ie_j+e_je_i=0 & i\neq j=1\ldots m \end{array}$$

An element of $\mathbb{R}_{p,q}$ is called a Clifford number; it has the form $a=\sum_{A\subset M}a_Ae_A$ where $a_A\in\mathbb{R}$ and $M=\{1,\cdots,m\}$, $e_A=e_{i_1}\cdots e_{i_k}$ for $A=\{i_1,\cdots,i_k\}$ with $1\leq i_1<\cdots<i_k\leq m$ and $e_\phi=1$. If A has k elements, e_A is called a k-vector. The space of k-vectors is denoted by $\mathbb{R}_{p,q}^{(k)}$. If $[a]_k$ is the projection of the Clifford number a on $\mathbb{R}_{p,q}^{(k)}$, then

$$a \;=\; \sum_{k=0}^{m}[a]_k\ , \quad \forall a\in\mathbb{R}_{p,q}\ .$$

The subspace $\mathbb{R}_{p,q}^{(+)}=\sum_{k\ \mathrm{even}}\oplus\mathbb{R}_{p,q}^{(k)}$ is a subalgebra of $\mathbb{R}_{p,q}$, called the *even subalgebra*; it is isomorphic with $\mathbb{R}_{q,p-1}\cong\mathbb{R}_{p,q-1}$. For two 1-vectors $\vec{x},\vec{y}\in\mathbb{R}_{p,q}^{(1)}$ - or vectors for short - we define the inner and outer product as follows :

$$\left\{\begin{array}{lcl} \vec{x}\cdot\vec{y} & = & \frac{1}{2}(\vec{x}\vec{y}+\vec{y}\vec{x}) \\ \vec{x}\wedge\vec{y} & = & \frac{1}{2}(\vec{x}\vec{y}-\vec{y}\vec{x}) \end{array}\right.$$

On the Clifford algebra $\mathbb{R}_{p,q}$ we have three important involutory (anti-)automorphisms. For all $a,b\in\mathbb{R}_{p,q}$ and $\lambda\in\mathbb{R}$ we define :

1. the main involution $a \mapsto \tilde{a}$

$$\tilde{e}_i = -e_i \ , \quad (ab)\tilde{} = \tilde{a}\tilde{b}$$

2. the reversion $a \mapsto a^*$

$$e_i^* = e_i \ , \quad (ab)^* = b^* a^*$$

3. the conjugation (also known as bar-map) $a \mapsto \bar{a}$

$$\bar{e}_i = -e_i \ , \quad \overline{(ab)} = \bar{b}\bar{a}$$

The following subgroups of $\mathbb{R}_{p,q}$ are of interest : the Clifford group $\Gamma(p,q)$ defined as the set of all invertible elements $g \in \mathbb{R}_{p,q}$ such that for all $\vec{x} \in \mathbb{R}^{(1)}_{p,q}$: $g\vec{x}(\tilde{g})^{-1} \in \mathbb{R}^{(1)}_{p,q}$, the Pin group $Pin(p,q)$ defined as the quotient group $\Gamma(p,q)/\mathbb{R}^+$ and the Spin group $Spin(p,q) = Pin(p,q) \cap \mathbb{R}^{(+)}_{p,q}$.

For each element $s \in Pin(p,q)$ the map $\chi(s) : \mathbb{R}^{p,q} \mapsto \mathbb{R}^{p,q} : \vec{x} \mapsto s\vec{x}\overline{s}$ induces a map from $\mathbb{R}^{p,q}$ onto itself. In this way $Pin(p,q)$ defines a double covering of the orthogonal group $O(p,q)$ whereas $Spin(p,q)$ defines a double covering of the orthogonal group $SO(p,q)$.

3 The Gegenbauer Functions

As we will frequently use Gegenbauer functions in this paper, we give a brief introduction to these special functions.

The Gegenbauer functions $C^\mu_\nu(z)$ and $D^\mu_\nu(z)$ are holomorphic functions in the z-plane cut along the real axis from $-\infty$ to 1, and solutions in this region of Gegenbauer's differential equation

$$(1-z^2)\frac{d^2 f}{dz^2} - (2\nu+1)z\frac{df}{dz} + \mu(\mu+2\nu)f = 0 \ , \quad \mu, \nu \in \mathbb{C} \ .$$

The Gegenbauer functions are defined in terms of the associated Legendre functions by :

$$C^\mu_\nu(z) = \pi^{\frac{1}{2}} 2^{-\mu+\frac{1}{2}} \frac{\Gamma(\nu+2\mu)}{\Gamma(\mu)\Gamma(1+\nu)} (z^2-1)^{\frac{1}{4}-\frac{\mu}{2}} P^{-\mu+\frac{1}{2}}_{\nu+\mu-\frac{1}{2}}(z) \qquad (1)$$

$$D^\mu_\nu(z) = \pi^{-\frac{1}{2}} e^{2i\pi(\mu-\frac{1}{4})} 2^{-\mu+\frac{1}{2}} \frac{\Gamma(\nu+2\mu)}{\Gamma(\mu)\Gamma(1+\nu)} (z^2-1)^{\frac{1}{4}-\frac{\mu}{2}} Q^{-\mu+\frac{1}{2}}_{\nu+\mu-\frac{1}{2}}(z) \qquad (2)$$

The Gegenbauer function $C^\mu_\nu(z)$ has zeroes for $\mu \in -\mathbb{N}$ and simple poles for $\nu+2\mu \in -\mathbb{N}$, while the Gegenbauer function $D^\mu_\nu(z)$ has zeroes for $\mu \in -\mathbb{N}$ and simple poles for $\nu + 2\mu \in -\mathbb{N}$ (see e.g. [6]).

The functions $D^\mu_\nu(z)$ and $C^\mu_\nu(z)$ satisfy the same recurrence relations, see e.g. [10]. Some of these relations that will be used in the sequel are listed here :

$$\frac{d}{dz}C^\mu_\nu(z) = 2\mu C^{\mu+1}_{\nu-1}(z)\ . \tag{3}$$

$$\nu C^\mu_\nu(z) = 2\mu[zC^{\mu+1}_{\nu-1}(z) - C^{\mu+1}_{\nu-2}(z)]\ . \tag{4}$$

$$(\nu+2\mu)C^\mu_\nu(z) = 2\mu[C^{\mu+1}_{\nu}(z) - zC^{\mu+1}_{\nu-1}(z)] \tag{5}$$

For $\nu = n \in \mathbb{N}$, the Gegenbauer function of the second kind $C^\mu_\nu(z)$ reduces to the classical Gegenbauer polynomial $C^\mu_n(t)$, defined as the coefficient of z^n in the power series expansion of $(1-2tz+z^2)^{-\mu}$:

$$(1-2tz+z^2)^{-\mu} = \sum_{n=0}^{\infty} C^\mu_n(t)z^n\ ,\quad |z| < |t \pm (t^2-1)^{\frac{1}{2}}| \tag{6}$$

4 The Euclidean Cauchy Kernel $E(\vec{x})$ on $\mathbb{R}_{0,m}$

Consider the Clifford algebra $\mathbb{R}_{0,m}$ generated by an orthonormal basis $(e_1, \cdots, e_m)$ for $\mathbb{R}^{0,m}$. The Dirac operator $\partial_{\vec{x}}$ on $\mathbb{R}^{0,m}$ is defined as $\sum_{i=1}^m e_i\partial_{x_i}$ and a polar decomposition for this operator is given by $\partial_{\vec{x}} = \vec{\xi}(\partial_r + \frac{1}{r}\Gamma_{\vec{\xi}})$ with $\vec{x} = r\vec{\xi}$, $\vec{\xi}$ belonging to the unit sphere S^{m-1} in $\mathbb{R}^m$, and with $\Gamma_{\vec{\xi}} = -\vec{x}\wedge\partial_{\vec{x}}$ the spherical Dirac operator on S^{m-1}. A function $f(\vec{x})$ defined in an open set $\Omega \subset \mathbb{R}^m$ such that $\partial_{\vec{x}}f = 0$ in Ω is called a monogenic function in Ω. The concept of monogenic functions lies at the very heart of Clifford analysis and has been studied in extenso, see e.g. [1] and [5].

A $\mathbb{R}_{0,m}$-valued C^∞ function $P_k(\vec{\xi})$ on S^{m-1} is called an *inner* spherical monogenic of order k if it is the restriction to the unit sphere of a polynomial monogenic function of order k on $\mathbb{R}^m$. Inner spherical monogenics are global eigenfunctions of the spherical Dirac operator, with eigenvalue $\Gamma_{\vec{\xi}}P_k = -kP_k$. A C^∞ function $Q_k(\vec{\xi})$ on S^{m-1} is an *outer* spherical monogenic of order k if it is the restriction to the unit sphere of a monogenic function on $R^m \setminus \{\vec{0}\}$ which is homogeneous of degree $(1-k-m)$. Outer spherical monogenics are also eigenfunctions of the spherical Dirac operator, satisfying $\Gamma_{\vec{\xi}}Q_k = (k+m-1)Q_k$. The set of inner (resp. outer) spherical monogenics provided with the obvious laws of addition and (right) multiplication with Clifford numbers is a right Clifford-module, denoted as $M^+(k)$ (resp. $M^-(k)$). Inner and outer spherical monogenics are related : $P_k(\vec{\xi}) \in M^+(k) \Rightarrow \vec{\xi}P_k(\vec{\xi}) \in M^-(k)$ and $Q_k(\vec{\xi}) \in M^-(k) \Rightarrow \vec{\xi}Q_k(\vec{\xi}) \in M^+(k)$.

The fundamental solution for the Euclidean Dirac operator $\partial_{\vec{x}}$ is the so-called Cauchy kernel $E(\vec{x})$, defined as :

$$E(\vec{x}) = \frac{1}{A_m}\frac{\vec{x}}{|\vec{x}|^m}\ ,$$

with $A_m = \frac{2\pi^{m/2}}{\Gamma(\frac{m}{2})}$ the area of the unit sphere S^{m-1} in $\mathbb{R}^m$. The Cauchy kernel is both left and right monogenic in $\mathbb{R}^m \setminus \{\vec{0}\}$ w.r.t. the Dirac operator $\partial_{\vec{x}}$ and $\partial_{\vec{x}} E(\vec{x}) = -\delta(\vec{x})$ in distributional sense. As the Dirac operator $\partial_{\vec{x}}$ is invariant under translations, we also have that $\partial_{\vec{x}} E(\vec{x}-\vec{y}) = -\delta(\vec{x}-\vec{y})$. A series representation for $E(\vec{x}-\vec{y})$ can easily be found as follows :

$$E(\vec{x}-\vec{y}) = \frac{1}{A_m}\frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|^m} = \frac{1}{A_m}\frac{1}{m-2}\,\partial_{\vec{y}}\,\frac{1}{|\vec{x}-\vec{y}|^{m-2}}\ .$$

With $\vec{x} = |\vec{x}|\ \vec{\xi}$, $\vec{y} = |\vec{y}|\ \vec{\eta}$ and putting $r = |\vec{x}|/|\vec{y}|$ and

$$t = < \vec{\xi}, \vec{\eta} > = -\vec{\xi}\cdot\vec{\eta} = \sum_{j=1}^{m} \xi_j \eta_j$$

the standard Euclidian inner product of $\vec{\xi}$ and $\vec{\eta} \in S^{m-1}$, one can easily verify that for $|\vec{x}| < |\vec{y}|$

$$\frac{1}{|\vec{x}-\vec{y}|^{m-2}} \quad = \quad |\vec{y}|^{2-m}\ (1-2tr+r^2)^{-(\frac{m}{2}-1)}\ .$$

In view of (6), we find :

$$\partial_{\vec{y}}\ \frac{1}{|\vec{x}-\vec{y}|^{m-2}} \quad = \quad \vec{\eta}\sum_k \left\{(2-m-k)C_k^{\frac{m}{2}-1}(t) + \Gamma_{\vec{\eta}}\ (t)\frac{d}{dt}\left(C_k^{\frac{m}{2}-1}(t)\right)\right\}\frac{|\vec{x}|^k}{|\vec{y}|^{m+k-1}}\ .$$

Using relations (4) and (5) and the fact that $\Gamma_{\vec{\eta}}\ (t) = \vec{\xi}\wedge\vec{\eta}$, we obtain :

$$\begin{aligned} E(\vec{x}-\vec{y}) &= -\frac{1}{A_m}\sum_{k=0}^{\infty}\frac{|\vec{x}|^k}{|\vec{y}|^{k+m-1}}\ \left\{C_k^{\frac{m}{2}}(t)\vec{\eta} - C_{k-1}^{\frac{m}{2}}(t)\vec{\xi}\right\} \\ &= -\frac{1}{A_m}\sum_{k=0}^{\infty}\frac{|\vec{x}|^k}{|\vec{y}|^{k+m-1}}\ C_k(\vec{\eta},\vec{\xi})\ , \end{aligned} \tag{7}$$

where we have introduced the functions $C_k(\vec{\eta},\vec{\xi})$ on $S^{m-1}\times S^{m-1}$ as

$$C_k(\vec{\eta},\vec{\xi}) = C_k^{\frac{m}{2}}(<\vec{\xi},\vec{\eta}>)\vec{\eta} - C_{k-1}^{\frac{m}{2}}(<\vec{\xi},\vec{\eta}>)\vec{\xi}\ . \tag{8}$$

For all $k \in \mathbb{N}$ the function $C_k(\vec{\eta},\vec{\xi})$ is an inner spherical monogenic of order k with respect to the Dirac operator $\partial_{\vec{x}}$ and an outer spherical monogenic of order k with respect to the Dirac operator $\partial_{\vec{y}}$, whence :

$$\begin{aligned} \Gamma_{\vec{\xi}}\, C_k(\vec{\eta},\vec{\xi}) &= -kC_k(\vec{\eta},\vec{\xi}) \\ \Gamma_{\vec{\eta}}\, C_k(\vec{\eta},\vec{\xi}) &= (k+m-1)C_k(\vec{\eta},\vec{\xi}) \end{aligned}$$

Each $f \in L_2(S^{m-1})$ can be decomposed as

$$f(\vec{\xi}) \quad = \quad \sum_{k=0}^{\infty} P(k)f(\vec{\xi}) + Q(k)f(\vec{\xi})$$

where the series converges in L_2-sense on S^{m-1}. The projections $P(k)f$ and $Q(k)f$ of the function f on the spaces $M_+(k)$ and $M_-(k)$ of inner and outer spherical monogenics of order k are given by :

$$\begin{aligned} P(k)f(\vec{\eta}) &= -\frac{1}{A_m}\vec{\eta}\int_{S^{m-1}} C_k(\vec{\eta},\vec{\xi})f(\vec{\xi})dS(\vec{\xi}) \\ Q(k)f(\vec{\eta}) &= -\frac{1}{A_m}\int_{S^{m-1}} C_k(\vec{\eta},\vec{\xi})\vec{\xi}f(\vec{\xi})dS(\vec{\xi}) \end{aligned}$$

For further details, we refer the reader to [5].

5 Hyperbolic Space

In this section a model for the m-dimensional hyperbolic unit ball is introduced. Consider the real orthogonal space $\mathbb{R}^{1,m}$ of signature $(1,m)$ with an orthonormal basis $(\epsilon, e_1, \cdots, e_m)$. Note that we prefer to make a clear distinction between the time unit vector ϵ and the spatial unit vectors e_i. Space-time vectors will be denoted by $X = \epsilon T + \vec{X}$, again making a clear distinction between the time co-ordinate T and the spatial co-ordinates $\vec{X} = (X_1, \cdots, X_m)$. The quadratic form associated with the real orthogonal space $\mathbb{R}^{1,m}$ is given by :

$$Q(X) = T^2 - |\vec{X}|^2 \ , \quad \text{for all } X \in \mathbb{R}^{1,m} \ .$$

The norm $|X|$ of a space-time vector X is defined as $Q(X)^{\frac{1}{2}} = (T^2 - |\vec{X}|^2)^{\frac{1}{2}}$. With each space-time vector $X \in \mathbb{R}^{1,m}$ we associate the unit space-time vector ξ, defined as

$$\xi = \frac{X}{|X|} = \frac{\epsilon T + \vec{X}}{(T^2 - |\vec{X}|^2)^{\frac{1}{2}}} \ .$$

The null cone NC is then defined as the set of all space-time vectors X satisfying $Q(X) = 0$, and this NC separates the *time-like* region TLR (space-time vectors X for which $Q(X) > 0$) from the *space-like* region SLR (space-time vectors X for which $Q(X) < 0$). The TLR is the union of the future cone $FC = \{X : Q(X) > 0, T > 0\}$ and the past cone $PC = \{X : Q(X) > 0, T < 0\}$. In what follows we will often encounter FC_T, defined as $FC_T = \{X = \epsilon T + \vec{X} \in FC : \vec{X} \neq \vec{0}\}$ or the future cone FC minus the time-axis. The hyperboloid H_+ is defined as $\{\xi \in FC : |\xi| = 1\}$.

A *projective* model for the m-dimensional hyperbolic unit ball is obtained by identifying the rays inside FC with points on the hyperbolic unit ball. Other models for the m-dimensional hyperbolic unit ball are then readily obtained by intersecting the manifold of rays

$$\text{ray}(FC) = \{\lambda X : X \in FC, \lambda \in \mathbb{R}^+\}$$

inside FC with an arbitrary surface Σ inside FC, such that each ray intersects Σ in a unique point.

6 Clifford Analysis on the Hyperbolic Unit Ball

Consider the real Clifford algebra $\mathbb{R}_{1,m}$ generated by $(\epsilon, e_1, \cdots, e_m)$, the orthonormal basis for the real orthogonal space $\mathbb{R}^{1,m}$. The Dirac operator on $\mathbb{R}^{1,m}$ is defined as $\partial_X = \epsilon\partial_T - \partial_{\vec{X}}$ and it can be decomposed as $\partial_X = \xi(\partial_{|X|} + \frac{1}{|X|}\Gamma)$ with $X = |X|\xi$, ξ belonging to H_+ and with $\Gamma = X \wedge \partial_X$ the hyperbolic angular operator tangent to H_+. This operator satisfies $\Gamma(\xi \cdot \eta) = \xi \wedge \eta$ and $\xi\Gamma\xi + \Gamma = m$ (see e.g. [8]). Introducing the Euler operator $\mathbb{E}_X$ on $\mathbb{R}_{1,m}$ as $\mathbb{E}_X = T\partial_T + \sum_{i=1}^m X_i\partial_{X_i}$ we also have that $X\partial_X = \mathbb{E}_X + \Gamma$.

Due to the projective nature of our model for the hyperbolic unit ball, nullsolutions for the Dirac operator ∂_X on the hyperbolic unit ball have to be defined in such a way that they correspond to an invariant object on ray(FC), which is the true hyperbolic space. This can be done by considering functions satisfying a fixed homogeneity condition of the form $f(\lambda X) = \lambda^\alpha f(X)$, $\alpha \in \mathbb{R}$. Such functions are sections of homogeneous bundles over the manifold of rays issuing from the origin, defined as the equivalence classes of the equivalence relation $(X, c) \sim (\lambda X, \lambda^\alpha c)$ on $(\mathbb{R}^{1,m} \setminus \{0\}) \times \mathbb{R}_{1,m}$, with $\lambda > 0$. Hence we define *hyperbolic monogenic functions* as nullsolutions for ∂_X which are α-homogeneous.

Each hyperbolic monogenic function $F(X)$ after restriction to H_+ gives rise to an eigenfunction $F(\xi)$ of Γ and vice versa : each eigenfunction $F(\xi)$ of the hyperbolic angular operator can be extended to a hyperbolic monogenic function $F(X)$. Indeed, if $F(X)$ is an α-homogeneous solution of the Dirac operator ∂_X for all $X \in FC$

$$\begin{cases} \partial_X F(X) &= 0 \\ \mathbb{E}_X F(X) &= \alpha F(X) \end{cases}$$

the restriction of $F(X)$ to H_+ yields an eigenfunction of the angular operator Γ :

$$\xi(\Gamma + \alpha)F(\xi) = 0$$

Conversely, an eigenfunction $F(\xi)$ for Γ with eigenvalue α gives a hyperbolic monogenic function $F(X) = |X|^\alpha F(\xi)$ which is homogeneous of degree α.

Let us therefore introduce the following definition :

Definition 1 : *Let $\Omega \subset H_+$ be open, let*

$$\mathbb{R}_+\Omega = \{X \in FC : X = \lambda\xi, \lambda \in \mathbb{R} \textit{ and } \xi \in \Omega\}$$

be the open half cone over Ω and let $\alpha \in \mathbb{C}$. Then one puts :

$$\begin{aligned} \mathcal{H}^\alpha(\Omega) &= \{F \in C^1(\Omega) : \xi(\Gamma + \alpha)F = 0 \text{ in } \Omega\} \\ \mathcal{H}^\alpha(\mathbb{R}_+\Omega) &= \{F \in C^1(\mathbb{R}_+\Omega) : \mathbb{E}F = \alpha F \text{ and } \partial_X F = 0 \text{ in } \mathbb{R}_+\Omega\} \end{aligned}$$

Provided with the obvious laws for addition and multiplication with Clifford numbers both sets are right $\mathbb{R}_{1,m}$-modules. Elements of $\mathcal{H}^\alpha(\mathbb{R}_+\Omega)$ are $\mathbb{R}_{1,m}$-valued hyperbolic monogenic functions in $\mathbb{R}_+\Omega$ and elements of $\mathcal{H}^\alpha(\Omega)$ are the restrictions to $\Omega \subset H_+$ of hyperbolic monogenic functions in $\mathbb{R}_+\Omega$. From now on we will label these latter functions as *hyperbolic monogenics* :

Definition 2 : *Elements of $\mathcal{H}^\alpha(\Omega)$ are called hyperbolic monogenics in $\Omega \subset H_+$.*

In [7] it was proved that each inner spherical monogenic on $\mathbb{R}_{0,m}$ can be used to construct an element of $\mathcal{H}^\alpha(FC)$, whereas each outer spherical monogenic on $\mathbb{R}_{0,m}$ can be used to construct an element of $\mathcal{H}^\alpha(FC_T)$. Before rephrasing this theorem, let us introduce the following definition :

Definition 3 : *For all $X = \epsilon T + \vec{X} \in FC$ the function $Mod(\alpha, k, X)$ is defined as*

$$\mathrm{Mod}\,(\alpha, k, X) \;=\; F_1\left(\frac{|\vec{X}|^2}{T^2}\right) + \frac{k-\alpha}{2k+m}\,\frac{\vec{X}\epsilon}{T}\,F_2\left(\frac{|\vec{X}|^2}{T^2}\right)$$

where

$$F_1(t) \;=\; F\left(\frac{1+k-\alpha}{2}, \frac{k-\alpha}{2}, k+\frac{m}{2}; t\right)$$

and

$$F_2(t) \;=\; F\left(\frac{1+k-\alpha}{2}, \frac{2+k-\alpha}{2}, 1+k+\frac{m}{2}; t\right)$$

For the proof of the following theorem we refer to [7] :

Theorem 1 : *Let $P_k(\vec{\xi}) \in M_+(k)$ be an inner spherical monogenic on $\mathbb{R}_{0,m}$ and let $\alpha \in \mathbb{C}$. Then the function $P_{\alpha,k}(X)$ given for all $X = \epsilon T + \vec{X} \in FC$ by*

$$P_{\alpha,k}(X) \;=\; T^{\alpha}\mathrm{Mod}\,(\alpha, k, X)\,P_k\left(\frac{\vec{X}}{T}\right)$$

belongs to $\mathcal{H}^{\alpha}(FC)$.

Using the definition for the Gegenbauer function $C^{\mu}_{\nu}(z)$, and writing

$$X = |X|\xi = |X|\left(\tau\epsilon + (\tau^2-1)^{\frac{1}{2}}\vec{\xi}\right) \;\text{ with } \tau = \frac{T}{(T^2-|\vec{X}|^2)^{\frac{1}{2}}} \;\text{ and }\; \vec{\xi} \in S^{m-1}$$

Theorem 1 can be reformulated as follows :

Theorem 1(bis) : *Let $P_k(\vec{\xi}) \in M_+(k)$ be an inner spherical monogenic on $\mathbb{R}_{0,m}$ and let $\alpha \in \mathbb{C}$. Then the function $P_{\alpha,k}(\xi)$ for all $\xi \in H_+$ given by*

$$P_{\alpha,k}(\xi) \;=\; \frac{\Gamma(1+\alpha-k)\Gamma(2k+m)}{\Gamma(\alpha+k+m)}(\tau^2-1)^{\frac{k}{2}}\left\{C^{k+\frac{m+1}{2}}_{\alpha-k}(\tau) - C^{k+\frac{m+1}{2}}_{\alpha-k-1}(\tau)\xi\epsilon\right\}P_k\left(\vec{\xi}\right)$$

belongs to $\mathcal{H}^{\alpha}(H_+)$

Note that in the above expression the poles of $\Gamma(\alpha+k+m)$ are cancelled by the poles of the Gegenbauer functions and that the poles of $\Gamma(1+\alpha-k)$ are cancelled by the zeroes of the Gegenbauer functions, whence no restrictions on the complex parameter α are to be made.

Recalling the definition of FC_T as the future cone minus the time-axis, we have a similar result for the outer spherical monogenics on $\mathbb{R}_{0,m}$:

Theorem 2 : *Let $Q_k(\vec{\xi}) \in M_-(k)$ be an outer spherical monogenic on $\mathbb{R}_{0,m}$ and let $\alpha \in \mathbb{C}$. Then the function $Q'_{\alpha,k}(X)$ for all $X = \epsilon T + \vec{X} \in FC$ given by*

$$Q'_{\alpha,k}(X) \;=\; T^{\alpha}\mathrm{Mod}\,(\alpha, 1-k-m, X)\, Q_k\left(\frac{\vec{X}}{T}\right)$$

belongs to $\mathcal{H}^{\alpha}(FC_T)$.

Note that $Q'_{\alpha,k}(X) \in \mathcal{H}^{\alpha}(FC_T)$, constructed by means of the outer spherical monogenic $Q_k(\vec{\xi}) \in M_-(k)$ on $\mathbb{R}_{0,m}$, is not unique in the sense that one can always add an arbitrary element of $\mathcal{H}^{\alpha}(FC)$ without changing the singular behaviour of $Q'_{\alpha,k}(X)$ for $\vec{X} = \vec{0}$.

This enables us to add a particular null-solution $P'_{\alpha,k}(X) \in \mathcal{H}^{\alpha}(FC)$ to $Q'_{\alpha,k}(X)$ in order to obtain a function $Q_{\alpha,k}(X) \in \mathcal{H}^{\alpha}(FC_T)$ satisfying a boundary condition "at infinity". We will explain what is meant by that in what follows.

This particular null-solution $P'_{\alpha,k}(X)$ is constructed by means of the inner spherical monogenic[1] $P_k(\vec{\xi}) = \epsilon\vec{\xi}Q_k(\vec{\xi})$ associated with $Q_k(\vec{\xi}) \in M_-(k)$, hereby using Theorem 1 :

$$P'_{\alpha,k}(X) \;=\; T^{\alpha}\mathrm{Mod}\,(\alpha, k, X)\left[\left(\frac{|X|}{T}\right)^{k} \epsilon\vec{\xi}Q_k(\vec{\xi})\right]$$

We then introduce $Q_{\alpha,k}(X) \in \mathcal{H}^{\alpha}(FC_T)$ as follows :

$$Q_{\alpha,k}(X) \;=\; Q'_{\alpha,k}(X) - 2^{1-2k-m}\frac{\Gamma\left(1-k-\frac{m}{2}\right)}{\Gamma\left(k+\frac{m}{2}\right)}\frac{\Gamma(\alpha+k+m)}{\Gamma(\alpha-k+1)}P'_{\alpha,k}(X)$$

Theorem 2 can now also be reformulated :

Theorem 2(bis) : *Let $Q_k(\vec{\xi}) \in M_-(k)$ be an outer spherical monogenic on $\mathbb{R}_{0,m}$ and let $\alpha \in \mathbb{C}$. Then the function $Q_{\alpha,k}(X)$ given for all $X = \epsilon T + \vec{X} \in FC$ and $\alpha \notin -\mathbb{N} - k - m$ by*

$$Q_{\alpha,k}(\xi) \;=\; \frac{2\pi^{\frac{1}{2}}}{e^{i\pi\left(k+\frac{m+1}{2}\right)}}\frac{\Gamma\left(k+\frac{m+1}{2}\right)}{\Gamma\left(k+\frac{m}{2}\right)}(\tau^2-1)^{\frac{k}{2}}\left\{D^{k+\frac{m+1}{2}}_{\alpha-k-1}(\tau)\xi\epsilon - D^{k+\frac{m+1}{2}}_{\alpha-k}(\tau)\right\}\vec{\xi}\epsilon Q_k\left(\vec{\xi}\right)$$

belongs to $\mathcal{H}^{\alpha}(FC_T)$.

Note that the above expression has simple poles at $\alpha = -m-k-n$, $n \in \mathbb{N}$, whence these values for the parameter α are excluded.

[1] Notice the fact that we have multiplied the inner spherical monogenic $P_k(\vec{\xi})$ on $\mathbb{R}_{0,m}$ with ϵ, but this does not change monogeneity w.r.t. the operator $\partial_{\vec{x}}$

From Theorem 2(bis) it is clear that for $\tau \to \infty$ the function $Q_{\alpha,k}(\xi)$ behaves like

$$Q_{\alpha,k}(\xi) \sim \left\{ \frac{\xi\epsilon}{\tau^{\alpha+m}} - \frac{1}{\tau^{\alpha+m+1}} \right\} \vec{\xi}\epsilon Q_k\big(\vec{\xi}\big)$$

Because $\tau \to \infty$ is equivalent to $|\vec{X}| \to T$, it becomes clear what is meant by the boundary condition at infinity : for $\alpha + m > 0$, the function $Q_{\alpha,k}(\xi)$ disappears on the null-cone NC. This condition must be interpreted as the hyperbolic counterpart of the demand that the Cauchy kernel on $\mathbb{R}_{0,m}$ tends to zero at infinity in order to be uniquely determined.

Choosing $k = 0$, which is equivalent with saying that we choose $Q_k(\vec{x}) = E(\vec{x})$ to be the Cauchy kernel on $\mathbb{R}_{0,m}$, we find the fundamental solution for the Dirac equation on the hyperbolic unit ball (see references [8] and [9]), defined for all $\alpha + m \notin -\mathbb{N}$:

$$E_\alpha(X) \;=\; |X|^\alpha \frac{e^{-i\pi\frac{m+1}{2}}}{\pi^{\frac{m-1}{2}}} \Gamma\left(\frac{m+1}{2}\right) \left[D_{\alpha-1}^{\frac{m+1}{2}}(\tau)\xi - D_{\alpha}^{\frac{m+1}{2}}(\tau)\epsilon \right] .$$

Hence, we have for all $X = |X|\xi \in FC$:

$$E_\alpha(X) \;=\; |X|^\alpha E_\alpha(\xi, \epsilon)$$

where for arbitrary ξ and $\eta \in H_+$, we define the function $E_\alpha(\xi, \eta)$, $\alpha + m \notin -\mathbb{N}$, by

$$E_\alpha(\xi, \eta) \;=\; \frac{e^{-i\pi\frac{m+1}{2}}}{\pi^{\frac{m-1}{2}}} \Gamma\left(\frac{m+1}{2}\right) \left[D_{\alpha-1}^{\frac{m+1}{2}}(\xi\cdot\eta)\xi - D_{\alpha}^{\frac{m+1}{2}}(\xi\cdot\eta)\eta \right]$$

Note that $E_\alpha(\xi, \epsilon)$ can also be written as

$$E_\alpha(\xi, \epsilon) \;=\; \frac{e^{-i\pi\frac{m-1}{2}}}{2\pi^{\frac{m-1}{2}}} \Gamma\left(\frac{m-1}{2}\right) \xi \left(\Gamma + 1 + \alpha\right) D_{\alpha+1}^{\frac{m-1}{2}}(\tau) \ , \tag{9}$$

a relation that will be used in the sequel.

The function $E_\alpha(\xi, \epsilon)$ is an element of $\mathcal{H}^\alpha(H_+ \setminus \{\epsilon\})$ and as a fundamental solution of the operator $\xi(\Gamma + \alpha)$ it satisfies

$$\xi(\Gamma_\xi + \alpha)E_\alpha(\xi, \epsilon) \;=\; \delta(\xi - \epsilon) \tag{10}$$

where the delta-function $\delta(\xi - \epsilon)$ on the hyperboloid H_+ in ϵ has to be interpreted as the delta-function on the plane tangent to H_+ in ϵ, when considering the *radial* projection on this tangent plane as a local co-ordinate system for the hyperbolic manifold in the neighbourhood of $\epsilon \in H_+$. This radial projection is defined as the map sending an arbitrary vector $X = \epsilon T + \vec{X} \in FC$ to the intersection $(1, \frac{\vec{X}}{T})$ of the tangent plane to H_+ in ϵ and the ray through X.

7 Integral Formulae for the Operator $\xi(\Gamma + \alpha)$

Consider two arbitrary vectors ξ and $\eta \in H_+$. In what follows we will use, depending on the problem, two different expansions for these hyperbolic unit vectors :

$$\begin{array}{rcccl} \xi & = & \tau\epsilon + (\tau^2-1)^{1/2}\vec{\xi} & = & \epsilon\cosh\theta + \vec{\xi}\sinh\theta \\ \eta & = & \sigma\epsilon + (\sigma^2-1)^{1/2}\vec{\eta} & = & \epsilon\cosh\varphi + \vec{\eta}\sinh\varphi \end{array}$$

with $\vec{\xi}, \vec{\eta} \in S^{m-1}$ and $\theta, \varphi \in \mathbb{R}$.

According to (10) we have for arbitrary ξ and $\eta \in H_+$ and $\alpha \notin -m - \mathbb{N}$:

$$\xi(\Gamma_\xi + \alpha)E_\alpha(\xi, \eta) = \delta(\xi - \eta) . \tag{11}$$

Let us now put $\beta = -\alpha - m$. Using the following property of the Gegenbauer function $D^\mu_\nu(\tau)$ (see e.g. [6]),

$$D^\mu_\nu(\tau) = D^\mu_{-\nu-2\mu}(\tau) + \frac{\sin(\nu+\mu)\pi}{\sin(\nu\pi)} e^{i\mu\pi} C^\mu_\nu(\tau) ,$$

together with the fact that

$$\xi(\Gamma_\xi + \beta)\left[C^{\frac{m+1}{2}}_{\beta-1}(\xi\cdot\eta)\xi - C^{\frac{m+1}{2}}_{\beta}(\xi\cdot\eta)\eta\right] = 0$$

one can use the conjugation on $\mathbb{R}_{1,m}$ to deduce from (11) that

$$E_\alpha(\xi, \eta)(\Gamma_\eta - \beta)\eta = \delta(\xi - \eta) . \tag{12}$$

Note that the foregoing relation implies that $E_\alpha(X, Y)$ is monogenic with respect to the Dirac operator ∂_Y acting from the right, and homogeneous of degree β : $E_\alpha(\xi, Y) = |Y|^\beta E_\alpha(\xi, \eta)$.

In order to prove integral formulae for the Dirac operator, Stokes' theorem is fundamental. To prove Stokes' theorem for the Dirac operator $\partial_{\vec{x}}$ on the Clifford algebra $\mathbb{R}_{0,m}$, one starts from the following identity on $\mathbb{R}^m$ (see [5]) :

$$d(f\ d\sigma_{\vec{x}}\ g) = ((f\partial_{\vec{x}})g + f(\partial_{\vec{x}}g))dV(\vec{x}) , \tag{13}$$

with $dV(\vec{x})$ the volume element on $\mathbb{R}^m$, $\partial_{\vec{x}}$ the Dirac operator on the orthogonal space $\mathbb{R}^{0,m}$ and $d\sigma = \partial_{\vec{x}} \rfloor dV(\vec{x})$ the oriented surface element (the symbol $\rfloor$ denoting the contraction). What we need here is a *homogeneous* version of this identity, valid on $\mathbb{R}^{1,m}$. Therefore, we first define the Leray-form $L(X, dX)$ and a homogeneous version of the $d\sigma$-form as contractions of respectively the volume-form $dX_0 \cdots dX_m$ and the $d\sigma_X$-form on $\mathbb{R}^{1,m}$ with the Euler-operator on $\mathbb{R}^{1,m}$ (in the following formulae, X_0 is to be replaced by the time-variable T) :

Definition 4 : *The Leray-form $L(X, dX)$ is defined as*

$$\begin{aligned} L(X, dX) &= \mathbb{E}\rfloor dX_0 dX_1 \cdots dX_m \\ &= \sum_{j=0}^{m} (-1)^j X_j dX_{\hat{j}} \end{aligned}$$

Definition 5 : *The homogeneous version of the $d\sigma$-form is given by*

$$\begin{aligned} d\Sigma_X &= \mathbb{E}\rfloor d\sigma_X \\ &= \sum_{j<k}(-1)^{1+j+k}(e_jX_k - e_kX_j)dX_{\hat{j},\hat{k}} \end{aligned}$$

where the notation $\hat{j}$ indicates that this index is omitted in the summation.

Under the transformation $X \to \lambda X$, $dX \to \lambda dX + Xd\lambda$ both objects transform in a homogeneous manner, which means they are well-defined on the hyperboloid H_+. The homogeneous version of identity (13) on $\mathbb{R}^m$ is then given by Cauchy-Pompeju's Theorem (see e.g. [3]) :

Theorem 3 (*Cauchy-Pompeju*) **:** *For two C^1-functions $F(X)$ and $G(X)$ on FC, with $F(X) = |X|^\beta F(\xi)$, $G(X) = |X|^\alpha G(\xi)$ and $\alpha + \beta + m = 0$, one has :*

$$d(Fd\Sigma_X G) = -\Big[(F\partial_X)G + F(\partial_X G)\Big]L(X, dX)$$

As both sides of this equation are homogeneous of degree zero if $\alpha + \beta + m = 0$, this result is essentially valid on $\mathrm{Ray}(FC)$ and thus can be realized on an arbitrary surface inside FC, in particular on H_+. Let us therefore consider an open subset Ω of H_+ and let $C \subset \Omega$ be compact with smooth boundary ∂C. We then have the following theorems :

Theorem 4 (*Stokes*) **:** *Consider two homogeneous functions $F, G \in C^1(\Omega)$, with $F(X) = |X|^\beta F(\xi)$, $G(X) = |X|^\alpha G(\xi)$ and let $\alpha + \beta + m = 0$. Then :*

$$\begin{aligned} \int_{\partial C} F d\Sigma_\xi G &= \int_C \Big[(F\Gamma_\xi)\xi G + F\Gamma_\xi(\xi G)\Big]L(\xi, d\xi) \\ &= \int_C \Big[(F(\Gamma_\xi - \beta))\xi G - F\xi(\Gamma_\xi + \alpha)G\Big]L(\xi, d\xi) \end{aligned}$$

Theorem 5 (*Cauchy*) **:** *Let $F \in \mathcal{H}^\alpha(\Omega)$ and let $\alpha + \beta + m = 0$. Then :*

$$\begin{aligned} \int_{\partial C} E_\alpha(\eta, \xi) d\Sigma_\xi F(\xi) &= \int_C (E_\alpha(\eta, \xi)(\Gamma_\xi - \beta))\xi F(\xi) L(\xi, d\xi) \\ &= \begin{cases} F(\eta) & \text{if} \quad \eta \in \overset{\circ}{C} \\ 0 & \text{if} \quad \eta \in \Omega \setminus C \end{cases} \end{aligned}$$

Note that we have swapped the roles of ξ and η in Cauchy's theorem.

8 The Taylor Series on the Hyperbolic Unit Ball

In this section the Taylor series for hyperbolic monogenics on $SO(m)$-invariant subdomains Ω_ϵ of H_+ is established. An $SO(m)$-invariant subdomain Ω_ϵ of H_+ is defined as an open subset $\Omega_\epsilon \subset H_+$ such that the subgroup $SO(m)_\epsilon$ of $SO(1, m)$ fixing $\epsilon \in H_+$, leaves the subset Ω_ϵ invariant. We will introduce a decomposition for the fundamental solution $E_\alpha(\xi, \eta)$ using the Cauchy kernel on $\mathbb{R}_{0,m}$ and Theorems

1(bis) and 2(bis) and we will then prove this formula using an addition formula for the Gegenbauer functions. We will eventually use Cauchy's theorem to find the Taylor series for functions belonging to $\mathcal{H}^\alpha(\Omega_\epsilon)$.

From the previous section it is clear that the fundamental solution for the operator $\eta(\Gamma_\eta + \alpha)$ is the restriction to the hyperboloid H_+ of a function $E_\alpha(Y, X)$ which is α-homogeneous in Y and monogenic with respect to the operator ∂_Y acting from the left, and β-homogeneous in X and monogenic with respect to the operator ∂_X acting from the right (with $\alpha + \beta + m = 0$). It is therefore natural to consider the Cauchy kernel $E(\vec{y} - \vec{x})$ on $\mathbb{R}_{0,m}$ and to modulate this function, by means of Theorems 1(bis) and 2(bis), to a function $E_\alpha(\eta, \xi)$.

Consider the series expansion (7) for $E(\vec{y} - \vec{x})$, valid for all $|\vec{y}| > |\vec{x}|$. Since $C_k(\vec{\eta}, \vec{\xi})$ is an outer spherical monogenic with respect to the variable $\vec{\eta}$ for each $k \in \mathbb{N}$, we can use Theorem 2(bis) to obtain a hyperbolic monogenic on $H_+ \setminus \{\epsilon\}$. Denoting $\eta \in H_+$ as $\sigma\epsilon + (\sigma^2 - 1)^{\frac{1}{2}}\vec{\eta}$, this function is given, for all $\eta \neq \epsilon$, by

$$\frac{2\pi^{\frac{1}{2}}}{e^{i\pi\left(k+\frac{m+1}{2}\right)}} \frac{\Gamma\left(k + \frac{m+1}{2}\right)}{\Gamma\left(k + \frac{m}{2}\right)} (\sigma^2 - 1)^{\frac{k}{2}} \left\{ D_{\alpha-k-1}^{k+\frac{m+1}{2}}(\sigma)\eta\epsilon - D_{\alpha-k}^{k+\frac{m+1}{2}}(\sigma) \right\} \vec{\eta}\epsilon C(\vec{\eta}, \vec{\xi})$$

Using the recurrence relations (4) and (5) this can also be written as

$$\eta \frac{\pi^{\frac{1}{2}}}{e^{i\pi\left(k+\frac{m-1}{2}\right)}} \frac{\Gamma\left(k + \frac{m-1}{2}\right)}{\Gamma\left(k + \frac{m}{2}\right)} (\sigma^2 - 1)^{\frac{k}{2}} \times$$
$$\left\{ (1 + \alpha - k) D_{1+\alpha-k}^{k+\frac{m-1}{2}}(\sigma) + (2k + m - 1)(\sigma^2 - 1)^{1/2} D_{\alpha-k}^{k+\frac{m+1}{2}}(\sigma)\vec{\eta}\epsilon \right\} \vec{\eta} C(\vec{\eta}, \vec{\xi})$$

On the other hand we also know that the function $C(\vec{\eta}, \vec{\xi})$ is an inner spherical monogenic with respect to the variable $\vec{\xi}$. We may then use a slightly modified version of Theorem 1(bis) to find a hyperbolic monogenic with respect to the operator $(\Gamma - \beta)\xi$ *acting from the right* :

$$\frac{\Gamma(1 + \beta - k)\Gamma(2k + m)}{\Gamma(\beta + k + m)} (\tau^2 - 1)^{\frac{k}{2}} C(\vec{\eta}, \vec{\xi}) \left\{ C_{\beta-k}^{k+\frac{m+1}{2}}(\tau) - C_{\beta-k-1}^{k+\frac{m+1}{2}}(\tau)\overline{\xi\epsilon} \right\}$$

As the Gegenbauer function $C_\nu^\mu(z)$ satisfies

$$C_\nu^\mu(z) \quad = \quad -\frac{\sin(\nu\pi)}{\sin(\nu + 2\mu)\pi} C_{-\nu-2\mu}^\mu(z)$$

this can also be written as

$$-\frac{\Gamma(1 + \alpha - k)}{\Gamma(\alpha + k + m)} \Gamma(2k + m - 1)(\tau^2 - 1)^{\frac{k}{2}} \times$$
$$C(\vec{\eta}, \vec{\xi}) \left\{ (2k + m - 1)(\tau^2 - 1)^{1/2} \overline{\vec{\xi}\epsilon} C_{\alpha-k}^{k+\frac{m+1}{2}}(\tau) + (1 + \alpha - k) C_{1+\alpha-k}^{k+\frac{m-1}{2}}(\tau) \right\}$$

When modulating the Euclidean Cauchy kernel $E(\vec{y}-\vec{x})$ to the hyperbolic bi-monogenic function $E_\alpha(\eta,\xi)$, with $\alpha+m\notin-\mathbb{N}$ and

$$\begin{array}{rclcl} Y & = & |Y|(\sigma\epsilon+(\sigma^2-1)^{\frac{1}{2}}\vec{\eta}) & = & \epsilon S+\vec{Y} \\ X & = & |X|(\tau\epsilon+(\tau^2-1)^{\frac{1}{2}}\vec{\xi}) & = & \epsilon T+\vec{X} \end{array} ,$$

we have identified $\vec{y}$ with $\vec{Y}/S$ and $\vec{x}$ with $\vec{X}/T$. Since $|\vec{y}|>|\vec{x}|$ iff $\sigma>\tau$ we propose the following decomposition for the function $E_\alpha(\eta,\xi)$, valid for all $\alpha+m\notin-\mathbb{N}$ and $\sigma>\tau$:

$$\begin{aligned} E_\alpha(\eta,\xi) \;=\; & \frac{1}{A_m}\sum_{k=0}^{\infty}\eta(-1)^k 2^{2k+m-2}e^{-i\pi\frac{m-1}{2}}\Gamma\left(k+\frac{m-1}{2}\right)^2\frac{\Gamma(1+\alpha-k)}{\Gamma(\alpha+k+m)} \\ & (\sigma^2-1)^{\frac{k}{2}}\left\{(1+\alpha-k)D_{1+\alpha-k}^{k+\frac{m-1}{2}}(\sigma)+(2k+m-1)(\sigma^2-1)^{\frac{1}{2}}D_{\alpha-k}^{k+\frac{m+1}{2}}(\sigma)\vec{\eta}\epsilon\right\} \\ & \left\{C_k^{\frac{m}{2}}(<\vec{\xi},\vec{\eta}>)+C_{k-1}^{\frac{m}{2}}(<\vec{\xi},\vec{\eta}>)\vec{\eta}\vec{\xi}\right\} \\ & (\tau^2-1)^{\frac{k}{2}}\left\{(1+\alpha-k)C_{1+\alpha-k}^{k+\frac{m-1}{2}}(\tau)+(2k+m-1)(\tau^2-1)^{\frac{1}{2}}\overline{\vec{\xi}\epsilon}C_{\alpha-k}^{k+\frac{m+1}{2}}(\tau)\right\} \end{aligned}$$

with a region of convergence that will be determined later.

Let us now put $<\vec{\eta},\vec{\xi}>=\cos\psi\in[-1,1]$ such that

$$\eta\cdot\xi=\sigma\tau-\Big((\sigma^2-1)(\tau^2-1)\Big)^{1/2}<\vec{\eta},\vec{\xi}>=\cosh\varphi\cosh\theta-\sinh\varphi\sinh\theta\cos\psi\ .$$

In order to prove the proposed series expansion we will use formula (9) and the following addition formula for the Gegenbauer function $D_{\alpha+1}^{\frac{m-1}{2}}(\eta\cdot\xi)$ (see reference [6]), valid for $\sigma>\tau>1$:

$$D_{\alpha+1}^{\frac{m-1}{2}}(\eta\cdot\xi) \;=\; \frac{\Gamma(m-1)}{\Gamma\left(\frac{m-1}{2}\right)^2}\sum_{k=0}^{\infty}a_k(\alpha,m)c_k(\theta,\varphi,\psi)$$

with

$$\begin{aligned} a_k(\alpha,m) &= (-1)^k 4^k\Gamma\left(\frac{m-1}{2}+k\right)^2\frac{\Gamma(2+\alpha-k)}{\Gamma(\alpha+k+m)} \\ c_k(\theta,\varphi,\psi) &= (\sinh\varphi\sinh\theta)^k D_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh\varphi)\frac{m-2+2k}{m-2}C_k^{\frac{m}{2}-1}(\cos\psi)C_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh\theta) \end{aligned}$$

This series converges in the region where

$$|\cos\psi+(\cos\psi^2-1)^{\frac{1}{2}}|<\left|\frac{(\cosh\varphi\mp1)(\cosh\theta+1)}{(\cosh\varphi\pm1)(\cosh\theta-1)}\right|^{\frac{1}{2}}\ .$$

One can easily verify that the hyperbolic angular operator $\Gamma_\eta=Y\wedge\partial_Y$ on H_+ has the following representation in space-time co-ordinates $(S,\vec{Y})$:

$$\Gamma_\eta \;=\; \vec{Y}\epsilon\partial_S-S\epsilon\partial_{\vec{Y}}+\Gamma_{\vec{\eta}}\ ,$$

where $\Gamma_{\vec{\eta}}$ is the spherical Dirac operator on S^{m-1}. With respect to ϵ as a privileged direction, a co-ordinate system on the hyperbolic unit ball is obtained by choosing $\varphi \in \mathbb{R}$ and $\vec{\eta} \in S^{m-1}$ as the co-ordinates on H_+. With respect to this co-ordinate system, the angular hyperbolic Dirac operator Γ_η is given by

$$\Gamma_\eta = \left(1 + \vec{\eta}\epsilon\frac{\cosh\varphi}{\sinh\varphi}\right)\Gamma_{\vec{\eta}} + \vec{\eta}\epsilon\frac{\partial}{\partial\varphi} .$$

Hence we have by equation (9), for all $\alpha + m \notin -\mathbb{N}$:

$$E_\alpha(\eta,\xi) =$$
$$\frac{e^{-i\pi\frac{m-1}{2}}}{2\pi^{\frac{m-1}{2}}}\frac{\Gamma(m-1)}{\Gamma\left(\frac{m-1}{2}\right)}\eta\left[\left(1 + \vec{\eta}\epsilon\frac{\cosh\varphi}{\sinh\varphi}\right)\Gamma_{\vec{\eta}} + \vec{\eta}\epsilon\frac{\partial}{\partial\varphi} + 1 + \alpha\right]\sum_{k=0}^{\infty} a_k(\alpha,m)c_k(\theta,\varphi,\psi)$$

with $a_k(\alpha, m)$ and $c_k(\theta,\varphi,\psi)$ as above. Before further calculating this, let us introduce the following definition :

Definition 6 : *For two arbitrary vectors $\vec{\eta},\vec{\xi} \in S^{m-1}$ one puts, for all $k \in \mathbb{N}$*

$$\begin{aligned} Z_k(\vec{\eta},\vec{\xi}) &= C_k^{\frac{m}{2}}(<\vec{\eta},\vec{\xi}>) + \vec{\eta}\vec{\xi}C_{k-1}^{\frac{m}{2}}(<\vec{\eta},\vec{\xi}>) \\ B_k(\vec{\eta},\vec{\xi}) &= \begin{cases} -\vec{\eta}Z_{k-1}(\vec{\eta},\vec{\xi})\vec{\xi} & k \geq 1 \\ 0 & k = 0 \end{cases} \end{aligned}$$

The motivation for introducing the functions Z_k and B_k on $S^{m-1} \times S^{m-1}$ lies in the following :

$$\frac{m-2+2k}{m-2}C_k^{\frac{m}{2}-1}(<\vec{\eta},\vec{\xi}>) = Z_k(\vec{\eta},\vec{\xi}) - B_k(\vec{\eta},\vec{\xi}) .$$

Using the fact that $\Gamma_{\vec{\eta}}Z_k(\vec{\eta},\vec{\xi}) = -kZ_k(\vec{\eta},\vec{\xi})$ and $\Gamma_{\vec{\eta}}B_k(\vec{\eta},\vec{\xi}) = (m+k-2)B_k(\vec{\eta},\vec{\xi})$, one can now verify that, with the expression for Γ_η in terms of the co-ordinates $(\varphi,\vec{\eta})$, we have :

$$\begin{aligned} &\left(\Gamma_\eta + 1 + \alpha\right)\left[(\sinh\varphi)^k D_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh\varphi)\frac{m-2+2k}{m-2}C_k^{\frac{m}{2}-1}(<\vec{\eta},\vec{\xi}>)\right] \\ =\ & (\sinh\varphi)^{k-1}\left\{\begin{matrix} (1+\alpha-k)\sinh\varphi D_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh\varphi) \\ + \\ (2k+m-1)\sinh^2\varphi D_{\alpha-k}^{\frac{m+1}{2}+k}(\cosh\varphi)\vec{\eta}\epsilon \end{matrix}\right\} Z_k \\ +\ & (\sinh\varphi)^{k-1}\left\{\begin{matrix} (m+\alpha+k-1)\sinh\varphi D_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh\varphi) \\ + \\ \frac{(m+\alpha+k-1)(\alpha+2-k)}{(m+2k-3)}D_{2+\alpha-k}^{\frac{m-3}{2}+k}(\cosh\varphi)\vec{\eta}\epsilon \end{matrix}\right\} B_k \end{aligned}$$

Both the left-hand side and the right-hand side of the previous equation must then be multiplied with

$$a_k(\alpha,m)\frac{\Gamma(m-1)}{\Gamma\left(\frac{m-1}{2}\right)^2}(\sinh\theta)^k C_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh\theta)$$

and summed over the parameter k, in order to obtain an expression for the operator $(\Gamma_\eta + 1 + \alpha)$ acting on the Gegenbauer function $D_{1+\alpha}^{\frac{m-1}{2}}(\eta \cdot \xi)$. This expression can then be cast into the form

$$\sum_{k=0}^{\infty} c_k \left(S_1 + S_2 \vec{\eta}\epsilon \right) \left(S_3 + S_4 \vec{\eta}\vec{\xi} \right) \left(S_5 + S_6 \overrightarrow{\xi\epsilon} \right)$$

with c_k a constant, depending on k, and S_i a scalar function ($i = 1, 2, \cdots, 6$). This goes as follows : first of all we write B_k as $-\vec{\eta} Z_{k-1} \vec{\xi}$. Since $Z_{-1} \equiv 0$, the second series starts from $k = 1$. Rewriting this series, by putting $k' = k - 1$, one finds :

$$(\Gamma_\eta + 1 + \alpha) D_{1+\alpha}^{\frac{m-1}{2}}(\eta \cdot \xi) \;\; = \;\; \Sigma_1 + \Sigma_2 \; ,$$

where we have put

$$\begin{aligned} \Sigma_1 \;\; = \;\; & \frac{\Gamma(m-1)}{\Gamma\left(\frac{m-1}{2}\right)^2} \sum_{k=0}^{\infty} a_k(\alpha, m) (\sinh\theta)^k (\sinh\varphi)^k C_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh\theta) \times \\ & \left[(1+\alpha-k) D_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh\varphi) + (2k+m-1)\sinh\varphi D_{\alpha-k}^{\frac{m+1}{2}+k}(\cosh\varphi) \vec{\eta}\epsilon \right] Z_k \end{aligned}$$

and

$$\begin{aligned} \Sigma_2 \;\; = \;\; & \frac{\Gamma(m-1)}{\Gamma\left(\frac{m-1}{2}\right)^2} \sum_{k=0}^{\infty} \frac{(2k+m-1)^2}{(1+\alpha-k)} a_k(\alpha, m) (\sinh\theta)^{k+1} C_{\alpha-k}^{\frac{m+1}{2}+k}(\cosh\theta) \times \\ & (\sinh\varphi)^k \left[\sinh\varphi D_{\alpha-k}^{\frac{m+1}{2}+k}(\cosh\varphi) \vec{\eta} + \frac{1+\alpha-k}{2k+m-1} D_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh\varphi) \epsilon \right] Z_k \vec{\xi} \; . \end{aligned}$$

Eventually, gathering the terms in $D_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh\varphi)$ (resp. $D_{\alpha-k}^{\frac{m+1}{2}+k}(\cosh\varphi)$) and making use of definition 6 to rewrite Z_k, we get :

$$\begin{aligned} \left(\Gamma_\eta + 1 + \alpha \right) D_{1+\alpha}^{\frac{m-1}{2}}(\eta \cdot \xi) \;\; = \;\; & \frac{\Gamma(m-1)}{\Gamma\left(\frac{m-1}{2}\right)^2} \sum_{k=0}^{\infty} (-1)^k 4^k \frac{\Gamma(\alpha+1-k)}{\Gamma(\alpha+m+k)} \Gamma\left(\frac{m-1}{2} + k \right)^2 \\ & \left[(1+\alpha-k) D_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh\varphi) + (2k+m-1)\sinh\varphi D_{\alpha-k}^{\frac{m+1}{2}+k}(\cosh\varphi) \vec{\eta}\epsilon \right] \\ & (\sinh\varphi \sinh\theta)^k \left[C_k^{\frac{m}{2}}(<\vec{\eta}, \vec{\xi}>) + \vec{\eta}\vec{\xi} C_{k-1}^{\frac{m}{2}}(<\vec{\eta}, \vec{\xi}>) \right] \\ & \left[(1+\alpha-k) C_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh\theta) + (2k+m-1)\sinh\theta C_{\alpha-k}^{\frac{m+1}{2}+k}(\cosh\theta) \overrightarrow{\xi\epsilon} \right] \end{aligned}$$

In view of (9) this yields the following expression for the fundamental solution $E_\alpha(\eta, \xi)$, for all $\alpha + m \notin -\mathbb{N}$:

$$\begin{aligned} E_\alpha(\eta, \xi) \;\; = \;\; & \frac{e^{-i\pi\frac{m-1}{2}}}{2\pi^{\frac{m-1}{2}}} \frac{\Gamma(m-1)}{\Gamma\left(\frac{m-1}{2}\right)} \eta \sum_{k=0}^{\infty} (-1)^k 4^k \frac{\Gamma(\alpha+1-k)}{\Gamma(\alpha+m+k)} \Gamma\left(\frac{m-1}{2} + k \right)^2 \\ & \left[(1+\alpha-k) D_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh\varphi) + (2k+m-1)\sinh\varphi D_{\alpha-k}^{\frac{m+1}{2}+k}(\cosh\varphi) \vec{\eta}\epsilon \right] \\ & (\sinh\varphi \sinh\theta)^k \left[C_k^{\frac{m}{2}}(<\vec{\eta}, \vec{\xi}>) + \vec{\eta}\vec{\xi} C_{k-1}^{\frac{m}{2}}(<\vec{\eta}, \vec{\xi}>) \right] \qquad (14) \\ & \left[(1+\alpha-k) C_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh\theta) + (2k+m-1)\sinh\theta C_{\alpha-k}^{\frac{m+1}{2}+k}(\cosh\theta) \overrightarrow{\xi\epsilon} \right] \end{aligned}$$

Invoking the relation $\pi^{1/2}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2)$ and using the explicit form of A_m, this yields precisely the series proposed earlier, at the beginning of this section, using the Cauchy kernel on $\mathbb{R}_{0,m}$. Before we determine the region of convergence of the series expansion for the fundamental solution $E_\alpha(\eta,\xi)$, we introduce :

Definition 7 : *For an arbitrary $R > 1$ and for each $\zeta \in H_+$ one defines $HC(R,\zeta)$ as*

$$HC(R,\zeta) \;=\; \{\xi \in H_+ : R \;>\; \xi\cdot\zeta \;\geq\; 1\}$$

The notation HC is inspired by the fact that this subset of H_+ is a *hyperbolic cap*, by analogy with the term spherical cap. Notice that $HC(R,\zeta)$ is invariant under the subgroup $SO(m)_\zeta$ of $SO(1,m)$ fixing ζ.

Recalling our notations

$$\begin{array}{rclcl} \eta &=& \sigma\epsilon + (\sigma^2-1)^{\frac{1}{2}}\vec{\eta} &=& \epsilon\cosh\varphi + \vec{\eta}\sinh\varphi \\ \xi &=& \tau\epsilon + (\tau^2-1)^{\frac{1}{2}}\vec{\xi} &=& \epsilon\cosh\theta + \vec{\xi}\sinh\theta \end{array}$$

the series expansion for $E_\alpha(\eta,\xi)$ converges normally on each hyperbolic cap $HC(R,\epsilon)$, for all $\sigma > R \geq \tau$ with σ kept fixed.

We then have the following :
Definition 8 : *For two arbitrary elements η and ξ in H_+ one defines, for all $k \in \mathbb{N}$ and $\alpha + m \notin -\mathbb{N}$:*

$$\begin{aligned} E^k_\alpha(\eta,\xi) \;&=\; \eta(-1)^k 2^{2k+m-2} e^{-i\pi\frac{m-1}{2}}\Gamma\left(k+\frac{m-1}{2}\right)^2 \frac{\Gamma(1+\alpha-k)}{\Gamma(\alpha+k+m)} \\ &(\sigma^2-1)^{\frac{k}{2}}\left\{(1+\alpha-k)D^{k+\frac{m-1}{2}}_{1+\alpha-k}(\sigma) + (2k+m-1)(\sigma^2-1)^{\frac{1}{2}}D^{k+\frac{m+1}{2}}_{\alpha-k}(\sigma)\vec{\eta}\epsilon\right\} \\ &\left\{C^{\frac{m}{2}}_k(<\vec{\eta},\vec{\xi}>) + C^{\frac{m}{2}}_{k-1}(<\vec{\eta},\vec{\xi}>)\vec{\eta}\,\vec{\xi}\right\} \\ &(\tau^2-1)^{\frac{k}{2}}\left\{(1+\alpha-k)C^{k+\frac{m-1}{2}}_{1+\alpha-k}(\tau) + (2k+m-1)(\tau^2-1)^{\frac{1}{2}}C^{k+\frac{m+1}{2}}_{\alpha-k}(\tau)\overline{\vec{\xi}\epsilon}\right\} \end{aligned}$$

The axial decomposition of the hyperbolic fundamental solution is expressed in the following Theorem :

Theorem 6 : *For all $\alpha + m \notin -\mathbb{N}$, the fundamental solution $E_\alpha(\eta,\xi)$ for the operator $\eta(\Gamma+\alpha)$ on H_+ can be decomposed as*

$$E_\alpha(\eta,\xi) \;=\; \frac{1}{A_m}\sum_{k=0}^{\infty} E^k_\alpha(\eta,\xi)\ .$$

For σ fixed this series converges normally on each closed hyperbolic cap $\overline{HC}(R,\epsilon)$, $\sigma > R \geq \tau$. By construction, one also has for each $k \in \mathbb{N}$:

$$\begin{gathered} E^k_\alpha(\eta,\xi) \in \mathcal{H}^\beta(H_+ \setminus \{\epsilon\}) \\ \overline{E}^k_\alpha(\eta,\xi) \in \mathcal{H}^\alpha(H_+ \setminus \{\epsilon\}) \end{gathered}$$

respectively with respect to the operators ∂_Y and ∂_X.

To establish a Taylor expansion, we need to switch variables first. Indeed : the expansion for $E_\alpha(\eta,\xi)$ above is valid for $\sigma > \tau$, so we need an integration over η instead of ξ (cfr. Cauchy's Theorem) if we are to obtain a Taylor expansion around $\epsilon \in H_+$. Since $E_\alpha(\eta,\xi) = -E_\beta(\xi,\eta)$, up to a nullsolution for the hyperbolic angular operator, Stokes' Theorem for functions $F \in \mathcal{H}^\alpha$ yields immediately :

$$\begin{aligned} \eta \in HC(R,\epsilon) \quad \Longrightarrow \quad F(\eta) &= \int_{\partial HC(R,\epsilon)} E_\alpha(\eta,\xi) d\Sigma_\xi F(\xi) \\ &= -\int_{\partial HC(R,\epsilon)} E_\beta(\xi,\eta) d\Sigma_\xi F(\xi) \ . \end{aligned}$$

Equivalently, hereby making use of the fact that $\overline{E}_{\bar\beta}(\xi,\eta) = -E_\beta(\xi,\eta)$, we get :

$$\xi \in HC(R,\epsilon) \quad \Longrightarrow \quad F(\xi) = \int_{\partial HC(R,\epsilon)} \overline{E}_{\bar\beta}(\eta,\xi) d\Sigma_\eta F(\eta)$$

This gives rise to the following Taylor expansion :

Theorem 7 *(Taylor)* **:** *Consider an arbitrary $\alpha \in \mathbb{C} \setminus (-m - \mathbb{N})$ and let $R > 1$ be fixed. Let $F \in \mathcal{H}^\alpha(HC(R,\epsilon))$. There exists a sequence of functions $(F_\epsilon^{(k)}(\xi))_{k\in\mathbb{N}}$ such that the function $\xi \mapsto F_\epsilon^{(k)}(\xi)$ belongs to $\mathcal{H}^\alpha(H_+)$ for each $k \in \mathbb{N}$ and such that the following expansion holds in $HC(R,\epsilon)$:*

$$F(\xi) = \sum_{k=0}^{\infty} F_\epsilon^{(k)}(\xi) \ , \tag{15}$$

where for each $k \in \mathbb{N}$ the function $F_\epsilon^{(k)}(\xi)$ has the following integral representation :

$$\begin{aligned} F_\epsilon^{(k)}(\xi) &= \int_{\sigma=r} \overline{E}^k_{\bar\beta}(\eta,\xi) d\Sigma_\eta F(\eta) \\ &= \frac{(-1)^k 2^{2k+m-2}}{e^{i\pi\frac{m-1}{2}}} \frac{\Gamma(1+\beta-k)}{\Gamma(m+\beta+k)} \Gamma\left(\frac{m-1}{2}+k\right)^2 (\sigma^2-1)^k (r^2-1)^{\frac{k+m-1}{2}} \\ &\quad \left[(\beta+m+k-1) C_{\beta-k}^{k+\frac{m-1}{2}}(\tau) + (2k+m-1)(\tau^2-1)^{\frac{1}{2}} C_{\beta-k-1}^{k+\frac{m+1}{2}}(\tau) \vec{\xi}\epsilon\right] \\ &\quad P(k)\left(\left[\begin{array}{c} (1+\beta-k) D_{1+\beta-k}^{k+\frac{m-1}{2}}(r)\vec{\eta}\epsilon \\ + \\ (2k+m-1)(r^2-1)^{\frac{1}{2}} D_{\beta-k}^{k+\frac{m+1}{2}}(r) \end{array}\right] F\left(r\epsilon + (r^2-1)^{1/2}\vec{\eta}\right)\right) \end{aligned}$$

with $r \in]1,R[$ arbitrarily and $P(k)f$ the projection of $f(\vec{\eta})$ onto the space of inner spherical monogenics of order k on S^{m-1}. This series expansion converges normally on each hyperbolic cap $\overline{HC}(\rho,\epsilon)$, with $r > \rho \geq \tau$.

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