A structure sheaf on the projective spectrum of a graded fully bounded noetherian ring

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Abstract

In this note, we show how abstract localization and graded versions of the Artin-Rees property may be applied to construct structure sheaves over the projective spectrum $Proj(R)$ of a graded fully bounded noetherian ring R.

Introduction.

The main purpose of this note is to construct structure sheaves on the projective spectrum $Proj(R)$ of a graded fully bounded noetherian ring R, generalizing the analogous construction in the commutative case. The principal tool to realize this will be abstract localization (in the sense of [5, 6, 7, 16, 19, et al]) in the category $R-\mathbf{gr}$ of graded left R-modules.

Actually, if I is a graded ideal contained in $R_+ = \bigoplus_{n>0} R_n$, then for any graded left R-module M, one associates to the Zariski open subset $D_+(I) \subseteq Proj(R)$ the graded R-module of quotients $Q_I^g(M)$, obtained by localizing M with respect to the torsion at positive powers I^n of I. In general, however, this construction only yields a separated presheaf on $Proj(R)$, which is not necessarily a sheaf. Of course, one may derive from this a sheaf on $Proj(R)$, by applying the usual sheafification process. Nevertheless, this is highly unsatisfactory in the present context, since passing over to the associated sheaf enlarges the modules of global sections, which, in particular, makes the sheaf thus obtained rather useless for any representational aims, for example.

If we want to obtain a sheaf directly, we may either restrict the class of graded R-modules, we wish to represent or change the topology on $Proj(R)$ in a reasonable

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way. In this note, we show how the latter approach gives satisfactory results over graded fully bounded noetherian (FBN) rings.

The starting point of our approach is the realization that the sheaf axiom is essentially equivalent to some compatibility property between torsion and localization at radicals. In fact, if σ and τ are radicals in $R-\mathbf{gr}$, the category of graded left R-modules, then σ and τ are compatible, i.e., $\sigma Q_\tau = Q_\tau\sigma$ (where Q_τ is the localization functor associated to τ) if and only if the following sequence of functors in $R-\mathbf{gr}$ is exact:

$$0 \to Q_{\sigma\wedge\tau} \to Q_\sigma \oplus Q_\tau \to Q_{\sigma\vee\tau}.$$

On the other hand, one may show that any pair of stable radicals in $R-\mathbf{gr}$ (i.e., whose associated torsion class is closed under taking injective hulls) are compatible. Now, it is rather easy to prove that the radical in $R-\mathbf{gr}$ associated to a twosided graded ideal I is stable exactly when I has the (graded) Artin-Rees property. An obvious solution to our problem thus seems to be to restrict the Zariski topology on $Proj(R)$ to the set $\mathcal{T}_+(R)$ consisting of all open sets $D_+(I)$, with I a graded Artin-Rees ideal. Unfortunately, for arbitrary rings, $\mathcal{T}_+(R)$ fails to be a topology, in general. However, if R is a graded FBN ring, one may prove that the Artin-Rees condition is equivalent to the so-called weak Artin-Rees condition, and, using this, one easily shows that for any graded FBN ring R, the set $\mathcal{T}_+(R)$ is a topology, indeed.

This note is organized as follows. In the first two sections, we collect some necessary background and new material on graded rings and modules, as well as on localization in the category of graded left R-modules. In the next section, we concentrate on graded FBN rings and, in particular, on the relationship between stability and the (weak) Artin-Rees property. These results are used in the following section, in order to construct the announced structure sheaves on $Proj(R)$ and to calculate their stalks in the module-finite case. We conclude with some remarks about the generalization of the previous results to graded rings, which are no longer FBN, but only FBN up to torsion at R_+. It appears that our constructions may be generalized easily to this set-up. Since this requires rather technical results stemming from abstract localization theory and the theory of relative invariants, cf. [22, 23], we preferred to limit ourselves to a brief sketch of this approach and leave details to the reader.

1 Graded rings and modules.

Throughout, R will denote a positively graded ring and $R-\mathbf{gr}$ the category of graded left R-modules. If no ambiguity arises, we will usually just speak of graded R-modules. For generalities about graded rings and modules, we refer to [15]. Let us recall here, however, some of the ideas we will need in the sequel.

If $M \in R-\mathbf{gr}$ and if n is an integer, then we denote by $M(n)$ the n-th shifted version of M, i.e., the graded R-module $M(n)$ coincides with M as an ungraded R-module and has gradation given by $M(n)_p = M_{n+p}$ for any integer p. We denote by $Hom^n_R(M, N)$ the additive group of R-linear maps $u : M \to N$ of degree n, i.e., with the property that $u(m) \in N_{n+p}$ for any $m \in M_p$. Morphisms in $R-\mathbf{gr}$ are given by putting $Hom_{R-\mathbf{gr}}(M,N) = Hom_R^0(M,N)$, for any pair of graded R-modules M and N. In particular,

$$Hom_R^n(M,N) = Hom_{R-\mathbf{gr}}(M,N(n)) = Hom_{R-\mathbf{gr}}(M(-n),N),$$

for any $n \in \mathbf{Z}$. We will also write $HOM_R(M,N) = \bigoplus_{n\in\mathbf{Z}} Hom_R^n(M,N)$. It has

been proved in [21] that if $M \in R-\mathbf{gr}$ is a graded essential extension of a graded submodule $N \subseteq M$, then M is an essential extension of N in $R-\mathbf{mod}$ as well. If $M \in R-\mathbf{gr}$ and if we let $E(M)$ denote the injective hull of M in $R-\mathbf{mod}$, then, for any integer m, we define $E_m^g(M)$ to consist of all $x \in E(M)$ with the property that there exists some graded essential left ideal L of R with the property that $L_n x \subseteq M_{m+n}$ for all integers n. One may then prove that $E^g(M) = \bigoplus_{m\in\mathbb{Z}} E_m^g(M)$ is the maximal submodule of $E(M)$, with a gradation such that it contains M as a graded submodule. This in turn amounts to saying that $E^g(M)$ is (up to isomorphism) the injective hull of M in $R-\mathbf{gr}$.

Lemma 1.1 *For all $M \in R-\mathbf{gr}$ and all integers n, we have $E^g(M(n)) = E^g(M)(n)$.*

Proof : Clearly $E^g(M)(n)$ contains $M(n)$ as a graded submodule and is a graded essential extension of the latter, so $E^g(M)(n)$ injects into $E^g(M(n))$. On the other hand, if $x \in E^g(M(n))_q$, say, then we may find some graded essential left ideal L of R with the property that $L_p x \subseteq M(n)_{p+q} = M_{n+q+p}$, for all integers p. This shows that $x \in E^g(M)_{n+q} = E^g(M)(n)_q$, thus proving the assertion. ∎

We will also need the following result, whose proof may be found in [21]:

Lemma 1.2 *If $M \in R-\mathbf{gr}$, then $Ass(M) = Ass^g(M(n))$, for all integers n.*

Let us briefly recall here some generalities about the *graded rank* $rank^g(M)$ of a (finitely generated) graded left R-module. For full details, we refer to [15].

We say that $M \in R-\mathbf{gr}$ has *finite graded rank*, if its injective hull $E^g(M)$ is a finite direct sum of indecomposable graded injectives. In particular, any graded uniform left R-module M has finite graded rank, as its injective hull $E^g(M)$ is by definition indecomposable. It is actually easy to see that M has finite graded rank if and only if M is an essential extension of a finite direct sum $\bigoplus_{i=1}^n N_i$ of graded uniform left R-modules. The integer n, which is also the number of components of the decomposition of $E^g(M)$ into a direct sum of indecomposable injectives, is the *graded (uniform) rank* of M.

Amongst its main properties, let us mention that

$$rank^g(M_1 \oplus M_2) = rank^g(M_1) + rank^g(M_2),$$

for any M_1, $M_2 \in R-\mathbf{gr}$.

Note also that. if M has finite graded rank and if N is a graded left R-submodule of M, then $rank^g(N) \leq rank^g(M)$, with equality if and only if N is essential in M.

Lemma 1.3 *Every non-zero graded left R-module of finite rank has a graded uniform submodule.*

Proof : If $M \in R\mathbf{-gr}$ has finite graded rank, then its graded injective hull $E^g(M)$ is a finite direct sum $\bigoplus_{i=1}^n E_i$ of non-zero indecomposable graded submodules E_i. Clearly the graded submodules $0 \neq M_i = M \cap E_i$ of M are independent. Since each M_i is essential within the corresponding E_i, obviously $\bigoplus_{i=1}^n M_i$ is graded essential within $\bigoplus_{i=1}^n E_i = E^g(M)$, hence within M. This proves the assertion. ∎

A graded ring R is graded semisimple, if every $M \in R\mathbf{-gr}$ is semisimple, i.e., if $R\mathbf{-gr}$ is a semisimple category. This is equivalent to the existence of a direct sum decomposition $R = L_1 \oplus \ldots \oplus L_n$ of R into minimal graded left R-ideals. We say that R is graded simple if the decomposition is such that $HOM_R(L_i, L_j) \neq 0$ for every $1 \leq i, j \leq n$. It follows that for each couple i, j, there exists some integer $n_{ij} \in \mathbf{Z}$ with $L_i = L_j(n_{ij})$. If, moreover, $n_{ij} = 0$ for each choice of i and j, then R is said to be *uniformly simple*. There is a graded version of Wedderburn's Theorem: if R is graded simple (resp. uniformly simple), then there exists a graded division ring D and an n-tuple $\underline{d} \in \mathbf{Z}^n$ such that $R \cong M_n(D)(\underline{d})$ (resp. $R \cong M_n(D)$), the notation $M_n(D)(\underline{d})$ being explained in [15]. We call R a *graded (left) Goldie ring*, if

R has finite graded rank and if R satisfies the ascending chain condition on graded left annihilators. If R admits a graded semisimple artinian ring of fractions, then R is a graded Goldie ring. The converse does not hold true, in general, unless we add some extra (reasonably mild) conditions, cf. [15]:

Proposition 1.4 *Let R be a semiprime graded Goldie ring satisfying one of the following conditions:*

1. *R has a central homogeneous element of positive degree;*
2. *no minimal prime ideal of R contains $R_+ = \bigoplus_{n \geq 1} R_n$;*
3. *R has a regular homogeneous element of positive degree;*
4. *all homogeneous elements of positive degree are nilpotent.*

Then R has a graded semisimple artinian ring of fractions.

Using the property that graded left ideals in a prime graded Goldie ring are essential (in the graded sense), if and only if they contain a regular element (a fact, which may be proved in essentially the same way as its non-graded analogue), one easily gets:

Corollary 1.5 *Any prime graded Goldie ring admits a graded simple artinian ring of fractions.*

Lemma 1.6 *Let R be a semiprime graded left Goldie ring and M a graded left R-module. If M is not torsion, then M contains a graded uniform submodule, which is isomorphic to a graded left ideal of R.*

Proof : Pick an element $x \in M$, which is homogeneous and not torsion. Then the left ideal $Ann_R(x)$ is not essential in R, so there exists a non zero graded left ideal I of R with the property that $I \cap Ann_R(x) = 0$. Since R has finite rank, I contains a graded uniform ideal J. As $J \cap Ann_R(x) = 0$, it follows that J is isomorphic to Jx, so the submodule Jx of M satisfies our requirements. ∎

Lemma 1.7 *Let R be a prime graded left noetherian ring. Consider a finitely generated graded left R-module M, endowed with a chain of graded R-submodules*

$$M_0 \supset M_1 \supset \ldots \supset M_n,$$

such that all quotients are torsionfree. Then $rank^g(M) \geq n$.

Proof : Let us argue by induction, the case $n = 0$ being trivial. So, assume the statement to be valid for chains of length strictly smaller than n. ¿From the above chain, we may thus infer that $rank^g(M_1) \geq n-1$. Since M_0/M_1 is torsionfree, M_1 is not essential in M_0, so there exists a nonzero graded R-submodule N of M_0 with the property that $N \cap M_1 = 0$. We thus obtain:

$$rank^g(M) \geq rank^g(N \oplus M_1) > rank^g(M_1) \geq n-1,$$

i.e., $rank^g(M) \geq n$. ∎

2 Graded Localization

The notion of torsion and localization may be introduced in any Grothendieck category, cf. [16, 19, et al], so applies to the category $R-\mathbf{gr}$, as well. In particular, a *radical* (or: idempotent kernel functor, cf. [7]) is a left exact subfunctor σ of the identity in $R-\mathbf{gr}$, with the property that for any graded R-module M, we have $\sigma(M/\sigma M) = 0$. To any radical σ, we may associate a *torsion class* $\mathcal{T}_\sigma$ resp. a *torsionfree class* $\mathcal{F}_\sigma$, which consists of all graded R-modules M with $\sigma M = M$ resp. $\sigma M = 0$, each of which completely determines σ.

To any radical σ, we may also associate a *localization functor* Q_σ in $R-\mathbf{gr}$. This functor yields for any graded R-module M a *graded module of quotients* (with respect to σ), denoted by $Q^g_\sigma(M)$ and endowed with a canonical morphism $j_\sigma : M \to Q^g_\sigma(M)$ with σ-torsion kernel and cokernel (a so-called σ-isomorphism). The graded module of quotients $Q^g_\sigma(M)$ satisfies the universal property that any σ-isomorphism of graded R-modules $N_1 \to N_2$ induces a bijection

$$Hom_{R-\mathbf{gr}}(N_2, Q^g_\sigma(M)) \to Hom_{R-\mathbf{gr}}(N_1, Q^g_\sigma(M)).$$

If the canonical morphism $j_\sigma : M \to Q^g_\sigma(M)$ is an isomorphism, then we say that M is *σ-closed*. We denote by $(R,\sigma)-\mathbf{gr}$ the full subcategory of $R-\mathbf{gr}$ consisting of all σ-closed graded R-modules.

As in [21], let us call a radical σ in $R-\mathbf{gr}$ *rigid* if $(\sigma M)(n) = \sigma(M(n))$, for any integer n and any $M \in R-\mathbf{gr}$. For example, if τ is a graded radical in $R-\mathbf{mod}$ (i.e., for any $M \in R-\mathbf{gr}$ the torsion $\tau M \subseteq M$ is a graded left R-submodule), then τ induces a radical σ in $R-\mathbf{gr}$ in the obvious way, and it is clear that σ is rigid.

To each radical σ in $R-\mathbf{gr}$, we associate the set $\mathcal{L}^g(\sigma)$, consisting of all graded left ideals L of R with $R/L \in \mathcal{T}_\sigma$. The set $\mathcal{L}^g(\sigma)$ possesses the following properties:

1. if $L \in \mathcal{L}^g(\sigma)$ and L_1 is a graded left ideal of R such that $L \subseteq L_1$, then $L_1 \in \mathcal{L}^g(\sigma)$;

2. if $L_1, L_2 \in \mathcal{L}^g(\sigma)$, then $L_1 \cap L_2 \in \mathcal{L}^g(\sigma)$;

3. if $L \in \mathcal{L}$, then for all homogeneous $x \in R$, we have $(L : x) \in \mathcal{L}^g(\sigma)$;

4. if $L_1 \in \mathcal{L}^g(\sigma)$ and $(L : x)\ \mathcal{L}^g(\sigma)$ for all homogeneous $x \in L$, then $L \in \mathcal{L}^g(\sigma)$.

Conversely, any set of graded left ideals $\mathcal{L}$ possessing the above properties is said to be a *graded Gabriel filter*.

In general, σ is *not* uniquely determined by $\mathcal{L}^g(\sigma)$. However, one easily verifies the following sets to correspond bijectively:

1. rigid radicals in $R-\mathbf{gr}$;

2. graded Gabriel filters over R.

In general, if $M \in R-\mathbf{gr}$, then $m \in (\sigma M)_n$ if and only if there exists $I \in \mathcal{L}^g(R(-n), \sigma)$, the set of all graded left submodules L of $R(-n)$ with the property that $R(-n)/L \in \mathcal{T}_\sigma$, such that $Im = 0$. If σ is rigid, then I belongs to $\mathcal{L}^g(R(-n), \sigma)$ if and only if $I(n) \in \mathcal{L}^g(R, \sigma) = \mathcal{L}^g(\sigma)$. Indeed, $I \in \mathcal{L}^g(R(-n), \sigma)$ is equivalent to $R(-n)/I \in \mathcal{T}_\sigma$, and this just says that $R/I(n) = [R(-n)/I](n) \in \mathcal{T}_\sigma$. Let us call a radical in $R-\mathbf{gr}$

symmetric if it is rigid and if for any $L \in \mathcal{L}^g(\sigma)$, we may find some twosided graded ideal $I \in \mathcal{L}^g(\sigma)$, with $I \subseteq L$. As an example, any finitely generated twosided graded ideal I of R yields a graded radical σ_I in $R-\mathbf{mod}$, defined on any $M \in R-\mathbf{mod}$, by letting $\sigma_I M$ consist of all $m \in M$ with the property that $I^n m = 0$, for some positive integer n. The induced radical in $R-\mathbf{gr}$, denoted by σ_I^g is a symmetric radical. Other examples will be considered below. Symmetric radicals possess very nice features. For example, if σ is a symmetric radical in $R-\mathbf{gr}$, then a finitely generated $M \in R-\mathbf{gr}$ is σ-torsion if and only if $Ann_R(M) \in \mathcal{L}^g(\sigma)$. Let us denote

by $Spec^g(R)$ the set of all graded prime ideals of R. Clearly, $P \in Spec^g(R)$ if and only if $aRb \subseteq P$ for some $a, b \in h(R)$ (the set of homogeneous elements of R) implies that $a \in P$ or $b \in P$. If P is a prime ideal of R, then $< h(P) >$ is a graded prime ideal of R. In particular, the (prime) radical of a graded ideal of R is a graded ideal of R as well.

Let σ be a radical in $R-\mathbf{gr}$. Then σ defines the subset $\mathcal{K}^g(\sigma) \subseteq Spec^g(R)$ consisting of all graded prime ideals P with $\sigma(R/P) = 0$. If σ is symmetric, then $\mathcal{K}^g(\sigma)$ is closed under generization, i.e., if $P \in \mathcal{K}^g(\sigma)$ and $Q \subseteq P$ is a graded prime ideal, then $Q \in \mathcal{K}^g(\sigma)$ as well. To see this, let us verify that if $P \in Spec^g(R)$, then either $P \in \mathcal{L}^g(R, \sigma)$ or $P \in \mathcal{K}^g(\sigma)$. Indeed, if $\sigma(R/P) \neq 0$, pick $r \in R_n$ for some positive integer n, with $r \notin P$ and $\bar{r} = r + P \in \sigma(R/P)$. We may then find $L \in \mathcal{L}^g(R(-n), \sigma)$ with $L\bar{r} = \bar{0}$, i.e., $Lr \subseteq P$. We may obviously assume L to be a twosided ideal of R, as σ is assumed to be symmetric, so $L \subseteq P$ (as $r \notin P$), hence $L(n) \subseteq P$ as graded R-submodules. Since $L \in \mathcal{L}^g(R(-n), \sigma)$ implies $L(n) \in \mathcal{L}^g(R, \sigma)$, it follows that $P \in \mathcal{L}^g(R, \sigma)$. *Note :* If R is assumed to be left noetherian,

we do not really need σ to be symmetric! Indeed, if $P \in Spec^g(R) - \mathcal{K}^g(\sigma)$, then $0 \neq \sigma(R/P) \subseteq R/P$ is a graded twosided ideal of R/P. It follows that $\sigma(R/P)$ contains a regular homogeneous element $\bar{r} = r + P$, as $\sigma(R/P)$ is then a graded essential ideal of R/P. Since $P = Ann_R(\bar{r})$, it follows that $P \in \mathcal{L}^g(\sigma)$, which proves the assertion. Let us assume from now R to be a left noetherian (positively graded) ring. Consider a subset $Y \subseteq Spec^g(R)$, which is closed under generization. We define a radical σ_Y in $R-\mathbf{gr}$, by letting $M \in R-\mathbf{gr}$ be σ_Y-torsion (or belonging to $\mathcal{T}_Y = \mathcal{T}_{\sigma_Y}$) if and only if it is torsion at every $P \in Y$, i.e., if for any $m \in M_n$ $(n \in \mathbb{Z})$, we may find some $I \in \mathcal{L}^g(R(-n), \sigma^g_{R-P})$ with $Im = 0$.

Here, σ^g_{R-P} is the symmetric radical in $R-\mathbf{gr}$, with graded Gabriel filter $\mathcal{L}^g(\sigma^g_{R-P})$ consisting of all graded left ideals L of R containing RsR for some $s \in R-P$, which may then be chosen in $h(R-P)$, the set of all homogeneous elements in $R-P$, as one easily verifies.

It is thus clear that σ_Y is rigid as well. We call radicals of this type *half centered*. In particular, if $\sigma = \sigma_Y$, then $\mathcal{K}^g(\sigma) = Y$. Since R is assumed to be noetherian, we have:

Proposition 2.1 *Every symmetric radical in* $R-\mathbf{gr}$ *is half-centered.*

Proof : Let σ be a symmetric radical in $R-\mathbf{gr}$ and put

$$Y = \{P \in Spec^g(R);\ \sigma(R/P) = 0\}.$$

We claim that $\sigma = \sigma_Y$. Indeed, it is easy to see that $I \in \mathcal{L}^g(R, \sigma_Y)$ if and only if $V^g(I) \cap Y = \emptyset$, where $V^g(I)$ consists of all $P \in Spec^g(R)$ with $I \subseteq P$. We thus have to show that $I \notin \mathcal{L}^g(R, \sigma)$ if and only if $V^g(I) \cap Y \neq \emptyset$, i.e., if there exists some $P \in Spec^g(R)$ with $I \subseteq P$ and $P \notin \mathcal{L}^g(R, \sigma)$.

Let $\mathcal{G}$ denote the set of all graded ideals $K \supseteq I$, with $K \notin \mathcal{L}^g(R, \sigma)$. Clearly, $\mathcal{G}$ is a non-empty, partially ordered set. Moreover, any chain

$$\mathcal{K}:\quad K_1 \subseteq K_2 \subseteq \ldots \subseteq K_n \subseteq \ldots$$

in $\mathcal{G}$ stabilizes (as R is left noetherian), so we may apply Zorn's Lemma to infer the existence of a maximal element P in $\mathcal{G}$. As P is now easily seen to be a graded prime ideal, this proves the assertion. ∎

3 Graded FBN Rings

It has been proved in [21] that $E^g(R/P)$ is P-cotertiary for any $P \in Spec^g(R)$, the set of all graded prime ideals of R. Let $\mathcal{E}^g(R)$ denote the set of equivalence classes (up to shift and isomorphism) of indecomposable injectives in $R-\mathbf{gr}$. The previous remark then shows the map $\Phi^g : \mathcal{E}^g(R) \to Spec^g(R)$, which associates to $[E] \in \mathcal{E}^g(R)$ the graded prime ideal $Ass(E)$, is surjective.

Note also that, just as in the ungraded case, if Φ^g is bijective for R, then Φ^g is bijective for each graded epimorphic image of R. Using this, it has been proved in [21], that the following assertions are equivalent:

1. Φ^g is a bijection;
2. for all $P \in Spec^g(R)$, the ring R/P is graded left bounded, i.e., every graded left essential ideal of R/P contains a non zero twosided graded ideal;

3. every cotertiary graded left R-module M is isotypic in $R-\mathbf{gr}$, i.e., $E^g(M)$ is a direct sum of equivalent indecomposable injectives in $R-\mathbf{gr}$.

We call a ring satisfying these conditions *graded left fully bounded noetherian* (or: graded left FBN). Graded FBN rings possess very nice features. Let us already mention here, for future reference, the following results, whose proof may be found in [21]:

Proposition 3.1 *Let R be a graded left noetherian ring. The following assertions are equivalent:*

1. *R is graded left fully bounded noetherian;*
2. *every rigid radical in $R-\mathbf{gr}$ is symmetric.*

Proposition 3.2 *[21] Let R be graded ring, let $M \in R-\mathbf{gr}$ and let $P \in Ass(M)$, then:*

1. *P is a graded ideal of R;*
2. *there exists some nonzero homogeneous $x \in M$, with $P = Ann_R(Rx)$;*
3. *if R is left fully bounded, then there exists some nonzero homogeneous $x \in M$ with $P = Ann_R(Rx) = Ann_R(M')$ for all nonzero $M' \subseteq Rx$.*

Lemma 3.3 *Let R be a graded left FBN ring and M a finitely generated graded left R-module. Then there exists a chain of graded submodules*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M,$$

with the property that for any $1 \leq i \leq n$:

1. *$P_i = Ann_R(M_i/M_{i-1}) \in Spec^g(R)$;*
2. *M_i/M_{i-1} is isomorphic to a graded uniform left ideal of R/P_i.*

Proof : Since M is graded left noetherian, we easily reduce the proof to showing that M contains a non zero graded submodule N of M with the property that $Ann_R(N) \in Spec^g(R)$ and that N is isomorphic to a graded uniform submodule of $R/Ann_R(N)$.

Consider a graded uniform submodule M' of M. Let $P = ass^g(M')$ and put $M'' = Ann_{M'}(P)$. Then $P \in Spec^g(R)$ and M'' is a fully faithful graded left R/P-module, with $P = Ann_R(M'')$. As M'' is finitely generated and uniform, it is torsionfree as a left R/P-module. So, it contains a graded submodule N, which is isomorphic to a graded uniform left ideal of R/P. Finally, since M'' is fully faithful, $Ann_R(N) = P$. ∎

Proposition 3.4 *The following assertions are equivalent:*

1. *R is a graded FBN ring;*

2. *for any graded left ideal L of R, there exist homogeneous elements $x_1, \ldots, x_n$ of R such that*

$$b_g(L) = \bigcap_{i=1}^{n} (L : x_i),$$

where $b_g(L)$ is the graded bound of L, i.e., the largest graded twosided ideal contained in L;

3. *any finitely generated graded left R-module M is finitely annihilated, i.e., there exist homogeneous elements $m_1, \ldots, m_n$ in M such that*

$$Ann_R(M) = \bigcap_{i=1}^{n} Ann_R(m_i).$$

Proof : The equivalence of (2) and (3) is fairly obvious, and the fact that (2) implies (1) has been proved in [24]. Let us conclude by showing that (1) implies (3).

Assume the statement is false and pick a graded left R-submodule $N \subseteq M$, which is maximal with the property that the statement fails for M/N. Up to replacing M by M/N, we may thus assume that $Ann_R(M)$ is not the annihilator of a finite subset, but that the annihilator of any proper quotient of M is. Replacing R by $R/Ann_R(M)$, we may also assume M to be faithful. ¿From 3.3, it follows that M contains a graded R-submodule U, which is isomorphic to a graded uniform left ideal L of R/P for some $P \in Spec^g(R)$. Due to the minimality of M, there exist $x_1, \ldots, x_k \in M$, such that

$$Ann_R(x_1 + U, \ldots, x_k + U) = Ann_R(M/U).$$

Let $I = Ann_R(x_1, \ldots, x_k)$, then $IM \subseteq U$, so $PIM = 0$. This implies that $PI = 0$, as M is faithful, so I is a finitely generated R/P-module. On the other hand, I strictly contains $Ann_R(M)$, as the statement is assumed to fails for M. So, there exists $x_{k+1} \in M$, with $Ix_{k+1} \neq 0$.

Let $I_1 = Ann_R(x_1, \ldots, x_{k+1}) = I \cap Ann_R(x_{k+1})$. Clearly,

$$I/I_1 \cong Ix_{k+1} \subseteq IM \subseteq U,$$

so I/I_1 is isomorphic to a nonzero graded ideal of R/P, i.e., I/I_1 is torsionfree.

We may thus pick $x_{k+2} \in M$, such that, with $I_2 = I_1 \cap Ann_R(x_{k+2})$, the nonzero quotient I_1/I_2 is torsionfree. Iterating this process yields an infinite chain

$$I \supset I_1 \supset I_2 \supset \ldots$$

such that each graded quotient I_{i-1}/I_i is torsionfree. This contradicts 1.7, however.

■

Note : The previous result, which may be viewed as a graded version of Gabriel's condition (H), also says that $R/Ann_R(M)$ embeds into a finite direct sum $\bigoplus_{i=1}^t M(d_i)$. Indeed, if $Ann_R(M) = Ann_R(m_1, \ldots, m_t)$, then we may obviously choose the m_i to be homogeneous (up to changing their number!). Consider the linear map

$$\phi : R \to \bigoplus_{i=1}^{t} M(d_i) : r \mapsto r(m_1, \ldots, m_t),$$

where $d_i = deg(m_i)$. Since $Ker(\phi) = Ann_R(m_1, \ldots, m_t) = Ann_R(M)$, this shows that ϕ induces an embedding

$$R/Ann_R(M) \hookrightarrow \bigoplus_{i=1}^{t} M(d_i).$$

The following result generalizes a similar statement in the non-graded case, proved in [1]:

Proposition 3.5 *Let σ be a rigid radical in $R-$**gr** and M a graded left R-module. Assume every finitely generated graded left R-submodule of M to be finitely annihilated. If for every $P \in \mathcal{K}^g(\sigma)$ and any $Q \in \mathcal{Z}^g(\sigma)$ there exists some $D \in \mathcal{L}^g(\sigma)$ with $PD \subseteq QP$, then σM is essentially closed in M.*

Proof : We may clearly assume M to be cyclic. So, suppose there is a graded essential extension N of σM in M, such that N is not σ-torsion (so σM is *not* essentially closed in M). Then $Ann_R(N) \notin \mathcal{L}^g(\sigma)$. Choose I maximal in

$$\mathcal{S} = \{Ann_R(N_1);\ N_1 \subseteq N,\ Ann_R(N_1) \notin \mathcal{L}^g(\sigma)\},$$

say $I = Ann_R(X)$. We will show below that I is a (graded) prime ideal. This will then finish the proof. Indeed, since every graded left R-submodule of M is finitely annihilated, there exists an embedding

$$f : R/I \hookrightarrow \bigoplus_{i=1}^{n} X(d_i),$$

if $I = Ann_R(X) = Ann_R(x_1, \ldots, x_n)$, with x_i homogeneous of degree d_i, say. As $\sigma M \subseteq N$ is essential, so is $\sigma X = \sigma M \cap X \subseteq X$ and hence also $\bigoplus_i (\sigma X)(d_i) \subseteq \bigoplus_i X(d_i)$, as σ is assumed to be rigid. This shows that R/I (identified with $f(R/I) \subseteq \bigoplus_i X(d_i)$) has a nonzero σ-torsion left R-submodule, i.e., $I \notin \mathcal{K}^g(\sigma)$. As this implies $I \in \mathcal{Z}^g(\sigma)$, this contradicts the choice of I.

So, let us now show that I is prime, to finish the proof. Suppose this is not the case. As R is graded left noetherian, there exists a maximal right annihilator ideal P of R/I, say $P = Ann(J/I)_R$ with $J \supset I$. It is easy to see that P is prime. Since $JPX \subseteq IX = 0$, with $PX \neq 0$ (as $P \supset I$), we may assume $J = Ann_R(PX)$. Moreover, $J \in \mathcal{L}^g(\sigma)$, as $J \supset I$ and I is assumed to be maximal in $\mathcal{S}$. It follows that $P \in \mathcal{K}^g(\sigma)$. Indeed, otherwise $P \in \mathcal{Z}^g(\sigma) \subseteq \mathcal{L}^g(\sigma)$, hence also $JP \in \mathcal{L}^g(\sigma)$. This yields that $I \in \mathcal{L}^g(\sigma)$, as $JP \subseteq I$ – a contradiction.

Since R is a graded left noetherian ring, J contains a product of prime ideals, each of which contains J, i.e., there exist $Q_1, \ldots, Q_n \in \mathcal{Z}^g(\sigma)$ with $Q_1 \ldots Q_n PX = 0$. Our assumption implies the existence of $D_1, \ldots, D_n \in \mathcal{L}^g(\sigma)$ with

$$P(D_1 D_2 \ldots D_n) \subseteq Q_1 P(D_2 \ldots D_n) \subseteq \ldots \subseteq Q_1 \ldots Q_n P \subseteq I.$$

Let $D_1 \dots D_n = D \in \mathcal{L}^g(\sigma)$, then $PD \subseteq JP$. If $DX = 0$, then $D \subseteq I$, implying $I \in \mathcal{L}^g(\sigma)$ – a contradiction. So, $C = Ann_R(DX)$ is a proper graded ideal of R, with $C \supseteq P \supset I$. But this implies that $C \in \mathcal{L}^g(\sigma)$, hence also that $CD \in \mathcal{L}^g(\sigma)$ – another contradiction. So I is a prime ideal indeed, which proves the assertion. ■

Corollary 3.6 *Let R be a graded left FBN ring and let σ be a rigid radical in $R-\mathbf{gr}$. The following assertions are equivalent:*

1. *σ is stable;*
2. *for every finitely generated graded left R-module M, every graded submodule $N \subseteq M$ and any $I \in \mathcal{L}^g(\sigma)$, there exists $J \in \mathcal{L}^g(\sigma)$ with $JM \cap N \subseteq IN$ ("the Artin-Rees property");*
3. *for every graded R-ideal K and any $I \in \mathcal{L}^g(\sigma)$, there exists $J \in \mathcal{L}^g(\sigma)$ with $KJ \subseteq IK$ ("the weak Artin-Rees property");*
4. *for every $P \in \mathcal{K}^g(\sigma)$ and $Q \in \mathcal{Z}^g(\sigma)$, there exists some $I \in \mathcal{L}^2(\sigma)$ with $PI \subseteq QP$.*

Proof : It is clear that (2) implies (3) and that (3) implies (4), whereas the previous result proves that (4) implies (1). To conclude, let us prove that (1) implies (2). Choose X to be a graded left R-submodule of M, which is maximal with respect to the property $X \cap N = IN$. As $N + X/X = N/X \cap N = N/IN$ and $I \in \mathcal{L}^g(\sigma)$, clearly $N + X/X \in \mathcal{T}_\sigma$. Since $N + X/X \subseteq M/X$ is essential, also $M/X \in \mathcal{T}_\sigma$. But then, $J = Ann_R(M/X) \in \mathcal{L}(\sigma)$, as M/X is finitely annihilated (by the FBN assumption!). So, $JM \cap N \subseteq X \cap N = IN$. ■

We say that a graded twosided ideal I of R satisfies the graded left Artin-Rees condition resp. the graded weak left Artin-Rees condition, if for any graded left ideal L of R, we may find a positive integer n with $I^n \cap L \subseteq IL$ resp. if for any graded twosided ideal L of R we may find a positive integer n with $LI^n \subseteq IL$. It is then clear that I has the graded (weak) left Artin-Rees property if and only if the radical σ_I^g in $R-\mathbf{gr}$ has the (weak) Artin-Rees property. The previous result thus states that the graded left Artin-Rees property and the graded weak left Artin-Rees property are equivalent for graded twosided ideals of a graded left FBN ring.

4 Structure Sheaves

We continue using notations as in the previous section. For simplicity's sake, we assume R to be a graded (left and right) FBN ring. We denote by R_+ the graded twosided R-ideal $\bigoplus_{n>0} R_n$, and we assume R_+ to be a graded Artin-Rees ideal (i.e., possessing the left and right Artin-Rees property).

The projective spectrum $Proj(R)$ of R is defined to consist of all $P \in Spec^g(R)$ with the property that $R_+ \not\subset P$. The Zariski topology $\mathcal{T}_+^{zar}(R)$ on $Proj(R)$ consists of all subsets $D_+(I)$ of $Proj(R)$, where I is a graded ideal of R, and where

$$D_+(I) = \{P \in Proj(R);\ I \not\subset P\}.$$

That $\mathcal{T}_+^{zar}(R)$ is a topology on $Proj(R)$, indeed, follows from the fact that for graded ideals I, J and $\{I_\alpha;\ \alpha \in A\}$ of R, we have

$$D_+(I) \cap D_+(J) = D_+(IJ),$$

resp.

$$\bigcup_{\alpha\in A} D_+(I_\alpha) = D_+(\sum_{\alpha\in A} I_\alpha).$$

Note also that, up to replacing I by IR_+ or $I \cap R_+$, we may always assume the graded ideals I used to define the open subsets in the Zariski topology on $Proj(R)$, to be contained in R_+. Clearly, any $D_+(I) \subseteq Spec^g(R)$ is closed under generization, so, as in 2, we may define the radical $\tau_I = \sigma_{D_+(I)}$ in $R-\mathbf{gr}$ and clearly $D_+(I) = \mathcal{K}^g(\tau_I)$. It is also obvious that $\tau_I = \sigma^g_{R_+} \vee \sigma^g_I$.

These radicals τ_I are in bijective correspondence with the open subsets $D_+(I)$ in $\mathcal{T}_+^{zar}(R)$. Actually, one easily verifies that $D_+(I) \subseteq D_+(J)$ if and only if $\tau_I \geq \tau_J$. In particular, it follows for all graded R-ideals I, J and $\{I_\alpha;\ \alpha \in A\}$, that $\tau_I \vee \tau_J = \tau_{IJ}$ resp. $\bigwedge_{\alpha\in A} \tau_{I_\alpha} = \tau_{\sum_{\alpha\in A} I_\alpha}$. For any $M \in R-\mathbf{gr}$, one may define a presheaf of graded R-modules on $Proj(R)$, by associating to any $D_+(I) \in \mathcal{T}_+^{zar}(R)$ the graded R-module of quotients $Q_I^+(M)$ of M at τ_I. Note that if $I \subseteq R_+$, then clearly $Q_I^+(M) = Q_I^g(M)$, for any $M \in R-\mathbf{gr}$, where Q_I^g is the usual localization functor at the radical σ_I^g induced in $R-\mathbf{gr}$ by the graded radical σ_I.

It is rather easy to see that this presheaf is separated. Indeed, if $D_+(I) = \bigcup_{\alpha\in A} D_+(I_\alpha)$ for some family of graded R-ideals $\{I_\alpha; \alpha \in A\}$, then we have to show that the map

$$\rho : Q_I^g(M) \to \prod_{\alpha\in A} Q^g_{I_\alpha}(M)$$

is injective. Now, for any $\alpha \in A$, there is an exact sequence of graded left R-modules

$$0 \to \sigma^g_{I_\alpha} Q_I^g(M) \to Q_I^g(M) \overset{\rho_\alpha}{\to} Q^g_{I_\alpha}(M).$$

So, we obtain that $q \in Ker(\rho)$ if and only if

$$q \in \bigcap_{\alpha\in A} Ker(\rho_\alpha) = \bigcap_{\alpha\in A} \sigma^g_{I_\alpha} Q_I^g(M) = (\bigwedge_{\alpha\in A} \sigma^g_{I_\alpha}) Q_I^g(M) = \sigma_I^g Q_I^g(M) = 0,$$

which proves our claim.

Unfortunately, in general, this presheaf is not a sheaf, however (it is, for example, if $M = R$ and R is prime). In order to remedy this, one may either restrict the class of graded R-modules, we wish to consider or, and this is our approach, restrict the topology on $Proj(R)$. Of course, the topology chosen on $Proj(R)$ should then still remain sufficiently fine for, otherwise, the stalks of the sheaves we construct will bear little local information and therefore be rather useless. Denote by $\mathcal{T}_+(R)$

the set of all $D_+(I) \in \mathcal{T}_+^{zar}(R)$, with the property that both τ_I and its analogue in the category of graded right R-modules satify the weak Artin-Rees property. Of course, this just says that $R_+ \cap I$ (or, equivalently, IR_+ or R_+I) is a graded weak Artin-Rees ideal. Note also that this is true, if I is a graded weak Artin-Rees ideal, due to our assumption on R_+.

On the other hand, since R is graded FBN, all this is equivalent to I and its right analogue satisfying the Artin-Rees property or of being stable. The reason why we gave the definition of $\mathcal{T}_+(R)$ in terms of the weak Artin-Rees property and not the (more natural) ordinary Artin-Rees condition, is that this *always* yields a topology on $Proj(R)$, even if R is not fully bounded.

Indeed, one easily reduces to verifying that for any graded ideals I, J and $\{I_\alpha;\ \alpha \in A\}$ satisfying the graded weak left Artin-Rees property, so do IJ and $\sum_{\alpha\in A} I_\alpha$ (and similarly on the right).

Consider a graded twosided ideal K of R. Since I and J are assumed to satisfy the weak left Artin-Rees condition, we may find positive integers p and q such that $KI^p \subseteq IK$ and $KJ^q \subseteq JK$. It is clear that $rad(I^pJ^q) = rad(IJ)$, so there exists some positive integer n, such that $(IJ)^n \subseteq I^pJ^q$. We thus find that

$$K(IJ)^n \subseteq KI^pJ^q \subseteq IKJ^q \subseteq IJK,$$

proving that IJ satisfies the weak left Artin-Rees condition as well.

On the other hand, put $\sum_{\alpha\in A} I_\alpha = I$, then for any $\alpha \in A$, we may find some positive integer $n(\alpha)$ with $KI_\alpha^{n(\alpha)} \subseteq I_\alpha K \subseteq IK$. Since $rad(\sum_\alpha I_\alpha^{n(\alpha)}) = rad(I)$, there exists some positive integer n with $I^n \subseteq \sum_\alpha I_\alpha^{n(\alpha)}$. Hence,

$$KI^n \subseteq K(\sum_{\alpha\in A} I_\alpha^{n(\alpha)}) \subseteq IK,$$

which proves that $I = \sum_{\alpha\in A} I_\alpha$ satisfies the weak left Artin-Rees condition. Recall

from [2, 18, 25] that two radicals σ and τ in a Grothendieck category $\mathcal{C}$ are said to be *compatible*, if $\sigma Q_\tau = Q_\tau\sigma$, where Q_τ is the localization functor in $\mathcal{C}$, associated to τ. Since this is equivalent to $\tau Q_\sigma = Q_\sigma\tau$, cf. [27], this notion is symmetric in σ and τ.

It has been proved in [14] that σ and τ are compatible, if and only if the canonical sequence of functors

$$0 \to Q_{\sigma\wedge\tau} \to Q_\sigma \oplus Q_\tau \to Q_{\sigma\vee\tau}$$

is exact. Moreover, if σ and τ are stable, then they are compatible. In this case, one may even show that the localization functors Q_σ and Q_τ commute, cf. [2, 25].

Applying this to $\mathcal{C} = R-\mathbf{gr}$, it thus follows for any pair of graded ideals I, J with the weak Artin-Rees property and any $M \in R-\mathbf{gr}$, that we have an exact sequence

$$0 \to Q_{I+J}^+(M) \to Q_I^+(M) \oplus Q_J^+(M) \to Q_{IJ}^+(M)$$

where $Q_I^+ = Q_{R_+}^g Q_I^g = Q_I^g Q_{R_+}^g$ (and similarly for the other terms), by the above remarks.

As before, associating $Q_I^+(M)$ to $D_+(I) \in \mathcal{T}_+(R)$ defines a separated presheaf $\mathcal{O}_M^+$ on $(Proj(R), \mathcal{T}_+(R))$. The next result shows that $\mathcal{O}_M^+$ is actually a sheaf, for any graded left R-module M:

Lemma 4.1 *Let P be a separated presheaf on a topological space X. Assume that every open subset of X is quasicompact and that P satisfies the sheaf axiom for coverings consisting of two open subsets. Then P is a sheaf.*

Proof : Let us first consider finite coverings and argue by induction, i.e., let us assume that P satisfies the sheaf axiom for coverings consisting of $n-1$ open subsets, for some positive integer $n > 2$, and let us show that it holds for open coverings by n open subsets. Since the sheaf axiom is assumed to hold for coverings consisting of two open subsets, this will prove that P will satisfy the sheaf axiom for finite coverings.

Let $U = U_1 \cup \ldots \cup U_n$ be an open covering of U and assume we are given $s_i \in P(U_i)$ with the property that $s_i|U_{ij} = s_j|U_{ij}$, for any $1 \le i, j \le n$, where $U_{ij} = U_i \cap U_j$. Put $V = U_2 \cup \ldots \cup U_n$. By induction, we know that there exists a (unique) $s' \in P(V)$ with $s'|U_i = s_i$ for $1 \le i \le n$. Let us write $p = s'|U_1 \cap V$ resp. $q = s_1|U_1 \cap V$, then we claim that $p = q$. Indeed, for any $2 \le i \le n$, we have

$$p|U_{1i} = s'|U_{1i} = s_i|U_{1i} = s_1|U_{1i} = q|U_{1i}.$$

Since $U_1 \cap V = \bigcup_{i=2}^n U_{1i}$ and since P is assumed to be separated, this proves our claim. So, there exists $s \in P(U)$ with $s|U_1 = s_1$ and $s|V = s'$, and since this implies that $s'|U_i = s_i$, for any $1 \le i \le n$, this shows that P satisfies the sheaf axiom for any finite covering.

Let us now consider an arbitrary covering $\{U_\alpha;\ \alpha \in A\}$ of an open subset U of X. Since U is assumed to be quasicompact, there exist indices $\alpha(1), \ldots, \alpha(n) \in A$ with $\bigcup_{i=1}^n U_{\alpha(i)} = U$. Assume we are given $s_\alpha \in P(U_\alpha)$ for each $\alpha \in A$, with $s_\alpha|U_{\alpha\beta} = s_\beta|U_{\alpha\beta}$, for any $\alpha, \beta \in A$. Since the sheaf axiom holds for finite coverings, by the first part of the proof, we may find a unique $s \in P(U)$ with $s|U_{\alpha(i)} = s_{\alpha(i)}$ for $1 \le i \le n$. Let us now take an arbitrary $\alpha \in A$ and write $s'_\alpha = s|U_\alpha$. We claim that $s'_\alpha = s_\alpha$, which then proves the assertion. Indeed, for any $1 \le i \le n$, we have

$$s'_\alpha|U_{\alpha,\alpha(i)} = s|U_{\alpha,\alpha(i)} = s_{\alpha(i)}|U_{\alpha,\alpha(i)} = s_\alpha|U_{\alpha,\alpha(i)},$$

hence $s'_\alpha = s_\alpha$, since P is assumed to be separated. This finishes the proof. ∎

The structure sheaf on $Proj(R)$ is now defined to be $\mathcal{O}_R = (\mathcal{O}_R^+)_0$, i.e., for each $D_+(I) \in \mathcal{T}_+(R)$, we put

$$\Gamma(D_+(I), \mathcal{O}_R) = Q_I^+(R)_0 = Q_{R_+}^g(Q_I^g(R))_0,$$

(or $Q_I^g(R)_0$, if I is chosen within R_+). For any $M \in R-\mathbf{gr}$, the sheaf of left $\mathcal{O}_R$-modules is defined similarly.

Put

$$\Gamma_*(D_+(I), \mathcal{E}) = \bigoplus_n \Gamma(D_+(I), \mathcal{O}_{R(n)} \otimes \mathcal{E}),$$

for any sheaf of left $\mathcal{O}_R$-modules $\mathcal{E}$. Clearly, for any integer n, we have $\mathcal{O}_{R(n)} \otimes \mathcal{O}_M = \mathcal{O}_{M(n)}$. So, since taking sections commutes with direct sums (as open subsets of $Proj(R)$ are quasicompact), we obtain

$$\Gamma_*(D_+(I), \mathcal{O}_M) = \Gamma(D_+(I), \bigoplus_n \mathcal{O}_{M(n)}) = \Gamma(D_+(I), \mathcal{O}_M^+) = Q_I^+(M).$$

Note also that locally, on T-sets (cf. [20]), $(Proj(R), \mathcal{O}_R)$ is affine, i.e., isomorphic to an affine scheme $(Spec(S), \mathcal{O}_S)$, as defined in [2]. This shows that the above construction allows to further develop projective algebraic geometry for graded FBN rings, along the lines of the affine approach in [2, 20, et al]. Let us assume from

now on R to be module finite over its noetherian center $C = Z(R)$. Note that this implies R to be FBN, in the usual, ungraded sense. Note also that C is graded itself, and that for every $P \in Spec^g(R)$, the intersection $\mathfrak{p} = P \cap C$ belongs to $Spec^g(C)$.

Denote by $Cl(P)$ the clique of P, cf. [8, 11, 12], and define

$$\sigma_{Cl(P)} = \bigwedge_{Q \in Cl(P)} \sigma_{R-Q}.$$

In particular, it follows that a twosided ideal I of R belongs to $\mathcal{L}^2(\sigma_{Cl(P)})$ if and only if $I \not\subset Q$, for all $Q \in Cl(P)$.

It has been proved in [3] that $\sigma_{Cl(P)} = \overline{\sigma_{C-\mathfrak{p}}}$, where the latter radical is induced in $R-$**mod** by the radical $\sigma_{C-\mathfrak{p}}$ in $C-$**mod**. Since $\mathfrak{p}$ is a graded prime ideal of C, clearly $\sigma_{C-\mathfrak{p}}$ is a graded radical, hence so is $\overline{\sigma_{C-\mathfrak{p}}}$. Let us now calculate the stalks of the sheaves $\mathcal{O}_M^+$:

Proposition 4.2 *Let $P \in Proj(R)$ and put $\mathfrak{p} = P \cap C$. Then, for any $M \in R-$**gr**, we have:*

$$\mathcal{O}_{M,P}^+ = M_{\mathfrak{p}}^g,$$

where $(-)_{\mathfrak{p}}^g$ denotes the usual graded localization at $\mathfrak{p}$.

Proof : Let us first note that for any $P \in Spec^g(R)$, we have $Cl(P) \subseteq Spec^g(R)$. Indeed, if Q is a prime ideal of R, then, from [8, 11.20], we know that $Q \in Cl(P)$ if and only if $Q \cap C = P \cap C = \mathfrak{p}$. Denote by Q_g the largest graded R-ideal contained in Q, then it is clear that $Q_g \in Spec^g(R)$. Moreover, $Q_g \cap C = \mathfrak{p} = Q \cap C$, as $\mathfrak{p}$ is a graded prime ideal of C. Since $Q_g \subseteq Q$, "incomparability" shows that $Q_g = Q$, i.e., $Q \in Spec^g(R)$, indeed.

On the other hand, from the above remarks, it follows that

$$Q_{Cl(P)}(M) = M_{\mathfrak{p}}^g = \varinjlim M_f^g,$$

where $Q_{Cl(P)}$ is the localization at $\sigma_{Cl(P)}$ and where f runs through the homogeneous elements of C, not contained in $\mathfrak{p}$, and where $(-)_f^g$ is the graded localization at f.

Let us now pick $D_+(I) \in \mathcal{T}_+(R)$, and assume that $P \in Proj(R)$ belongs to $D_+(I)$. We claim that $Q \in D_+(I)$, for any $Q \in Cl(P)$. Indeed, since all $Q \in Spec^g(R)$, we just have to show that if $P' \rightsquigarrow P''$, for graded prime ideals P' and P'' then $I \subseteq P'$ if and only if $I \subseteq P''$, if I satisfies the graded weak Artin-Rees condition. This may be proven exactly as in the ungraded case. Actually, suppose that the link

$P' \rightsquigarrow P''$ is given by a twosided ideal K of R, i.e., assume $P'P'' \subseteq K \subset P' \cap P''$, with $P' = Ann^{\ell}_R((P' \cap P'')/K)$ and $P'' = Ann^r_R((P' \cap P'')/K)$. If $I \subseteq P'$, then there exists a positive integer n with

$$(P' \cap P'')I^n \subseteq I(P' \cap P'') \subseteq P'(P' \cap P'') \subseteq P'P'' \subseteq K.$$

It follows that $I^n \subseteq Ann^r_R((P' \cap P'')/K) = P''$, hence $I \subseteq P''$. The other implication may be proved similarly.

It thus follows that there is a map

$$Q^g_I(M) \to Q^g_{Cl(P)}(M),$$

for any $D_+(I) \in \mathcal{T}_+(R)$ containing P. Passing over the limit of all of these, we obtain a map

$$\mathcal{O}^+_{M,P} \to Q^g_{Cl(P)}(M) = M^g_{\mathfrak{p}}.$$

Since for every $f \in h(C - \mathfrak{p})$, we have $D_+(Rf) \in \mathcal{T}_+(R)$, clearly this map is surjective. On the other hand, to prove injectivity (and thus finishing the proof), let $D_+(I) \in \mathcal{T}_+(R)$ and $P \in D_+(I)$, then we have an exact sequence

$$0 \to \sigma_{C-\mathfrak{p}} Q^+_I(M) \to Q^+_I(M) \to M^g_{\mathfrak{p}}$$

Pick $q \in \sigma_{C-\mathfrak{p}} Q^+_I(M)$, then $fq = 0$ for some $f \in h(C - \mathfrak{p})$. Let $J = Rf$ and denote by q' the image of q in $Q^+_{IJ}(M)$, then $IJq' = 0$, hence $q' = 0$, as $Q^+_{IJ}(M)^+$ is τ_{IJ}-torsionfree. Since both I and IJ satisfy the graded Artin-Rees property, it thus follows that

$$\varinjlim \sigma_{C-\mathfrak{p}} Q^+_I(M) = \sigma_{C-\mathfrak{p}} \varinjlim Q^+_I(M) = 0,$$

which proves our claim. ■

Note : Applying the techniques developed in [26, 27], the above constructions may be generalized to a much wider class of rings. Indeed, let us denote by $R-$**proj** the quotient category of $R-$**gr** with respect to torsion at R_+, i.e., $R-\mathbf{proj} = (R, \sigma^+_{R_+})-\mathbf{gr}$. If R is commutative, then it is well-known, that $R-$**proj** is equivalent to the category of quasicoherent sheaves on $Proj(R)$.

We assume R to be a noetherian object in $R-$**proj** (this is sufficient to imply every Zariski open subset of $Proj(R)$ to be quasicompact) and R_+ to satisfy the weak Artin-Rees condition. We then define the graded ring R to be left FBN over $Proj(R)$, if for every $P \in Proj(R)$, any essential graded left R-submodule $L \subseteq Q_{R_+}(R/P)$ is $R-$**proj** contains a nonzero twosided $I \in R-$**proj**. One may show that our assumptions imply this to be equivalent to the following relative version of Gabriel's condition (H): if M is torsionfree and finitely generated at R_+ (i.e., if $Q^g_{R_+}(M)$ is a finitely generated object in $R-$**proj**), then M is finitely annihilated. If the analogous property on the right is also valid, then we say that R is FBN over $Proj(R)$.

Let us say that a graded twosided ideal I of R satisfies the graded (left) Artin-Rees condition resp. the graded weak (left) Artin-Rees condition over $Proj(R)$, if for every twosided ideal K of R, we may find a positive integer n such that $I^n \cap K \subseteq Q_{R_+}(IK)$ resp. $KI^n \subseteq Q_{R_+}(IK)$. Adapting the techniques in [26, 27] to the present, graded situation, one may show that these two notions are equivalent for any graded twosided R-ideal I containing R_+, whenever R is left FBN over $Proj(R)$. Using this, it is rather easy to see that essentially the same method as before allows us to construct for any $M \in R-\mathbf{gr}$ a structure sheaf $\mathcal{O}_M^+$ on $(Proj(R), \mathcal{T}_+(R))$, with similar properties as before, for any graded ring R, which is left FBN over $Proj(R)$. We leave details to the reader.

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