The Cartan Product

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Abstract

The Cartan product is a generalization of the symmetric product of two vectors. We review its definition, present some examples, and exhibit its rôle in the construction of various natural algebras.

1 Definitions and Examples

Let G denote a compact Lie group (or $\mathfrak{g}$ a semisimple Lie algebra). Suppose V and W are finite-dimensional irreducible representations of G (or $\mathfrak{g}$). The tensor product $V \otimes W$ decomposes into irreducibles amongst which there is one of largest dimension and it occurs with multiplicity one. (Facts concerning the representation theory of semisimple Lie algebras may be found in [6] or [7].) This is the Cartan product [3] of V and W and we shall denote it by $V \circledcirc W$. The decomposition

$$V \otimes W = V \circledcirc W \oplus \cdots$$

affords a projection and inclusion $V \otimes W \overset{\leftarrow}{\hookrightarrow} V \circledcirc W$. The image of $v \otimes w \in V \otimes W$ under the projection $V \otimes W \to V \circledcirc W$ is the Cartan product $v \circledcirc w$ of $v \in V$ and $w \in W$. An alternative definition of the Cartan product is in terms of weights. The irreducible representations of $\mathfrak{g}$ are in 1–1 correspondence with dominant integral weights in the usual way. If V and W have highest weights λ and μ respectively, then $V \circledcirc W$ has highest weight $\lambda + \mu$ and is generated as a $\mathfrak{g}$-module by the tensor product $v \otimes w$ of highest weight vectors $v \in V$ and $w \in W$.

The basic example is if $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and $V = W = \mathbb{C}^n$, the defining representation. Then $V \circledcirc W = \bigodot^2 \mathbb{C}^n$ and the Cartan product of two vectors coincides with

*Support from the Australian Research Council is gratefully acknowledged
Received by the editors October 31 2003.
1991 *Mathematics Subject Classification* : 17B10, 15A69.
Key words and phrases : Lie group, representation, highest weight.

their symmetric product $v \odot w$. Written in a given basis therefore, $v^i w^j \mapsto v^{(i}w^{j)}$ where, following [9], round brackets denote symmetrization. In terms of Young tableau (see [9] for a discussion of Young tableau well suited to this article and [6] for proofs)

$$\square \circledcirc \square = \square\!\square.$$

If $V = \mathbb{C}^n$ is the defining representation of $\mathfrak{sl}(n, \mathbb{C})$ and $W = V^*$ is its dual, then $V \otimes W = \mathrm{End}(\mathbb{C}^n)$ whilst $V \circledcirc W = \mathrm{End}_\circ(\mathbb{C}^n)$, the trace-free endomorphisms. With indices, the Cartan product is

$$v^i w_j \mapsto v^i w_j - \tfrac{1}{n}\delta^i{}_j v^k w_k,$$

where $\delta^i{}_j$ is the Kronecker delta and repeated indices implicitly carry a sum.

If $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ acting on symmetric powers of the defining representation, the Cartan product again coincides with the symmetric product $v^{i\cdots j}w^{k\cdots l} \mapsto v^{(i\cdots j}w^{k\cdots l)}$. With Young tableau,

$$\underbrace{\square\!\square\cdots\square}_{p \text{ boxes}} \circledcirc \underbrace{\square\!\square\cdots\square}_{q \text{ boxes}} = \underbrace{\square\!\square\cdots\cdots\square}_{p+q \text{ boxes}}.$$

More generally, the Cartan product of two Young tableau is given by a tableau with row-lengths obtained by simple addition. For example,

$$\begin{array}{c}\square\\[-4pt]\square\end{array} \circledcirc \square = \begin{array}{l}\square\!\square\\[-4pt]\square\end{array}.$$

In this case the Cartan product is $v^{ij}w^k \mapsto v^{ij}w^k - v^{[ij}w^{k]}$ where, again following [9], square brackets denote skew symmetrization.

When $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$, the Cartan product also removes traces:–

$$v_i w_j \mapsto v_{(i}w_{j)} - \tfrac{1}{n} g_{ij} v^k w_k, \tag{1}$$

where g_{ij} is the quadratic form preserved by $\mathfrak{so}(n, \mathbb{C})$ and whose inverse is used to raise indices.

These simple examples may mislead. For tensors with slightly more complicated symmetries the Cartan product is difficult to express in terms of indices. If, for example, $v, w \in \Lambda^2(V)$, then $v \circledcirc w$ is given by

$$\tfrac{1}{3}v_{ij}w_{kl} + \tfrac{1}{3}v_{lk}w_{ji} + \tfrac{1}{6}v_{ik}w_{jl} - \tfrac{1}{6}v_{il}w_{jk} + \tfrac{1}{3}v_{lj}w_{ki} - \tfrac{1}{6}v_{kj}w_{li}$$

under $\mathfrak{sl}(V)$ whilst, under $\mathfrak{so}(V)$, several extra terms are needed:–

$$\begin{aligned}
&\tfrac{1}{3}v_{ij}w_{kl} + \tfrac{1}{3}v_{lk}w_{ji} + \tfrac{1}{6}v_{ik}w_{jl} - \tfrac{1}{6}v_{il}w_{jk} + \tfrac{1}{3}v_{lj}w_{ki} - \tfrac{1}{6}v_{kj}w_{li}\\
&\quad - \tfrac{1}{2(n-2)}\left(v_i{}^m w_{km}g_{jl} - v_j{}^m w_{km}g_{il} + v_j{}^m w_{lm}g_{ik} - v_i{}^m w_{lm}g_{jk}\right)\\
&\qquad - \tfrac{1}{2(n-2)}\left(v_k{}^m w_{im}g_{jl} - v_k{}^m w_{jm}g_{il} + v_l{}^m w_{jm}g_{ik} - v_l{}^m w_{im}g_{jk}\right)\\
&\qquad\quad + \tfrac{1}{(n-1)(n-2)}v^{pq}w_{pq}(g_{ik}g_{jl} - g_{jk}g_{il})
\end{aligned}$$

to ensure that the resulting tensor is totally trace-free.

2 Associativity

Suppose U, V, and W are $\mathfrak{g}$-modules and denote by $U \circledcirc V \circledcirc W$ the irreducible representation in $U \otimes V \otimes W$ generated by $u \otimes v \otimes w$ for highest weight vectors $u \in U$, $v \in V$, $w \in W$. Then, in the sense that $(U \circledcirc V) \circledcirc W = U \circledcirc V \circledcirc W = U \circledcirc (V \circledcirc W)$ and the following diagram evidently commutes,

$$\begin{array}{ccc} U \otimes V \otimes W & \to & U \circledcirc V \otimes W \\ \downarrow & & \downarrow \\ U \otimes V \circledcirc W & \to & U \circledcirc V \circledcirc W \end{array} \tag{2}$$

the Cartan product is associative on vectors: $(u \circledcirc v) \circledcirc w = u \circledcirc (v \circledcirc w)$. Beware, however, that even in the simplest cases, the operations

$$\begin{array}{ccccccc} U \otimes V \otimes W & \ni & u \otimes v \otimes w & \mapsto & u \circledcirc v \otimes w & \in & U \otimes V \otimes W \\ U \otimes V \otimes W & \ni & u \otimes v \otimes w & \mapsto & u \otimes v \circledcirc w & \in & U \otimes V \otimes W \end{array}$$

do not commute. In many cases, there is a kind of associativity on the level of representations. Specifically, we might expect that

$$((U \circledcirc V) \otimes W) \cap (U \otimes (V \circledcirc W)) = U \circledcirc V \circledcirc W \tag{3}$$

where the intersection is taken inside $U \otimes V \otimes W$. Manifestly, we have

$$((U \circledcirc V) \otimes W) \cap (U \otimes (V \circledcirc W)) \supseteq U \circledcirc V \circledcirc W$$

and, in many cases, we have equality. For example, it is clear that

$$((\mathbb{C}^n \circledcirc \mathbb{C}^n) \otimes \mathbb{C}^n) \cap (\mathbb{C}^n \otimes (\mathbb{C}^n \circledcirc \mathbb{C}^n)) = \mathbb{C}^n \circledcirc \mathbb{C}^n \circledcirc \mathbb{C}^n$$

under $\mathfrak{sl}(n, \mathbb{C})$ or $\mathfrak{so}(n, \mathbb{C})$. In the former case, this is the statement that a tensor T_{ijk} that is symmetric in the indices ij and jk is, in fact, completely symmetric. The latter case is simply the observation that if, in addition, T_{ijk} is trace-free in ij and jk then T_{ijk} is totally trace-free.

In [5], general equality was conjectured but it was pointed out to me by Vladimir Souček that this is too optimistic. He observed that

$$((V \circledcirc V^*) \otimes V) \cap (V \otimes (V^* \circledcirc V)) \neq V \circledcirc V^* \circledcirc V$$

as representations of $\mathfrak{sl}(V)$ since the left hand side consists of tensors $T_i{}^j{}_k$ that are trace free in ij and jk but the right hand side imposes, in addition, that $T_i{}^j{}_k$ be symmetric in ik. He further speculated that the reason behind this counterexample is that the representation in the middle has a highest weight that, when written as an integral linear combination of the fundamental weights, has a zero coefficient where the outside representations have a non-zero coefficient. This seems to be confirmed by other examples. Thus, one should expect

$$((\mathbb{C}^n \circledcirc \Lambda^2\mathbb{C}^n) \otimes \mathbb{C}^n) \cap (\mathbb{C}^n \otimes (\Lambda^2\mathbb{C}^n \circledcirc \mathbb{C}^n)) \neq \mathbb{C}^n \circledcirc \Lambda^2\mathbb{C}^n \circledcirc \mathbb{C}^n$$

and this is the case. Tensors T_{ijkl} in the left hand side are characterized by the symmetries

$$T_{ijkl} = T_{i[jk]l} \qquad T_{[ijk]l} = 0 \qquad T_{i[jkl]} = 0$$

which decompose into two irreducible parts according to

$$T_{ijkl} = T_{ljki} \quad \text{or} \quad T_{ijkl} = -T_{ljki}$$

only the first of which corresponds to the right hand side. For $\mathfrak{sl}(n, \mathbb{C})$ it is possible directly to verify that Souček's speculation is correct. For $\mathfrak{sl}(2, \mathbb{C})$ it is clear: if a tensor

$$T_{i_1 i_2 \cdots i_p j_1 j_2 \cdots j_q k_1 k_2 \cdots k_r}$$

is symmetric in the indices $i_1 i_2 \cdots i_p j_1 j_2 \cdots j_q$ and $j_1 j_2 \cdots j_q k_1 k_2 \cdots k_r$, then is is totally symmetric unless, of course, $q = 0$ and both p and r are non-zero. For $\mathfrak{sl}(3, \mathbb{C})$ and, more generally, for any complex semisimple Lie algebra, we may specify its irreducible representations by attaching non-negative integers to the nodes of the corresponding Dynkin diagram, defining a dominant integral weight as a linear combination of the fundamental weights. Following the conventions of [2] in this regard, Souček's statement for $\mathfrak{sl}(3, \mathbb{C})$ is as follows.

Theorem 1.

$$((\overset{p}{\bullet}\!\!-\!\!\overset{a}{\bullet} \circledcirc \overset{q}{\bullet}\!\!-\!\!\overset{b}{\bullet}) \otimes \overset{r}{\bullet}\!\!-\!\!\overset{c}{\bullet}) \cap (\overset{p}{\bullet}\!\!-\!\!\overset{a}{\bullet} \otimes (\overset{q}{\bullet}\!\!-\!\!\overset{b}{\bullet} \circledcirc \overset{r}{\bullet}\!\!-\!\!\overset{c}{\bullet})) = \overset{p}{\bullet}\!\!-\!\!\overset{a}{\bullet} \circledcirc \overset{q}{\bullet}\!\!-\!\!\overset{b}{\bullet} \circledcirc \overset{r}{\bullet}\!\!-\!\!\overset{c}{\bullet} \tag{4}$$

unless $p \neq 0$, $q = 0$, $r \neq 0$ *or* $a \neq 0$, $b = 0$, $c \neq 0$.

Proof. We may realize $\overset{p}{\bullet}\!\!-\!\!\overset{a}{\bullet}$ explicitly as those tensors

$$T_{i_1 i_2 \cdots i_p}{}^{j_1 j_2 \cdots j_a} = T_{(i_1 i_2 \cdots i_p)}{}^{(j_1 j_2 \cdots j_a)} \quad \text{such that} \quad T_{k i_2 \cdots i_p}{}^{k j_2 \cdots j_a} = 0.$$

An element of the left hand side of (4), therefore, is a tensor

$$T_{i_1 \cdots i_p i_{p+1} \cdots i_{p+q} i_{p+q+1} \cdots i_{p+q+r}}{}^{j_1 \cdots j_a j_{a+1} \cdots j_{a+b} j_{a+b+1} \cdots j_{a+b+c}} \tag{5}$$

that is, in particular,

- symmetric in $i_1 \cdots i_{p+q}$ and $i_{p+1} \cdots i_{p+q+r}$
- symmetric in $j_1 \cdots j_{a+b}$ and $j_{a+1} \cdots j_{a+b+c}$

and, therefore, totally symmetric in both its contravariant and covariant indices. In addition, this tensor is trace-free with respect to some pair of contravariant and covariant indices and, therefore, totally trace-free. This is precisely the remaining characterizing feature of

$$\overset{p}{\bullet}\!\!-\!\!\overset{a}{\bullet} \circledcirc \overset{q}{\bullet}\!\!-\!\!\overset{b}{\bullet} \circledcirc \overset{r}{\bullet}\!\!-\!\!\overset{c}{\bullet} = \overset{p+q+r}{\bullet}\!\!\longrightarrow\!\!\overset{a+b+c}{\bullet},$$

the right hand side of (4). ■

The method of proof suggests an inductive possibility for $\mathfrak{sl}(n, \mathbb{C})$ since the key deduction of total symmetry in the upper and lower indices of the tensor (5) is evidently no different from the $\mathfrak{sl}(2, \mathbb{C})$ case. This is, indeed, the case:–

Theorem 2. *Suppose that U, V, W are irreducible representations of $\mathfrak{sl}(n, \mathbb{C})$ and that if the integer corresponding to a particular node in the Dynkin diagram for V is zero then the same is true for at least one of U or W. Then* (3) *holds.*

Proof. To avoid notational awkwardness, the method of proof is best explained by means of a typical example, the general case being left to the reader. Let us consider the case $\mathfrak{sl}(5, \mathbb{C})$. The Cartan product is

$$\overset{a}{\bullet}-\overset{b}{\bullet}-\overset{c}{\bullet}-\overset{d}{\bullet} \circledcirc \overset{p}{\bullet}-\overset{q}{\bullet}-\overset{r}{\bullet}-\overset{s}{\bullet} = \overset{a+p}{\bullet}\!\!-\!\!-\!\!-\overset{b+q}{\bullet}\!\!-\!\!-\!\!-\overset{c+r}{\bullet}\!\!-\!\!-\!\!-\overset{d+s}{\bullet}.$$

If all three representations are of the form $\overset{p}{\bullet}-\overset{0}{\bullet}-\overset{0}{\bullet}-\overset{0}{\bullet}$, then the statement is the familiar one, namely that tensors that are symmetric in two overlapping clumps of indices are completely symmetric. Next we need to describe tensors in $\overset{0}{\bullet}-\overset{q}{\bullet}-\overset{0}{\bullet}-\overset{0}{\bullet}$. There are two ways, using either covariant tensors or contravariant indices:–

$$T_{i_1j_1i_2j_2\cdots i_qj_q} = T_{[i_1j_1][i_2j_2]\cdots[i_qj_q]} \quad \text{subject to } \underbrace{\begin{array}{|c|c|c|c|}\hline & & \cdots & \\ \hline & & \cdots & \\ \hline\end{array}}_{q \text{ columns}} \tag{6}$$

or

$$T^{i_1j_1k_1i_2j_2k_2\cdots i_qj_qk_q} = T^{[i_1j_1k_1][i_2j_2k_2]\cdots[i_qj_qk_q]} \quad \text{subject to } \underbrace{\begin{array}{|c|c|c|c|}\hline & & \cdots & \\ \hline & & \cdots & \\ \hline & & \cdots & \\ \hline\end{array}}_{q \text{ columns}}, \tag{7}$$

where the Young tableau specify the additional symmetries satisfied by T. These two realizations are easily related by raising the indices in clumps using the invariant volume form ϵ^{ijklm}. Though not usually included in their definition, it is well known that for such rectangular tableau, the corresponding tensors must be symmetric in their clumped skew indices. Indeed, $\circledcirc^q V \subseteq \bigodot^q V$ for any representation V and, in particular,

$$\overset{0}{\bullet}-\overset{q}{\bullet}-\overset{0}{\bullet}-\overset{0}{\bullet} = \circledcirc^q \overset{0}{\bullet}-\overset{1}{\bullet}-\overset{0}{\bullet}-\overset{0}{\bullet} \subseteq \bigodot^q \overset{0}{\bullet}-\overset{1}{\bullet}-\overset{0}{\bullet}-\overset{0}{\bullet}.$$

Using either (6) or (7), if all three representations are of the form $\overset{0}{\bullet}-\overset{q}{\bullet}-\overset{0}{\bullet}-\overset{0}{\bullet}$, the theorem follows easily: a tensor in the left hand side of (3) is completely symmetric in its indices clumped as skew pairs or triplets, respectively. The additional symmetries corresponding to a rectangular tableau follow immediately.

Using the contravariant realization (7), the representation $\overset{p}{\bullet}-\overset{q}{\bullet}-\overset{0}{\bullet}-\overset{0}{\bullet}$ is realized by tensors

$$T_{h_1h_2\cdots h_p}{}^{i_1j_1k_1i_2j_2k_2\cdots i_qj_qk_q}$$

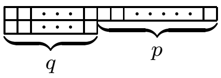
symmetric in $h_1h_2\cdots h_p$, subject to (7) in the remaining indices, and totally trace-free. Owing to the symmetry in the covariant and clumped contravariant indices, however, the trace-free condition is equivalent to any of the traces vanishing:–

$$T_{kh_2\cdots h_p}{}^{kj_1k_1i_2j_2k_2\cdots i_qj_qk_q} = 0.$$

At this point it is clear that by mimicking the proof of Theorem 1, but with the contravariant indices gathered in skew triplets, we may deduce Theorem 2 in case all three representations are of the form $\overset{p}{\bullet}-\overset{q}{\bullet}-\overset{0}{\bullet}-\overset{0}{\bullet}$. But we may also view this result as a statement about tensors with only covariant indices. Specifically, if we revert to (6), then $\overset{p}{\bullet}-\overset{q}{\bullet}-\overset{0}{\bullet}-\overset{0}{\bullet}$ may be realized as tensors

$$T_{h_1h_2\cdots h_p i_1 j_1 i_2 j_2\cdots i_q j_q} = T_{(h_1h_2\cdots h_p)[i_1j_1][i_2j_2]\cdots[i_qj_q]}$$

subject to some additional symmetries, namely those specified by the tableau


.

Now we are in a position to proceed to the case $\overset{p}{\bullet}-\overset{q}{\bullet}-\overset{r}{\bullet}-\overset{0}{\bullet}$. By writing

$$\overset{p}{\bullet}-\overset{q}{\bullet}-\overset{r}{\bullet}-\overset{0}{\bullet} = \overset{p}{\bullet}-\overset{q}{\bullet}-\overset{0}{\bullet}-\overset{0}{\bullet} \circledcirc \overset{0}{\bullet}-\overset{0}{\bullet}-\overset{r}{\bullet}-\overset{0}{\bullet}$$

and adopting the contravariant realization of $\overset{0}{\bullet}-\overset{0}{\bullet}-\overset{r}{\bullet}-\overset{0}{\bullet}$ we may view the Cartan product as the trace-free part of the tensor product. This allows us to combine with the result we already have for $\overset{p}{\bullet}-\overset{q}{\bullet}-\overset{0}{\bullet}-\overset{0}{\bullet}$ exactly as before. Of course, we may view this result as a statement about tensors with covariant indices. Finally, we bring in $\overset{0}{\bullet}-\overset{0}{\bullet}-\overset{0}{\bullet}-\overset{s}{\bullet}$ realized contravariantly

$$\overset{0}{\bullet}-\overset{0}{\bullet}-\overset{0}{\bullet}-\overset{s}{\bullet} = \{T^{i_1i_2\cdots i_s} = T^{(i_1i_2\cdots i_s)}\}$$

in order to reach the general representation $\overset{p}{\bullet}-\overset{q}{\bullet}-\overset{r}{\bullet}-\overset{s}{\bullet}$. The proof for $\mathfrak{sl}(n,\mathbb{C})$ proceeds similarly by induction. ■

This proof gives no clue for the general case of a semisimple Lie algebra $\mathfrak{g}$. If $u \in U$, $v \in V$, and $w \in W$ are highest weight vectors, then (3) is equivalent to saying that one can apply suitable raising operators from $\mathfrak{g}$ to any vector in the left hand side to achieve $u \otimes v \otimes w$. A direct proof of this would be much more satisfactory.

If $U = V = W$, then the hypotheses of Theorem 2 are certainly satisfied. More generally, since $\circledcirc^m \overset{p}{\bullet}-\overset{q}{\bullet}-\overset{r}{\bullet}-\overset{s}{\bullet} = \overset{mp}{\bullet}-\overset{mq}{\bullet}-\overset{mr}{\bullet}-\overset{ms}{\bullet}$ and so on, the following corollary is immediate.

Corollary 1. *For any irreducible representation V of $\mathfrak{sl}(n,\mathbb{C})$,*

$$((\circledcirc^p V \circledcirc \circledcirc^q V) \otimes \circledcirc^r V) \cap (\circledcirc^p V \otimes (\circledcirc^q V \circledcirc \circledcirc^r V)) = \circledcirc^{p+q+r} V. \tag{8}$$

It seems reasonable to suppose that this corollary is valid for any semisimple Lie algebra. The particular case of $\mathfrak{so}(n,\mathbb{C})$ acting on $\Lambda^2\mathbb{C}^n$, for example, is proved 'by hand' in [4, §4]. More generally, most cases of tensor representations of $\mathfrak{so}(n,\mathbb{C})$ follow from Corollary 1 since being trace-free with respect to the metric is usually the only extra requirement for irreducibility (but, for tensors whose symmetries are described by Young tableau of height $n/2$ in case n is even, there is the further decomposition into self-dual and anti-self-dual parts).

3 Algebras

As usual, suppose V is an irreducible representation of a semisimple Lie algebra $\mathfrak{g}$. The Cartan algebra $\circledcirc V$ associated to V is the direct sum

$$\circledcirc V = \bigoplus_{s=0}^{\infty} \circledcirc^s V \tag{9}$$

equipped with the algebra operation induced by the Cartan product:–

$$\circledcirc^r V \otimes \circledcirc^s V \xrightarrow{\circledcirc} \circledcirc^{r+s} V. \tag{10}$$

The commutativity of diagram (2) implies that $\circledcirc V$ is an associative algebra. Clearly we have an algebra homomorphism: $\bigotimes V \to \circledcirc V$ and, indeed, $\bigodot V \to \circledcirc V$, which is the identity if $\mathfrak{g} = \mathfrak{sl}(V)$. Let

$$\mathcal{I} = \operatorname{span}\{u \otimes v - u \circledcirc v \mid u, v \in V\} \subset \bigotimes\nolimits^2 V.$$

Evidently, $\mathcal{I}$ is in the kernel of $\bigotimes V \to \circledcirc V$. In the case of $\mathfrak{sl}(V)$ acting on V, the kernel of $\bigotimes V \to \circledcirc V$ is the two-sided ideal generated by $\mathcal{I}$. But this corresponds to such a familiar fact, namely that the symmetry group on finitely many letters is generated by adjacent transpositions, that it is rarely mentioned. The generalization of this result to the Cartan product requires proof:–

Theorem 3. *Suppose that, for all $r \geq 1$,*

$$(\circledcirc^2 V \otimes \circledcirc^r V) \cap (V \otimes \circledcirc^{r+1} V) = \circledcirc^{r+2} V,$$

where the intersection is taken inside $\bigotimes^{r+2} V$. Then the kernel of $\bigotimes V \to \circledcirc V$ is the two-sided ideal generated by $\mathcal{I}$.

Proof. By definition, $\mathcal{I}$ is the kernel of $\bigotimes^2 V \to \circledcirc^2 V$. Better is the canonical splitting

$$\bigotimes\nolimits^2 V = \circledcirc^2 V \oplus \mathcal{I}$$

obtained by decomposing $\bigotimes^2 V$ under the action of $\mathfrak{g}$. We shall deduce that

$$\bigotimes\nolimits^3 V = \circledcirc^3 V \oplus ((\mathcal{I} \otimes V) + (V \otimes \mathcal{I})).$$

More generally, we shall write

$$\bigotimes\nolimits^s V = \circledcirc^s V \oplus \mathcal{J}_s$$

and deduce that

$$\bigotimes\nolimits^{s+1} V = \circledcirc^{s+1} V \oplus ((\mathcal{I} \otimes \bigotimes\nolimits^{s-1} V) + (V \otimes \mathcal{J}_s)).$$

Then, it will be clear by induction that $\mathcal{J}_s$ is in the two-sided ideal generated by $\mathcal{I}$. Both of these deductions are special cases of the following. Suppose that (3) holds and we write

$$U \otimes V = (U \circledcirc V) \oplus \mathcal{I} \quad \text{and} \quad V \otimes W = (V \circledcirc W) \oplus \mathcal{J}.$$

Then

$$U \otimes V \otimes W = U \circledcirc V \circledcirc W \oplus ((\mathcal{I} \otimes W) + (U \otimes \mathcal{J})). \tag{11}$$

To see this, let us regard U, V, and W as representations of the compact real form G of $\mathfrak{g}$ and endow them with G-invariant Hermitian inner products (essentially this is Weyl's 'unitary trick'). Denote by P and Q, orthogonal projection onto

$$(U \circledcirc V) \otimes W \quad \text{and} \quad U \otimes (V \circledcirc W),$$

respectively. Our hypothesis (3) says that

$$U \circledcirc V \circledcirc W = \operatorname{im} P \cap \operatorname{im} Q$$

and we are required to show (11), which now reads

$$U \otimes V \otimes W = (\operatorname{im} P \cap \operatorname{im} Q) \oplus (\ker P + \ker Q).$$

This is a general property of orthogonal projections. Suppose $v \in (\operatorname{im} P \cap \operatorname{im} Q)^\perp$. We are required to show that $v \in \ker P + \ker Q$. Let $w \in \operatorname{im} P \cap \operatorname{im} Q$ and write $w = Pu$. Then

$$\langle Pv, w\rangle = \langle v, Pw\rangle = \langle v, P^2 u\rangle = \langle v, Pu\rangle = \langle v, w\rangle = 0.$$

Therefore, P preserves $(\operatorname{im} P \cap \operatorname{im} Q)^\perp$. This is also true of Q and, hence, their composition QP. Now P is strictly norm-decreasing except on its image. Similarly, for Q. Therefore QP, not only preserves $(\operatorname{im} P \cap \operatorname{im} Q)^\perp$, but is also strictly norm-decreasing there. Thus, $\mathrm{Id} - QP$ is invertible on this subspace. Therefore, we may write

$$\begin{aligned} v &= (\mathrm{Id} - QP)(\mathrm{Id} - QP)^{-1} v \\ &= \Big((\mathrm{Id} - P) + (\mathrm{Id} - Q)P\Big)(\mathrm{Id} - QP)^{-1} v \\ &= (\mathrm{Id} - P)(\mathrm{Id} - QP)^{-1} v + (\mathrm{Id} - Q)P(\mathrm{Id} - QP)^{-1} v, \end{aligned}$$

an expression evidently lying in $\ker P + \ker Q$, as required. ∎

Corollary 2. *For an arbitrary irreducible representation V of $\mathfrak{sl}(n, \mathbb{C})$, the kernel of $\bigotimes V \to \bigodot V$ is the two-sided ideal generated by $\mathcal{I}$.*

Proof. Corollary 1 ensures that the hypotheses of Theorem 3 are met. ∎

Though the procedure just described to obtain (11) from (3) is, in principle, quite feasible, in practise it can be extraordinarily complicated. The projections P and Q can be defined algebraically by Young projectors and the problem is to invert QP on $(U \circledcirc V \circledcirc W)^\perp$. To do this, we can break $(U \circledcirc V \circledcirc W)^\perp$ into irreducibles and employ Schur's lemma. But, since there can be many such irreducibles, perhaps occurring with multiplicity, this can be algebraically complicated. In simple cases, it is feasible. For example,

$$\begin{aligned} \phi_{ijk} \;=\; & \tfrac{1}{6}(\phi_{ijk} + \phi_{jki} + \phi_{kij} + \phi_{jik} + \phi_{ikj} + \phi_{kji}) \\ & + \tfrac{1}{6}(3\phi_{ijk} - \phi_{jki} + \phi_{kij} - 3\phi_{jik} + \phi_{ikj} - \phi_{kji}) \\ & + \tfrac{1}{6}(2\phi_{ijk} - 2\phi_{kij} + 2\phi_{jik} - 2\phi_{ikj}) \end{aligned}$$

explicitly describes the simplest possible decomposition

$$\bigotimes 3\mathbb{C}^n = \bigodot 3\mathbb{C}^n \oplus ((\Lambda^2\mathbb{C}^n \otimes \mathbb{C}^n) + (\mathbb{C}^n \otimes \Lambda^2\mathbb{C}^n).$$

In other cases it can be much more difficult. For example, if $V = \Lambda^2\mathbb{C}^n$ acted upon by $\mathfrak{so}(n,\mathbb{C})$, then we have already remarked that (8) is valid. The decomposition of $\bigotimes^3 V$ in this case is very complicated indeed.

According to (9) and (10), the algebra $\circledcirc V$ is graded. This is reflected in the generators of $\mathcal{I}$ being homogeneous (of degree 2). The main point of Theorem 3 is that it allows us to define some natural filtered algebras whose corresponding graded algebra may be identified as $\circledcirc V$.

Suppose, for example, we consider $\mathfrak{so}(n,\mathbb{C})$ with its defining action on $\mathbb{C}^n$ and denote by g_{ij} the invariant non-degenerate quadratic form. The Cartan algebra in this case is $\bigodot_\circ \mathbb{C}^n$, the trace-free symmetric tensors with algebra operation

$$v_{i_1 i_2\cdots i_p} w_{j_1 j_2\cdots j_q} \mapsto v_{(i_1 i_2\cdots i_p} w_{j_1 j_2\cdots j_q)} - \tfrac{pq}{n+2p+2q-4} g_{(i_1 j_1} v^k{}_{i_2\cdots i_p} w_{j_2\cdots j_q)k} + \cdots .$$

This is an example to which Theorem 3 applies:–

$$\bigodot_\circ \mathbb{C}^n = \bigotimes \mathbb{C}^n/(u \otimes v - u \odot_\circ v)$$

where

$$u \odot_\circ v = \tfrac{1}{2}(u \otimes v + v \otimes u) - \tfrac{1}{n}\langle u, v\rangle g,$$

as in (1). We could also take $\langle u, v\rangle$ and feed it into the zeroth gradation. Precisely, we may define a filtered algebra

$$\bigotimes \mathbb{C}^n/(u \otimes v - u \odot_\circ v + \langle u, v\rangle) \tag{12}$$

whose corresponding graded algebra is evidently $\bigodot_\circ \mathbb{C}^n$. The construction is similar to that of the Clifford algebra

$$\bigotimes \mathbb{C}^n/(u \otimes v - u \wedge v + \langle u, v\rangle)$$

whose corresponding graded algebra is the exterior algebra $\Lambda\mathbb{C}^n$. This is the point of view taken in [5].

Perhaps the most interesting constructions start with the adjoint representation of a semisimple Lie algebra $\mathfrak{g}$. In this case, not only is there an invariant inner product in the Killing form, but also an invariant skew form with values in $\mathfrak{g}$, namely the Lie bracket itself. This gives rise to a series of algebras:–

$$\mathcal{L}_\lambda(\mathfrak{g}) \equiv \bigotimes \mathfrak{g}/(X \otimes Y - X \circledcirc Y - \tfrac{1}{2}[X, Y] + \lambda\langle X, Y\rangle)$$

where $[\ ,\ ]$ is the Lie bracket and $\langle\ ,\ \rangle$ is the Killing form. The factor of $\frac{1}{2}$ is conventional and can be adjusted by scaling but, having fixed this, λ cannot simply be scaled away: different values of λ apparently give isomorphically distinct filtered algebras. In cases when Theorem 3 applies, certainly for $\mathfrak{sl}(n,\mathbb{C})$ or $\mathfrak{so}(n,\mathbb{C})$, the corresponding graded algebra is always $\circledcirc\,\mathfrak{g}$. The construction of $\mathcal{L}_\lambda$ is similar to that of the universal enveloping algebra

$$\mathfrak{U}(\mathfrak{g}) \equiv \bigotimes \mathfrak{g}/(X \otimes Y - X \odot Y - \tfrac{1}{2}[X, Y])$$

whose corresponding graded algebra is $\bigodot \mathfrak{g}$.

The algebra $\mathcal{L}_\lambda$ arises geometrically as follows. Let us say that a linear differential operator $\mathcal{D}$ on $\mathbb{R}$ is a symmetry of the Laplacian Δ if and only if $\Delta\mathcal{D} = \delta\Delta$ for some linear differential operator δ. Let us say that two symmetry operators are equivalent if and only if their difference is of the form $\mathcal{P}\Delta$ for some linear differential operator $\mathcal{P}$. The symmetry algebra $\mathcal{A}_n$ comprises the symmetries of Δ on $\mathbb{R}^n$ up to equivalence with algebra operation induced by composition. The main result of [4] is that $\mathcal{A}_n$ is isomorphic to a real form of $\mathcal{L}_\lambda$, specifically

$$\mathcal{A}_n \cong \mathcal{L}_{\frac{n-2}{4(n+1)}}(\mathfrak{so}(\mathfrak{n}+\mathfrak{1},\mathfrak{1})),$$

the filtering on $\mathcal{A}_n$ being induced by the degree of the symmetry.

The algebra $\mathcal{L}_\lambda(\mathfrak{g})$ is built from $\mathfrak{g}$-invariant ingredients. Also the representation $\circledcirc^k \mathfrak{g}$ occurs with multiplicity one in $\bigotimes^k \mathfrak{g}$. It follows that, as a vector space and, indeed, as a $\mathfrak{g}$-module, we have a canonical isomorphism

$$\mathcal{L}_\lambda(\mathfrak{g}) \cong \bigoplus_{s=0}^{\infty} \circledcirc^s \mathfrak{g}.$$

This is analogous to the enveloping algebra: as vector spaces $\mathfrak{U}(\mathfrak{g}) \cong \bigodot \mathfrak{g}$. It is difficult, though possible, to transfer the algebra structure on $\mathfrak{U}(\mathfrak{g})$ to $\bigodot(\mathfrak{g})$. The result is Kontsevich's $\star$-product [1, 8]. An analogous description of $\mathcal{A}_n$ has recently been given by Vasiliev [10] in terms of the Weyl (or Moyal) $\star$-product.

Presumably, a more elementary $\star$-product would arise in describing the algebra structure on (12) directly on the vector space $\bigodot_\circ \mathbb{C}^n$. As a commutative example, however, it is surely too elementary to shed much light on $\mathfrak{L}_\lambda(\mathfrak{g})$.

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