Integral formulas for hypermonogenic functions

Sirkka-Liisa Eriksson*

Abstract

Let $C\ell_n$ be the (universal) Clifford algebra generated by $e_1, ..., e_n$ satisfying $e_i e_j + e_j e_i = -2\delta_{ij}$, $i, j = 1, ..., n$. The Dirac operator in $C\ell_n$ is defined by $D = \sum_{i=0}^{n} e_i \frac{\partial}{\partial x_i}$, where $e_0 = 1$. The modified Dirac operator is introduced for $k \in \mathbb{R}$ by $M_k f = Df + k\frac{Q'f}{x_n}$, where $'$ is the main involution and Qf is given by the decomposition $f(x) = Pf(x) + Qf(x) e_n$ with $Pf(x), Qf(x) \in C\ell_{n-1}$. A continuously differentiable function $f : \Omega \to C\ell_n$ is called k-hypermonogenic in an open subset Ω of $\mathbb{R}^{n+1}$, if $M_k f(x) = 0$ outside the hyperplane $x_n = 0$. Note that 0-hypermonogenic functions are monogenic and $n-1$-hypermonogenic functions are hypermonogenic defined by the author and H. Leutwiler in [10]. The power function x^m is hypermonogenic. We prove integral formulas of hypermogenic functions.

1 Introduction

There are several approaches to generalize classical complex analysis to higher dimensions. One approach led to the theory of monogenic functions based on Euclidean space which became popular in the 1970s (see for example [1]). Another one led to the theory of hypermonogenic functions based on hyperbolic metric initiated by H. Leutwiler around 1990 ([18], [19]). The advantage of hypermonogenic functions is that positive and negative power functions are included to the theory which is not in the monogenic case The essential result of generalization of the Cauchy formula was provided very earlier for monogenic functions, but not for hypermonogenic functions. In this paper we prove a Cauchy formula for hypermogenic

*This research is supported by the Academy of Finland

Received by the editors September 2003; revised: March 29, 2004.

1991 *Mathematics Subject Classification :* Primary 30G35; Secondary 30A05, 30F45.

Key words and phrases : Monogenic, hypermonogenic, Dirac operator, hyperbolic metric .

functions. This formula gives an important tool for handling partial differential equations arising from hyperbolic Laplace-Beltrami type operators. A first step towards the Cauchy formula was obtained earlier in [11].

Hypermonogenic functions are also related to $\triangle^k$-monogenic functions, that is functions satisfying $D\triangle^k f = 0$, considered in complex Clifford algebras by J. Ryan in [22] and in real Clifford algebras under the name holomorphic Cliffordian functions by G. Laville and I. Ramadanoff in [17] . In the case n odd hypermonogenic functions are holomorphic Cliffordian and therefore they satisfy also a complicated integral formula given in [22] or in [17] (see [8]).

2 Preliminaries

Let $C\ell_n$ be the universal Clifford algebra generated by the elements $e_1, ..., e_n$ satisfying the relation $e_ie_j + e_je_i = -2\delta_{ij}$, where δ_{ij} is the usual Kronecker delta. We use the same notations as in [10] and [12]. The elements $x = x_0 + x_1e_1 + ... + x_ne_n$ for $x_0, ..., x_n \in \mathbb{R}$ are called *paravectors*. The set $\mathbb{R}^{n+1}$ is identified with the set of paravectors.

The main involution $' : C\ell_n \to C\ell_n$ is the algebra isomorphism defined by $e_0' = 1$ and $e_i' = -e_i$ for $i = 1, ..., n$. The involution $\hat{\ } : C\ell_n \to C\ell_n$ is defined by $\widehat{e_n} = -e_n$, $\widehat{e_i} = e_i$ for $i = 0, ..., n-1$ and $\widehat{ab} = \widehat{a}\widehat{b}$. It is easy to calculate that for arbitrary $a \in C\ell_n$

$$a'e_n = e_n\widehat{a} \qquad \text{and} \qquad e_na' = \widehat{a}e_n. \tag{1}$$

The antiautomorphism $* : C\ell_n \to C\ell_n$, called *reversion*, is defined by $e_i^* = e_i$ for $i = 0, ..., n$ and $(ab)^* = b^*a^*$. The conjugation $\overline{a}$ is given by $\overline{a} = (a')^*$.

Using the decomposition $a = b + ce_n$ of the element $a \in C\ell_n$ for $b, c \in C\ell_{n-1}$(the Clifford algebra generated by $e_1, ..., e_{n-1}$) we define the mappings $P : C\ell_n \to C\ell_{n-1}$ and $Q : C\ell_n \to C\ell_{n-1}$ by $Pa = b$ and $Qa = c$. Note that if $w \in Cl_n$, then

$$\begin{aligned} Qw &= \frac{e_nw' - we_n}{2} = \frac{\widehat{w} - w}{2}e_n, \\ Pw &= \frac{w - e_nw'e_n}{2} = \frac{w + \widehat{w}}{2}. \end{aligned} \tag{2}$$

The following calculation rules are proved in ([10, Lemma 2] and [11, Lemma 1])

$$P(ab) = (Pa)Pb + (Qa)Q(b'), \tag{3}$$
$$Q(ab) = (Pa)Qb + (Qa)P(b'), \tag{4}$$
$$Q(ab) = aQb + (Qa)b'. \tag{5}$$

3 Hypermonogenic functions

Let Ω be an open subset of $\mathbb{R}^{n+1}$. The left Dirac operator in $C\ell_n$ is defined by $D_lf = \sum_{i=0}^n e_i\frac{\partial f}{\partial x_i}$ and the right Dirac operator by $D_rf = \sum_{i=0}^n \frac{\partial f}{\partial x_i}e_i$ for a mapping $f : \Omega \to C\ell_n$, whose components are continuously differentiable. The operators

$\overline{D_l}$ and $\overline{D_r}$ are defined by $\overline{D_l}f = \sum_{i=0}^{n} \overline{e_i}\frac{\partial f}{\partial x_i}$ and $\overline{D_r}f = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}\overline{e_i}$. As usual we abbreviate $D_l f = Df$ if there is no confusion possible.

Let Ω be an open subset of $\mathbb{R}^{n+1} \backslash \{x_n = 0\}$ and k be a non-negative integer. The modified Dirac operators M_k^l , $\overline{M}_k^l$, M_k^r and $\overline{M}_k^r$ are introduced by

$$M_k^l f(x) = D_l f(x) + k\frac{Q'f}{x_n}$$
$$M_k^r f(x) = D_r f(x) + k\frac{Qf}{x_n}$$

and

$$\overline{M}_k^l f(x) = \overline{D}_l f(x) - k\frac{Q'f}{x_n},$$
$$\overline{M}_k^r f(x) = \overline{D}_r f(x) - k\frac{Qf}{x_n},$$

where $f \in \mathcal{C}^1(\Omega, C\ell_n)$. The operator M_k^l is also denoted by M_k and M_{n-1}^l by M.

Definition 1. *Let $\Omega \subset \mathbb{R}^{n+1} \backslash \{x_n = 0\}$ be an open set. A mapping $f : \Omega \to C\ell_n$ is called left k-hypermonogenic, if $f \in \mathcal{C}^1(\Omega)$ and $M_k^l f(x) = 0$ for any $x \in \Omega$. The $n-1$-hypermonogenic functions are called briefly hypermonogenic and the 0-hypermonogenic functions are called monogenic. The right hypermonogenic functions are defined similarly.*

Hypermonogenic functions were introduced in [10] and developed further in [11]. Paravector-valued hypermonogenic functions are H-solutions introduced by H. Leutwiler ([18], [19], [20], [21], [9]). The H-solutions are also studied by Cerejeiras [2], Cnops [3], Hempfling [13], [14], [15] and the author [4], [5],[6]. In the case $n = 2$ hypermonogenic functions called hyperholomorphic functions are investigated in [16]. For the general reference to the properties of monogenic functions we refer to [1].

The Dirac operator and the modified Dirac operator are related as follows.

Lemma 2. *Let Ω be an open subset of $\mathbb{R}^{n+1} \backslash \{x_n = 0\}$ and $f : \Omega \to C\ell_n$ be a $\mathcal{C}^1(\Omega, C\ell_n)$ function. Then*

$$D_l\left(\frac{f}{x_n^k}\right) = \frac{M_k^l f}{x_n^k} - k\frac{P'f}{x_n^{k+1}}e_n, \tag{6}$$

$$D_r\left(\frac{f}{x_n^k}\right) = \frac{M_k^r f}{x_n^k} - k\frac{Pf}{x_n^{k+1}}e_n, \tag{7}$$

$$P\left(M_k^l f\right) = x_n^k P\left(D_l\left(\frac{f}{x_n^k}\right)\right), \tag{8}$$

$$Q\left(M_k^l f\right) = Q\left(D_l f\right), \tag{9}$$

$$D_l\left(\frac{Qfe_n}{x_n^k}\right) = \frac{M_k^l\left(Qfe_n\right)}{x_n^k}. \tag{10}$$

Proof. We just compute

$$D_l\left(\frac{f}{x_n^k}\right)=\frac{D_lf}{x_n^k}-k\frac{e_nf}{x_n^{k+1}}=\frac{D_lf}{x_n^k}+k\frac{Q^{'}f}{x_n^{k+1}}-k\frac{P'fe_n}{x_n^{k+1}}$$
$$=\frac{M_k^lf}{x_n^k}-k\frac{P'fe_n}{x_n^{k+1}},$$

which implies the first equality. If we take P from the both sides of this equality we obtain the third equality. The fourth equality follows directly from the definition of M_k. The last equality follows from the first one if we replace f by $(Qf)\,e_n$. ■

The modified generalization of the Cauchy-Riemann equations is the following system of equations.

Theorem 3 ([10, Proposition 3]). *Let Ω be an open subset of $\mathbb{R}^{n+1}$ and $f:\Omega\to C\ell_n$ be a mapping with continuous partial derivatives. The equation $x_nD_lf+kQ'f=0$ is equivalent with the following system of equations*

$$\begin{aligned}&x_n\left(D_{n-1}^l\left(Pf\right)-\tfrac{\partial(Q'f)}{\partial x_n}\right)+kQ'f=0,\\&D_{n-1}^l\left(Qf\right)+\tfrac{\partial P'(f)}{\partial x_n}=0.\end{aligned}\tag{11}$$

where $D_{n-1}^l=\sum_{i=0}^{n-1}e_i\frac{\partial}{\partial x_i}$.

Hypermonogenic functions multiplied from the left by e_n are not any more hypermonogenic functions, but they satisfy a similar equation.

Proposition 4. *Let Ω be an open subset of $\mathbb{R}^{n+1}\backslash\{x_n=0\}$. A function $f:\Omega\to C\ell_n$ is left k-hypermonogenic if and only if the function fe_n satisfies the equation*

$$D_lg-\frac{kP'ge_n}{x_n}=0.$$

Similarly, a function $f:\Omega\to C\ell_n$ is right k-hypermonogenic if and only if the function e_nf satisfies the equation

$$D_rg-\frac{kPge_n}{x_n}=0.$$

Proof. Since $P\left(fe_n\right)=-Qf$, we obtain

$$D_l\left(fe_n\right)+k\frac{Q'fe_n}{x_n}=D_l\left(fe_n\right)-\frac{kP'\left(fe_n\right)e_n}{x_n}.$$

Hence the first assertion holds. Using $P\left(e_nf\right)=-Q'f$ we infer

$$D_r\left(e_nf\right)+k\frac{e_nQf}{x_n}=D_r\left(e_nf\right)+k\frac{Q'fe_n}{x_n}=D_r\left(e_nf\right)-\frac{kP\left(e_nf\right)e_n}{x_n}$$

which implies the second assertion. ■

The key idea for proving the Cauchy formula is the relation between the operators M_{-k}^l and M_k^l stated next.

Proposition 5. *Let Ω be an open subset of $\mathbb{R}^{n+1}\backslash\{x_n = 0\}$ and $f : \Omega \to C\ell_n$ be a $\mathcal{C}^1(\Omega, C\ell_n)$ function. If $k \in \mathbb{R}$, then*

$$M_{-k}^l\left(\frac{fe_n}{x_n^k}\right) = \frac{\left(M_k^l f\right)e_n}{x_n^k},$$
$$M_{-k}^r\left(\frac{e_n f}{x_n^k}\right) = \frac{e_n\left(M_k^r f\right)}{x_n^k}.$$

Moreover a function $f : \Omega \to C\ell_n$ is k-hypermonogenic if and only if the function $\frac{fe_n}{x_n^k}$ is $-k$-hypermonogenic.

Proof. We just compute

$$D_l\left(\frac{fe_n}{x_n^k}\right) = \frac{(D_l f)\,e_n}{x_n^k} - k\frac{e_n f e_n}{x_n^{k+1}}$$
$$= \frac{\left(M_k^l f\right)e_n}{x_n^k} + k\frac{P'f}{x_n^{k+1}}.$$

Since $Q\left(\frac{fe_n}{x_n^k}\right) = \frac{Pf}{x_n^k}$ we obtain the first equality. The other equality is proved similarly. ■

Let Ω be an open subset of $\mathbb{R}^{n+1}\backslash\{x_n = 0\}$ and K is an $n+1$-chain satisfying $\overline{K} \subset \Omega$. Define a real n-form by

$$d\check{x}_i = dx_0 \wedge ... \wedge dx_{i-1} \wedge dx_{i+1} \wedge ...dx_n$$

and a paravector valued n-form by

$$d\sigma_k = \frac{1}{x_n^k}\sum_{i=0}^{n}(-1)^i e_i d\check{x}_i$$

A real $n+1$-form is introduced by

$$dm_k = \frac{1}{x_n^k}dx_0 \wedge ... \wedge dx_n.$$

Recall that

$$\int_{\partial K} g d\sigma_0 f = \int_K \left(\left(D_r g\right) f + gD_l f\right) dm_0, \tag{12}$$

see [1, 9.2 Proposition, p.52]. We give a corresponding formula for M_k operators.

Theorem 6. *Let Ω be an open subset of $\mathbb{R}^{n+1}\backslash\{x_n = 0\}$ and K an $n+1$-chain satisfying $\overline{K} \subset \Omega$. If $f, g \in \mathcal{C}^1(\Omega, C\ell_n)$, then*

$$\int_{\partial K} g d\sigma_k f = \int_K \left(\left(M_k^r g\right) f + gM_k^l f - \frac{k}{x_n}P\left(gf'\right)e_n\right) dm_k.$$

Proof. Using (12) and Lemma 2 we obtain

$$\begin{aligned}
\int_{\partial K} g d\sigma_k f &= \int_{\partial K} \frac{g}{x_n^k} d\sigma_0 P f + \int_{\partial K} g d\sigma_0 \frac{Qfe_n}{x_n^k} \\
&= \int_K \left(D_r \left(\frac{g}{x_n^k} \right) Pf + \frac{g}{x_n^k} D_l Pf + g D_l \left(\frac{Qfe_n}{x_n^k} \right) + \frac{D_r g Q f e_n}{x_n^k} \right) dm_0 \\
&= \int_K \left(D_r \left(\frac{g}{x_n^k} \right) Pf + \frac{g}{x_n^k} D_l Pf + g \frac{M_k^l (Qfe_n)}{x_n^k} + \frac{D_r g Q f e_n}{x_n^k} \right) dm_0 \\
&= \int_K \left(\left(M_k^r g - \frac{kPge_n}{x_n} \right) Pf + g M_k^l f + D_r g Q f e_n \right) dm_k.
\end{aligned}$$

Since $D_r g = M_k^r g - k\frac{Qg}{x_n}$, we infer

$$\int_{\partial K} g d\sigma_k f = \int_K \left((M_k^r g) f + g M_k^l f - \frac{k}{x_n} (PgP'f + QgQf) e_n \right) dm_k.$$

Applying the product rule of P and the properties $P(f') = P'f$ and $Qf' = -Q'f$ we obtain

$$P(gf') = PgP'f - QgQ'f' = PgP'f + QgQf,$$

completing the proof. ∎

By taking P-part from the both sides of the equation of the previous theorem we obtain.

Theorem 7. *Let Ω be an open subset of $\mathbb{R}^{n+1} \setminus \{x_n = 0\}$ and K an $n+1$-chain satisfying $\overline{K} \subset \Omega$. If $f, g \in \mathcal{C}^1(\Omega, C\ell_n)$, then*

$$\int_{\partial K} P(g d\sigma_k f) = \int_K P\left((M_k^r g) f + g M_k^l f \right) dm_k.$$

Corollary 8. *Let Ω be an open subset of $\mathbb{R}^{n+1} \setminus \{x_n = 0\}$ and K an $n+1$-chain satisfying $\overline{K} \subset \Omega$. If f is left k-hypermonogenic and g is right k-hypermonogenic in Ω , then*

$$\int_{\partial K} P(g d\sigma_k f) = 0.$$

In order to prove Theorem 7 for the Q-part we first verify a formula for the measure σ_0 corresponding to Theorem 6.

Theorem 9. *Let Ω be an open subset of $\mathbb{R}^{n+1} \setminus \{x_n = 0\}$ and K an $n+1$-chain satisfying $\overline{K} \subset \Omega$. If $f, g \in \mathcal{C}^1(\Omega, C\ell_n)$, then*

$$\int_{\partial K} g d\sigma_0 f = \int_K \left(\left(M_{-k}^r g \right) f + g M_k^l f + \frac{k}{x_n} Q(gf') \right) dm_0.$$

Proof. Using (12) we calculate

$$\int_{\partial K} g d\sigma_0 f = \int_K (D_r g f + g D_l f)\, dm_0.$$

Since $D_r g = M_{-k}^r g + k \frac{Qg}{x_n}$ and $D_l f = M_k^l f - k \frac{Q' f}{x_n}$ we deduce further

$$\int_{\partial K} (g d\sigma_0 f) = \int_K \left(\left(M_{-k}^r g \right) f + g M_k^l f + \frac{k}{x_n} \left((Qg) f - g Q' f \right) \right) dm_0$$

The product rule of Q implies that

$$Q\left(g f'\right) = (Qg) f + g Q\left(f'\right) = (Qg) f - g Q' f.$$

Hence the equality holds. ■

Applying the operator Q to the previous result, we directly conclude the following result:

Theorem 10. *Let Ω be an open subset of $\mathbb{R}^{n+1} \backslash \{x_n = 0\}$ and K an $n+1$-chain satisfying $\overline{K} \subset \Omega$. If $f, g \in \mathcal{C}^1(\Omega, C\ell_n)$, then*

$$\int_{\partial K} Q\left(g d\sigma_0 f\right) = \int_K Q\left(\left(M_{-k}^r g \right) f + g M_k^l f \right) dm_0.$$

Corollary 11. *Let Ω be an open subset of $\mathbb{R}^{n+1} \backslash \{x_n = 0\}$ and K an $n+1$-chain satisfying $\overline{K} \subset \Omega$. If f is left k-hypermonogenic and g is right $-k$-hypermonogenic, then*

$$\int_{\partial K} Q\left(g d\sigma_0 f\right) = 0.$$

We have proved in [12] that the kernel for the P-part of a hypermonogenic function is the following.

Lemma 12. *The function*

$$\begin{aligned} p(x,y) &= \frac{x_n^{n-1}}{2y_n} \left(\frac{(x-y)^{-1} - (x-\widehat{y})^{-1}}{|x-y|^{n-1} |x-\widehat{y}|^{n-1}} \right) \\ &= x_n^{n-1} \frac{(x-y)^{-1}}{|x-y|^{n-1}} e_n \frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}} = \frac{1}{2^{2n-1} y_n^{n-1}} \overline{D}^x \left(\int_{\frac{|x-y|}{|x-\widehat{y}|}}^1 \frac{\left(1-s^2\right)^{n-1}}{s^n} ds \right) \end{aligned}$$

is left and right hypermonogenic on $\mathbb{R}^{n+1} \backslash \{y, \widehat{y}\}$ for each y with $y_n \neq 0$.

The integral formula of the P-part given next is proved in [12].

Theorem 13. *Let Ω be an open subset of $\mathbb{R}_+^{n+1}$ (or $\mathbb{R}_-^{n+1}$) and K an $n+1$-chain satisfying $\overline{K} \subset \Omega$. If f is hypermonogenic in Ω and $y \in K$, then*

$$
\begin{aligned}
Pf(y) &= \frac{(2y_n)^n}{\omega_{n+1}} \int_{\partial K} P\left(p(x,y)\, d\sigma_{n-1}(x)\, f(x)\right) \\
&= \frac{(2y_n)^n}{\omega_{n+1}} \int_{\partial K} P\left(\frac{(x-y)^{-1}}{|x-y|^{n-1}} e_n \frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}} d\sigma_0(x)\, f(x)\right) \\
&= \frac{2^{n-1} y_n^n}{\omega_{n+1}} \left(\int_{\partial K} p(x,y)\, d\sigma_{n-1}(x)\, f(x) + \int_{\partial K} \widehat{p(x,y)} d\widehat{\sigma}_{n-1}(x)\, \widehat{f}(x)\right)
\end{aligned}
$$

where

$$
p(x,y) = x_n^{n-1} \frac{(x-y)^{-1}}{|x-y|^{n-1}} e_n \frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}}
$$

and ω_{n+1} is the surface measure of the unit ball in $\mathbb{R}^{n+1}$.

The kernel for the Q-part is the conjugate gradient of the product function of two Newtonian kernels.

Lemma 14. *The function*

$$
\begin{aligned}
q(x,y) &= \frac{(x-y)^{-1} + (x-\widehat{y})^{-1}}{2\,|x-y|^{n-1}\,|x-\widehat{y}|^{n-1}} \\
&= \frac{(x-y)^{-1}}{|x-y|^{n-1}} (x - Py) \frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}} \\
&= -\frac{1}{2(n-1)} \overline{D}^x \left(\frac{1}{|x-y|^{n-1}\,|x-\widehat{y}|^{n-1}}\right)
\end{aligned}
$$

is left and right $-n+1$-hypermonogenic on $\mathbb{R}^{n+1} \setminus \{y, \widehat{y}\}$ for each y with $y_n \neq 0$.

Proof. Note first that the functions $\frac{1}{|x-y|^{n-1}}$ and $\frac{1}{|x-\widehat{y}|^{n-1}}$ are harmonic. Hence

$$
\begin{aligned}
D^x \overline{D}^x \left(\frac{1}{|x-y|^{n-1}\,|x-\widehat{y}|^{n-1}}\right) &= \triangle^x \left(\frac{1}{|x-y|^{n-1}\,|x-\widehat{y}|^{n-1}}\right) \\
&= 2\left(\operatorname{grad} \frac{1}{|x-y|^{n-1}}, \operatorname{grad} \frac{1}{|x-\widehat{y}|^{n-1}}\right) \\
&= 2(n-1)^2 \frac{\sum_{i=0}^{n-1} (x_i - y_i)^2 + x_n^2 - y_n^2}{|x-y|^{n+1}\,|x-\widehat{y}|^{n+1}}.
\end{aligned}
$$

Noting that $Q\left(\overline{D} f\right) = -\frac{\partial f}{\partial x_n}$ for any real function f, we obtain

$$
\frac{1}{n-1} Q\left(\overline{D}\left(\frac{1}{|x-y|^{n-1}\,|x-\widehat{y}|^{n-1}}\right)\right) = \frac{x_n - y_n}{|x-y|^{n+1}\,|x-\widehat{y}|^{n-1}} + \frac{x_n + y_n}{|x-y|^{n-1}\,|x-\widehat{y}|^{n+1}}.
$$

Since

$$
\frac{1}{|x-y|^2} - \frac{1}{|x-\widehat{y}|^2} = \frac{4 x_n y_n}{|x-y|^2\,|x-\widehat{y}|^2}
$$

and

$$\frac{1}{|x-y|^2}+\frac{1}{|x-\widehat{y}|^2}=\frac{2\left(\sum_{i=0}^{n-1}(x_i-y_i)^2+x_n^2+y_n^2\right)}{|x-y|^2\,|x-\widehat{y}|^2}$$

we may compute

$$\begin{aligned}\frac{1}{n-1}Q\left(\overline{D}\left(\frac{1}{|x-y|^{n-1}\,|x-\widehat{y}|^{n-1}}\right)\right)&=\frac{2x_n\left(\sum_{i=0}^{n-1}(x_i-y_i)^2+x_n^2+y_n^2\right)-4x_ny_n^2}{|x-y|^{n+1}\,|x-\widehat{y}|^{n+1}}\\&=\frac{2x_n\left(\sum_{i=0}^{n-1}(x_i-y_i)^2+x_n^2-y_n^2\right)}{|x-y|^{n+1}\,|x-\widehat{y}|^{n+1}}.\end{aligned}$$

Hence we have

$$D^x\overline{D}^x\left(\frac{1}{|x-y|^{n-1}\,|x-\widehat{y}|^{n-1}}\right)-(n-1)\frac{Q\left(\overline{D}\left(\frac{1}{|x-y|^{n-1}|x-\widehat{y}|^{n-1}}\right)\right)}{x_n}=0.$$

■

The Cauchy formula of the Q-part of a hypermonogenic functions is given next.

Theorem 15. *Let Ω be an open subset of $\mathbb{R}^{n+1}_+$ (or $\mathbb{R}^{n+1}_-$) and K an $n+1$-chain satisfying $\overline{K}\subset\Omega$. If f is hypermonogenic in Ω and $y\in K$, then*

$$\begin{aligned}Qf(y)&=\frac{2^ny_n^{n-1}}{\omega_{n+1}}\int_{\partial K}Q\left(q(x,y)\,d\sigma_0(x)\,f(x)\right)\\&=-\frac{2^{n-1}y_n^{n-1}}{\omega_{n+1}}\left(\int_{\partial K}q(x,y)\,d\sigma_0(x)\,f(x)-\int_{\partial K}\widehat{q(x,y)}d\widehat{\sigma}_0(x)\,\widehat{f}(x)\right)e_n\end{aligned}$$

where

$$q(x,y)=\frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}}(x-Py)\frac{(x-y)^{-1}}{|x-y|^{n-1}}$$

and ω_{n+1} is the surface measure of the unit ball in $\mathbb{R}^{n+1}$.

Proof. Using Theorem 10 we obtain

$$\begin{aligned}\int_{\partial K}Q\left(q(x,y)\,d\sigma_0f\right)&=\int_{\partial(K\setminus B_r(y))}Q\left(q(x)\,d\sigma_0f\right)+\int_{\partial B_r(y)}Q\left(q(x)\,d\sigma_0f\right)\\&=\int_{\partial B_r(y)}Q\left(q(x)\,d\sigma_0f\right).\end{aligned}$$

The preceding Lemma implies that the function $q(x)$ is paravector valued. Thus we have

$$q(x)=q(x)^*=\frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}}(x-Py)\frac{(x-y)^{-1}}{|x-y|^{n-1}}.$$

Hence

$$\begin{aligned}
&\int_{\partial K} Q\left(q\left(x,y\right)d\sigma_0 f\right)\\
&= Q\left(\int_{\partial B_r(y)} \frac{\left(x-\widehat{y}\right)^{-1}}{\left|x-\widehat{y}\right|^{n-1}}\left(x-Py\right)\frac{\left(x-y\right)^{-1}}{\left|x-y\right|^{n-1}}d\sigma_0\left(x\right)f\left(x\right)\right)\\
&= Q\left(\int_{\partial B_r(y)} \frac{\left(x-\widehat{y}\right)^{-1}}{\left|x-\widehat{y}\right|^{n-1}}\left(x-Py\right)\frac{\left(x-y\right)^{-1}}{\left|x-y\right|^{n-1}}\frac{\left(x-y\right)}{r}f\left(x\right)dS\left(x\right)\right),
\end{aligned}$$

where S is the usual surface measure of the ball $B_r(y)$.When $r \to 0$, we obtain the result. ■

Combining the previous results we obtain.

Theorem 16. *Let Ω be an open subset of $\mathbb{R}^{n+1}_+$ (or $\mathbb{R}^{n+1}_-$) and K an $n+1$-chain satisfying $\overline{K} \subset \Omega$. If f is hypermonogenic in Ω and $y \in K$, then*

$$f\left(y\right)=\frac{2^{n-1}y_n^{n-1}}{\omega_{n+1}}\left(\int_{\partial K}\frac{\left(x-y\right)^{-1}d\sigma_0\left(x\right)f\left(x\right)}{\left|x-y\right|^{n-1}\left|x-\widehat{y}\right|^{n-1}}-\int_{\partial K}\frac{\left(\widehat{x}-y\right)^{-1}d\widehat{\sigma_0}\left(x\right)\widehat{f}\left(x\right)}{\left|x-y\right|^{n-1}\left|x-\widehat{y}\right|^{n-1}},\right)$$

where ω_{n+1} is the surface measure of the unit ball in $\mathbb{R}^{n+1}$.

Proof. Applying Theorems 13 and 15 we deduce

$$\begin{aligned}
f\left(y\right) &= \frac{\left(2y_n\right)^n}{\omega_{n+1}}\int_{\partial K}P\left(p\left(x,y\right)d\sigma_{n-1}\left(x\right)f\left(x\right)\right)+\frac{2^n y_n^{n-1}}{\omega_{n+1}}\int_{\partial K}Q\left(q\left(x,y\right)d\sigma_0\left(x\right)f\left(x\right)\right)e_n\\
&= \frac{2^{n-1}y_n^n}{\omega_{n+1}}\left(\int_{\partial K}p\left(x,y\right)d\sigma_{n-1}\left(x\right)f\left(x\right)+\int_{\partial K}\widehat{p\left(x,y\right)}d\widehat{\sigma}_{n-1}\left(x\right)\widehat{f}\left(x\right)\right)\\
&+\frac{2^{n-1}y_n^{n-1}}{\omega_{n+1}}\left(\int_{\partial K}q\left(x,y\right)d\sigma_0\left(x\right)f\left(x\right)-\int_{\partial K}\widehat{q\left(x,y\right)}d\widehat{\sigma}_0\left(x\right)\widehat{f}\left(x\right)\right)\\
&= \frac{2^{n-1}y_n^{n-1}}{\omega_{n+1}}\int_{\partial K}\left(y_n p\left(x,y\right)+x_n^{n-1}q\left(x,y\right)\right)d\sigma_{n-1}\left(x\right)f\left(x\right)\\
&+\frac{2^{n-1}y_n^{n-1}}{\omega_{n+1}}\int_{\partial K}\left(y_n\widehat{p\left(x,y\right)}-x_n^{n-1}\widehat{q\left(x,y\right)}\right)d\widehat{\sigma}_{n-1}\left(x\right)\widehat{f}\left(x\right)
\end{aligned}$$

where

$$\begin{aligned}
p\left(x,y\right) &= x_n^{n-1}\frac{\left(x-y\right)^{-1}}{\left|x-y\right|^{n-1}}e_n\frac{\left(x-\widehat{y}\right)^{-1}}{\left|x-\widehat{y}\right|^{n-1}},\\
q\left(x,y\right) &= \frac{\left(x-y\right)^{-1}}{\left|x-y\right|^{n-1}}\left(x-Py\right)\frac{\left(x-\widehat{y}\right)^{-1}}{\left|x-\widehat{y}\right|^{n-1}}.
\end{aligned}$$

Since

$$y_n p\left(x,y\right)+x_n^{n-1}q\left(x,y\right)=x_n^{n-1}\frac{\left(x-y\right)^{-1}}{\left|x-y\right|^{n-1}\left|x-\widehat{y}\right|^{n-1}}$$

and

$$y_n p\left(x,y\right)-x_n^{n-1}q\left(x,y\right)=-x_n^{n-1}\frac{\left(x-\widehat{y}\right)^{-1}}{\left|x-y\right|^{n-1}\left|x-\widehat{y}\right|^{n-1}},$$

we conclude the assertion. ■

Applying the previous integral formula to a continuous function we obtain a hypermonogenic function.

Theorem 17. *Let* Ω *be an open subset of* $\mathbb{R}_+^{n+1}$ *and* K *an* $n+1$*-chain satisfying* $\overline{K}\subset\Omega$ *If a function* $f:\Omega\to C\ell_n$ *is continuous, then the function*

$$h(y)=\frac{2^{n-1}y_n^{n-1}}{\omega_{n+1}}\left(\int_{\partial K}\frac{(x-y)^{-1}\,d\sigma_0(x)\,f(x)}{|x-y|^{n-1}\,|x-\widehat{y}|^{n-1}}-\int_{\partial K}\frac{(\widehat{x}-y)^{-1}\,d\widehat{\sigma_0}(x)\,\widehat{f}(x)}{|x-y|^{n-1}\,|x-\widehat{y}|^{n-1}}\right)$$

is hypermonogenic in K.

Proof. In view of Proposition 5 it is enough to prove that the function $\frac{\omega_{n+1}}{2^{n-1}y_n^{n-1}}h(y)\,e_n$ is $-n+1$-hypermonogenic. Note first that in the proof of the previous theorem we proved that

$$\frac{\omega_{n+1}}{2^n y_n^{n-1}}h(y)=y_n\int_{\partial K}P\left(p(x,y)\,d\sigma_{n-1}(x)\,f(x)\right)+\int_{\partial K}Q\left(q(x,y)\,d\sigma_0(x)\,f(x)\right)e_n.$$

Hence denoting $C=\frac{\omega_{n+1}}{2^{n-1}}$ and using (1) we may compute

$$\begin{aligned}\frac{Q'\left(\frac{Ch(y)e_n}{y_n^{n-1}}\right)}{y_n}&=\int_{\partial K}2P'\left(p(x,y)\,d\sigma_{n-1}(x)\,f(x)\right)\\&=\int_{\partial K}2P'\left(\frac{(x-y)^{-1}}{|x-y|^{n-1}}e_n\frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}}d\sigma_0(x)\,f(x)\right)\\&=-\int_{\partial K}2P\left(\frac{(x-y)}{|x-y|^{n+1}}e_n\frac{(x-\widehat{y})}{|x-\widehat{y}|^{n+1}}d\sigma_0'(x)\,f'(x)\right)\\&=-\int_{\partial K}2P\left(\frac{(x-y)}{|x-y|^{n+1}}\frac{(\widehat{x}-y)^{-1}}{|x-\widehat{y}|^{n-1}}d\widehat{\sigma_0}(x)\,\widehat{f}(x)\,e_n\right).\end{aligned}$$

Applying (2) and noting that

$$\begin{aligned}\frac{(\widehat{x}-\widehat{y})}{|x-y|^{n+1}}\frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}}&=-e_n\frac{\overline{x-y}}{|x-y|^{n+1}}e_n\frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}}\\&=-e_n\frac{(x-y)^{-1}}{|x-y|^{n-1}}e_n\frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}}\end{aligned}$$

we infer

$$\begin{aligned}\frac{Q'\left(\frac{Ch(y)e_n}{y_n^{n-1}}\right)}{y_n}&=-\int_{\partial K}\frac{(x-y)}{|x-y|^{n+1}}\frac{(\widehat{x}-y)^{-1}}{|x-\widehat{y}|^{n-1}}d\widehat{\sigma_0}(x)\,\widehat{f}(x)\,e_n\\&\quad+\int_{\partial K}\frac{(\widehat{x}-\widehat{y})}{|x-y|^{n+1}}\frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}}d\sigma_0(x)\,f(x)\,e_n\\&=-\int_{\partial K}\frac{(x-y)}{|x-y|^{n+1}}\frac{(\widehat{x}-y)^{-1}}{|x-\widehat{y}|^{n-1}}d\widehat{\sigma_0}(x)\,\widehat{f}(x)\,e_n\\&\quad-\int_{\partial K}e_n\frac{(x-y)^{-1}}{|x-y|^{n-1}}e_n\frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}}d\sigma_0(x)\,f(x)\,e_n.\end{aligned}$$

Since $p(x,y) = x_n^{n-1}\frac{(x-y)^{-1}}{|x-y|^{n-1}}e_n\frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}}$ is a paravector and therefore $p(x,y) = p(x,y)^*$, we obtain

$$\begin{aligned}\frac{Q'\left(\frac{Ch(y)e_n}{y_n^{n-1}}\right)}{y_n} = &-\int_{\partial K}\frac{(x-y)}{|x-y|^{n+1}}\frac{(\widehat{x}-y)^{-1}}{|x-\widehat{y}|^{n-1}}d\widehat{\sigma_0}(x)\widehat{f}(x)e_n \\ &-\int_{\partial K}e_n\frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}}e_n\frac{(x-y)^{-1}}{|x-y|^{n-1}}d\sigma_0(x)f(x)e_n.\end{aligned}$$

Since the functions $(x-y)^{-1}|x-y|^{1-n}$ and $(\widehat{x}-y)^{-1}|x-\widehat{y}|^{1-n} = (\widehat{x}-y)^{-1}|\widehat{x}-y|^{1-n}$ are monogenic we observe

$$\begin{aligned}D_l^y\left(\frac{Ch(y)e_n}{y_n^{n-1}}\right) &= \int_{\partial K}D_l^y\left(\frac{(x-y)^{-1}}{|x-y|^{n-1}|y-\widehat{x}|^{n-1}}\right)d\sigma_0(x)f(x)e_n \\ &\quad-\int_{\partial K}D_l^y\left(\frac{(\widehat{x}-y)^{-1}}{|x-y|^{n-1}|x-\widehat{y}|^{n-1}}\right)d\widehat{\sigma_0}(x)\widehat{f}(x)e_n \\ &= -(n-1)\int_{\partial K}\frac{(y-\widehat{x})(x-y)^{-1}}{|x-\widehat{y}|^{n+1}|x-y|^{n-1}}d\sigma_0(x)f(x)e_n \\ &\quad+(n-1)\int_{\partial K}\frac{(x-y)(\widehat{x}-y)^{-1}}{|x-y|^{n+1}|x-\widehat{y}|^{n-1}}d\widehat{\sigma}_0(x)\widehat{f}(x)e_n.\end{aligned}$$

Noting that

$$\begin{aligned}\frac{(y-\widehat{x})(x-y)^{-1}}{|x-\widehat{y}|^{n+1}|x-y|^{n-1}} &= -\frac{e_n\left(\overline{\widehat{y}-x}\right)e_n(x-y)^{-1}}{|x-\widehat{y}|^{n+1}|x-y|^{n-1}} \\ &= e_n\frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}}e_n\frac{(x-y)^{-1}}{|x-y|^{n-1}}\end{aligned}$$

we infer

$$\begin{aligned}D_l^y\left(\frac{Ch(y)e_n}{y_n^{n-1}}\right) &= -(n-1)\int_{\partial K}e_n\frac{(x-\widehat{y})^{-1}}{|x-\widehat{y}|^{n-1}}e_n\frac{(x-y)^{-1}}{|x-y|^{n-1}}d\sigma_0(x)f(x)e_n \\ &\quad-(n-1)\int_{\partial K}\frac{(x-y)(\widehat{x}-y)^{-1}}{|x-y|^{n+1}|x-\widehat{y}|^{n-1}}d\widehat{\sigma}_0(x)\widehat{f}(x)e_n \\ &= (n-1)\frac{Q'\left(\frac{Ch(y)e_n}{y_n^{n-1}}\right)}{y_n}.\end{aligned}$$

Hence $M_{-n+1}^l\left(\frac{h(y)e_n}{y_n^{n-1}}\right) = 0$ which implies by Proposition 5 that h is hypermonogenic.

■

References

[1] Brackx, R. Delanghe and F. Sommen, *Clifford Analysis*, Pitman, Boston, London, Melbourne, 1982.

[2] Cerejeiras, P., Decomposition of analytic hyperbolically harmonic functions. *Proceedings of Clifford Algebras and their Applications Aachen,* Kluwer, Dordrecht (1998), 45–51.

[3] Cnops, J., *Hurwitz pairs and applications of Möbius transformation*, Habilitation thesis, Univ. Gent, 1994.

[4] Eriksson-Bique, S.-L., Comparison of quaternionic analysis and its modification*,* Harlow: Longman. Pitman Res. Notes Math. Ser. 394, (1998), 109-121.

[5] Eriksson-Bique, S.-L, On modified Clifford analysis, *Complex Variables*,Vol. 45, (2001), 11–32..

[6] Eriksson-Bique, S.-L, *R*eal analycity on modified Clifford analysis, Submitted for publication, 1–12.

[7] Eriksson-Bique, S.-L., The binomial theorem for hypercomplex numbers, *Ann. Acad. Sci. Fenn. 24* (1999), 225–229.

[8] Eriksson-Bique, S.-L., $k-$hypermonogenic functions*, In Progress in Analysi*s, Vol I, World Scientific (2003), 337–348.

[9] Eriksson-Bique, S.-L. and H. Leutwiler, On modified quaternionic analysis in $\mathbb{R}^3$, *Arch. Math. 70 (*1998), 228–234.

[10] Eriksson-Bique, S.-L. and H. Leutwiler, Hypermonogenic functions, In *Clifford Algebras and their Applications in Mathematical Physics*, Vol. 2, Birkhäuser, Boston, 2000, 287–302.

[11] Eriksson-Bique, S.-L. and H. Leutwiler, Hypermonogenic functions and Möbius transformations, *Advances in Applied Clifford algebras* Vol 11 (S2), December (2001), 67–76.

[12] Eriksson, S.-L. and H. Leutwiler, Hypermonogenic functions and their Cauchy-type theorems, 1–18, to appear in Proceedings of Clifford analysis and applications in Macao 2002.

[13] Hempfling, Th., Beiträge zur Modifizierten Clifford-Analysi*s, Dissertation, Univ. Erlangen-Nürnberg*, August, 1997.

[14] Hempfling, Th., Multinomials in modified Clifford analysis, *C. R. Math. Rep. Acad. Sci. Canada* (1996), no. 18 (2,3), 99–102.

[15] Hempfling, Th., H. Leutwiler, Modified quaternionic analysis in $\mathbb{R}^4$. *In Proceedings of Clifford algebras and their application in mathematical physics*, Aachen 1996, Fundamental theories of physics, 94, Kluwer, Dordrecht (1998), 227–237.

[16] Hengartner, W., H. Leutwiler, Hyperholomorphic functions in $\mathbb{R}^3$,*Advances in Applied Clifford algebras* Vol 11 (S1), November (2001), 247–265.

[17] Laville, G.; Ramadanoff, I., Holomorphic Cliffordian functions. *Adv. Appl. Clifford Algebr.* 8, No.2, (1998), 323-340.

[18] Leutwiler, H., Modified Clifford analysis, *Complex Variables* **17** (1992) 153–171.

[19] Leutwiler, H., Modified quaternionic analysis in $\mathbb{R}^3$. *Complex Variables 20* (1992), 19–51.

[20] Leutwiler, H., More on modified quaternionic analysis in $\mathbb{R}$, *Forum Math.* **7** (1995), 279–305.

[21] Leutwiler, H., Rudiments of a function theory in $\mathbb{R}^3$, *Expo. Math* 14 (1996), 097-123.

[22] Ryan, J. Iterated Dirac operators in C_n, *Z. Anal. Anwendungen* 9 (1990), no. 5, 385–401

Tampere University of Technology
Institute of Mathematics
P.O.Box 553
FT-33101 Tampere
Finland
sirkka-liisa.eriksson@tut.fi