Flag-transitive extensions of dual projective spaces

Cécile Huybrechts[*†] Antonio Pasini

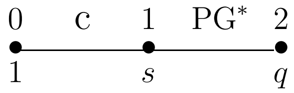
Abstract

We classify the flag–transitive circular extensions of line–point systems of finite projective geometries.

1 Introduction

We consider geometries belonging to the following diagram of rank 3, where $0, 1, 2$ are the types, q, s are finite orders with $q > 1$ and $s + 1 = (q^n - 1)/(q - 1)$ for some integer $n > 1$, the label c denotes the class of circular spaces and PG* stands for the class of dual projective spaces, namely geometries of lines and points of a projective geometry.

(c.PG*)



We call these geometries c.PG*–*geometries*. Given a c.PG*–geometry Γ with orders $1, s, q$ as above, we call q the *order* of Γ. As $s + 1 = (q^n - 1)/(q - 1)$, the residues of the elements of Γ of type 0 are dual n–dimensional projective spaces of order q. We call n the *residual dimension* of Γ.

A c.PG*–geometry of residual dimension 2 is a finite extended projective plane. It is well–known that just two finite extended projective planes exist, namely AG(3, 2)

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and the Witt design $S(22,6,3)$ for the Mathieu group M_{22} (Hughes [12]). Thus, we only consider c.PG*–geometries of residual dimension $n > 2$ in this paper. As $n > 2$, the order q is a prime power and the residues of the elements of type 0 are isomorphic to the dual point–line system of $\mathrm{PG}(n,q)$.

In the next section we shall describe a flag–transitive c.PG*–geometry of order 2 and residual dimension n, for any $n > 2$. We call that geometry Γ_n. It is a subgeometry of the D_{n+1}–building over GF(2) and it is related to the alternating form graph. One more flag–transitive example arises from the D_4–building over GF(2) (see §2.2). It has order 2 and residual dimension 3. We denote it Γ'_3. In this paper we prove the following:

Theorem 1 *The geometry* Γ_n *is the unique flag–transitive* c.PG*–*geometry of residual dimension* $n > 3$ *and there are just two flag–transitive* c.PG*–*geometries of residual dimension* 3, *namely* Γ_3 *and* Γ'_3.

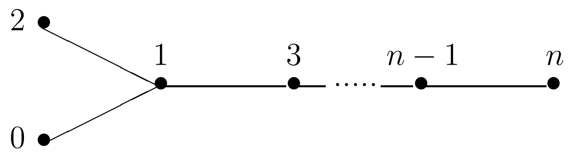
The paper is organized as follows. In Section 2 we decribe the flag–transitive examples and also some non flag–transitive ones. Section 3 is devoted to the proof of Theorem 1.

It will be useful for the forthcoming descriptions to have stated some terminology. Given a c.PG*–geometry Γ, the elements of Γ of type 0, 1, 2 are called *points*, *lines* and *planes*, respectively. We say that two distinct points are *collinear* when there is a line incident with both of them. The *collinearity graph* of Γ is the graph with the points of Γ as vertices and the collinearity relation as the adjacency relation.

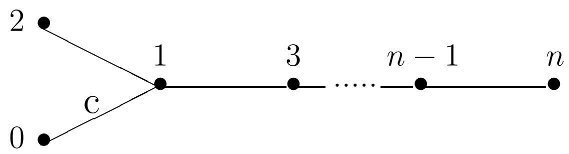
2 The known examples

2.1 The geometry Γ_n

Let Δ_{n+1} be the building of type D_{n+1} over GF(2), with $n > 2$. Having marked the nodes of the D_{n+1}–diagram as follows



we choose an element a of type 0 if n is odd and of type 2 if n is even. Let H the set of elements of Δ_{n+1} of type 0 at non-maximal distance from a, the distance between two elements of Δ_{n+1} being defined as the minimal length of a gallery stretched between them, as in [20]. Then H is a geometric hyperplane of the partial linear space having as points and lines the elements of Δ_{n+1} of type 0 and 1, respectively. For every element x of Δ_{n+1}, let $\sigma(x)$ be the 0–shadow of x, namely the set of elements of Δ_{n+1} of type 0 that are incident with x. (Note that $\sigma(x) = \{x\}$ for all elements of type 0.) If we remove from Δ_{n+1} all elements x with $\sigma(x) \subseteq H$, what is left is a flag–transitive geometry with diagram as follows and order 2 at all types $i > 0$.



Next, we truncate to $\{0, 1, 2\}$, namely we remove all elements of type $i > 2$. Thus, we obtain a c.PG*–geometry, say Γ_n, of order 2 and residual dimension n.

The collinearity graph. The collinearity graph of Γ_n is the alternating form graph $\mathrm{Alt}(n+1, 2)$ (Munemasa, Pasechnick and Shpectorov [15]; also [3, 9.5.B]).

The automorphism group. Let G be the stabilizer of a in $\mathrm{Aut}(\Delta_{n+1})$. The group G acts flag–transitively and faithfully on Γ_n. It consists of the square matrices of order $2(n+1)$ of the following shape

$$\begin{pmatrix} A & AB \\ O & (A^t)^{-1} \end{pmatrix}$$

with A a non-singular square matrix of order $n+1$, B an antisymmetric matrix of order $n+1$ and O the null square matrix of order $n+1$. Thus,

$$G = (V \wedge V){:}L_{n+1}(2) = 2^{(n+1)n/2}{:}L_{n+1}(2)$$

(where $V = V(n+1, 2)$). It is known that G is the full automorphism group of the alternating form graph [3, 9.5.3]. Hence $G = \mathrm{Aut}(\Gamma_n)$. (We will obtain the same conclusion in the case of $n > 3$ as a by-product of the proof of Theorem 1; see Proposition 17.)

When $n > 3$, none of the proper subgroups of G is flag-transitive on Γ_n. On the other hand, when $n = 3$ there is a flag-transitive proper subgroup of G of the form $2^6{:}A_7$.

Non-existence of covers and quotients. Munemasa, Pasechnik and Shpectorov [15] have proved that the collinearity graph of Γ_n, namely $\mathrm{Alt}(n+1, 2)$, does not admit any proper cover. Hence Γ_n is simply connected. (We cannot obtain this from our Theorem 1, as the simple connectedness of Γ_n will be exploited to finish the proof of that theorem.)

As Γ_n is simply connected, a proper flag–transitive quotient of Γ_n, if any, arises from a non-trivial subgroup H of $G = \mathrm{Aut}(\Gamma_n)$ acting semi–regularly on the set of elements of Γ_n and such that $N_G(H)$ acts flag–transitively on Γ_n ([18], Chapter 12). However, comparing the above description of G, it is straigthforward to check that no such subgroups of G exist. Thus, Γ_n does not admit any flag–transitive proper quotient.

An alternative description. Let $m = \binom{n+1}{2}$. Then Γ_n is the affine expansion to $\mathrm{AG}(m, 2)$ of the grassmannian of lines of $\mathrm{PG}(n, 2)$ naturally embedded in $\mathrm{PG}(m-1, 2)$ (see [5, Section 4] for affine expansions). Indeed, that affine expansion is a flag–transitive c.PG*–geometry of order 2 and residual dimension n and it has as many points as Γ_n. Thus, in view of Theorem 1, it is isomorphic to Γ_n.

2.2 The geometry Γ'_3

When $n = 3$, the partial linear space of 0– and 1–elements of Δ_4 is the point–line system of the hyperbolic quadric $Q_7^+(2)$ and the hyperplane H we remove from Δ_4 when constructing Γ_3 is just a tangent hyperplane H of $Q_7^+(2)$. However, in this case, we can imitate the above conctruction by chosing a secant hyperplane of $Q_7^+(2)$ as H instead of a tangent one. Thus, let H be a secant hyperplane of $Q_7^+(2)$ and let Γ'_3 be the subgeometry of Δ_4 obtained by removing H and all elements of Δ_4 of type 3. Clearly, Γ'_3 is a c.PG*–geometry of order 2 and residual dimension 3. It has 72 points (whereas 64 is the number of points of Γ_3).

Simple connectedness. The complement $\Delta_4 \setminus H$ of H in Δ_4 is 2–simply connected [18, Proposition 12.51]. Hence Γ'_3 is simply–connected, by [17, Theorem 1].

The automorphism group. We will see later (§3.3) that $\Delta_4 \setminus H$ can be recovered from Γ'_3. In turn, Δ_4 can be recovered from $\Delta_4 \setminus H$ (Cohen and Shult [7]). Consequently, the automorphism group of Γ'_3 is the stabilizer of H in $O_8^+(2)$, namely, $\mathrm{Aut}(\Gamma'_3) = S_6(2)$. It acts flag–transitively on Γ'_3.

Non-existence of proper quotients. As Γ'_3 is simply connected and $\mathrm{Aut}(\Gamma'_3)$ is isomorphic to $S_6(2)$, which is a simple group, Γ'_3 does not admit any flag–transitive proper quotient.

2.3 Some non flag–transitive examples

In this subsection we briefly describe the non flag–transitive c.PG*–geometries we are aware of.

More geometries from Δ_{n+1}. The construction of §2.1 can be repeated with H any hyperplane of the partial linear space of 0– and 1–elements of Δ_{n+1}, provided that the complement $\Delta_{n+1} \setminus H$ of H in Δ_{n+1} is connected. (The structure $\Delta_{n+1} \setminus H$ is connected when $n = 3$ for both choices of H and when $n > 3$ with H as in §2.1; maybe, the same is true for any n and any H, but we are not aware of any proof of this claim.)

In this way, when $n = 3$ we obtain Γ'_3. When $n > 3$, we still obtain a c.PG*–geometry of order 2 and residual dimension n. However, by our Theorem 1, no new flag–transitive examples arise.

Gluings. It is well known that a finite complete graph amits a 1–factorization if and only if its number of vertices is even. An n–dimensional finite projective space admits a parallelism only if n is odd (Buekenhout, Huybrechts, Pasini [5, 5.4]). On the other hand, all odd dimensional projective spaces of order 2 and all n–dimensional projective spaces with $n+1$ a power of 2, admit a parallelism (Baker [1], Buetelspacher [2], Denniston [10]).

Let $\mathcal{P}$ be a finite n–dimensional projective space of order q, admitting a parallelism, and let $\mathcal{K}$ be a complete graph with $v = 2 + q + ... + q^{n-1}$ vertices. As noticed above, n is odd. Hence v is even and $\mathcal{K}$ admits a 1–factorization. Thus, we can glue $\mathcal{K}$ with $\mathcal{S}$ (Buekenhout, Huybrechts and Pasini [5]). A c.PG*–geometry of order q and residual dimension n is obtained in this way. However, by Theorem 1, that geometry is not flag–transitive.

2.4 Remarks on the graphs $\mathrm{Alt}(n+1, 2)$ and $\mathrm{Quad}(n, 2)$

As we have noticed in §2.1, the alternating form graph $\mathrm{Alt}(n+1, 2)$ is the collinearity graph of Γ_n. The quadratic form graph $\mathrm{Quad}(n, 2)$ is considered by Munemasa, Pasechnick and Shpectorov [15] in combination with $\mathrm{Alt}(n+1, 2)$. These two graphs have the same number of vertices and the same local structure. However, the graph $\mathrm{Quad}(n, 2)$ does not give rise to any c.PG*–geometry. Indeed, there is no way of picking up a family of cliques from $\mathrm{Quad}(n, 2)$ to be taken as planes. This is implicit in Munemasa, Pasechnik and Shpectorov [16] (also in §3.3 of the present paper).

3 Proof of Theorem 1

In the sequel Γ is a c.PG*–geometry of order q and residual dimension $n > 2$. We assume that Γ is flag–transitive and G is a flag–transitive subgroup of $\mathrm{Aut}(\Gamma)$. (However, for some of the lemmas we are going to state in this section there is no need to assume flag–transitivity.)

3.1 Point–stabilizers

Given an element x of Γ, let G_x be its stabilizer in G. By K_x we denote the elementwise stabilizer in G_x of the residue of x and we set $\overline{G}_x = G_x/K_x$. The following is a special case of [11, Lemma 2.8]:

Lemma 2 *We have $K_a = 1$ (hence, $\overline{G}_a = G_a$) for any point a of Γ.*

The next statement is an assembling of results of Kantor [13] and Cameron and Kantor [6].

Lemma 3 *Given a point a of Γ, either $G_a \leq L_{n+1}(q)$ or $(n, q) = (3, 2)$ and $G_a = A_7$.*

3.2 The properties (LL) and (T)

We firstly state some notation to be used in the sequel. Given an element x of Γ, we denote its residue by Γ_x, as usual. When x is a point, Γ^*_x stands for the dual of Γ_x.

Given two distinct points a, b, we write $a \perp b$ to mean that they are collinear. By $a^{\perp}$ we mean the set of points collinear with or equal to a. We denote by $\delta(a, b)$ the distance between two points a, b in the collinearity graph of Γ. Accordingly, given a point a and a set of points A, the distance of a from A will be denoted by $\delta(a, A)$.

Lemma 4 *The following holds in* Γ*:*
(LL) *distinct lines are incident with distinct pairs of points.*

Proof. Given a point a, the relation 'having the same points' is an equivalence relation on the set of lines of Γ^*_a and G_a permutes the equivalence classes of that relation. However, by Lemma 3, G_a acts primitively on the set of lines of Γ^*_a. Therefore, either (LL) holds or all lines of Γ have the same points. The latter being impossible, (LL) holds. ∎

According to (LL), given two collinear points a, b, there is a unique line incident with both of them. We shall denote it by the symbol ab.

As the (LL) property holds in Γ, the Intersection Property also holds [18, Lemma 7.25]. Hence, no two distinct planes of Γ are incident with the same triple of points. Distinct planes of Γ being incident with distinct sets of points, the planes of Γ may be regarded just as sets of points. Accordingly, we write $a \in A$ (resp. $a \notin A$) to say that a point a and a plane A are (not) incident, we write $A \cap b^\perp$ to denote the set of points of A that are collinear with a given point b, and so on.

Lemma 5 *The following holds:*
(T) *every* 3*–clique of the collinearity graph of* Γ *is incident with a* (*unique*) *plane.*

Proof. Assume the contrary and let $\{a, b, c\}$ be a triple of mutually collinear points of Γ not contained in a common plane of Γ. The lines ab and ac are skew in Γ^*_a. Two cases are to examine.

Case 1. $G_a \geq L_{n+1}(q)$. Then G_a is transitive on the set of pairs of skew lines of Γ^*_a. Consequently, given any two lines $l = ax$, $m = ay$ through a skew in Γ^*_a, the points x, y are collinear in Γ. Clearly, the same conclusion holds if l and m are coplanar. Therefore, by the transitivity of G on the set of points of Γ, any two points of Γ are collinear. Consequently,

$$N \;=\; 1 + \frac{(1+q+...+q^n)(q+q^2+...+q^n)}{(1+q)q}$$

is the number of points of Γ. The number of planes of Γ is

$$N\frac{1+q+...+q^n}{2+s} \;=\; \frac{N(1+q+...+q^n)}{2+q+...+q^n}$$

By comparing the previous two equalities we see that $2+q+...+q^n$ divides the following:

$$1+q+...+q^n + \frac{(1+q+...+q^n)^2(1+q+...+q^{n-1})}{1+q}$$

It is straightforward to see that this contradicts the assumption $n > 2$. Thus, (T) holds in this case.

Case 2. $(n, q) = (3, 2)$ and $G_a = A_7$. A model of Γ^*_a can be constructed on $S = \{1, 2, ...7\}$ as follows [19, chapter 6] (also [18, p. 279]): the lines of Γ^*_a are the 3–subsets of S, two such subsets X, Y corresponding to skew (concurrent) lines of

Γ_a^* when $|X \cap Y| = 0$ or 2 (respectively, 1). The points of Γ_a^* are 15 out of the 30 projective planes that can be drawn on S, forming one orbit for A_7.

The stabilizer of ab in G_a has two orbits of size 12 and 4 respectively on the set of lines of Γ_a^* skew with ab. Assuming that ab corresponds to the subset $\{1, 2, 3\}$ of S, one orbit, say O_1, corresponds to the family of 3–subsets of S meeting $\{1, 2, 3\}$ in two points. The four 3–subsets of S exterior to $\{1, 2, 3\}$ contribute the other orbit, say O_2. Every point of Γ_a^* (plane of Γ through a) non-incident with ab is incident with exactly three lines of O_1, to one line of O_2 and to exactly three lines concurrent with ab.

Let $\{i, j\} = \{1, 2\}$ with $ac \in O_i$. If for some $l \in O_j$ the point of l different from a is collinear with b, then the same holds for all lines of O_j and a contradiction is reached as in Case 1. Therefore, given a point $x \in a^\perp \setminus \{a, b\}$, we have $x \perp b$ if and only if $ax \in O_i$. Thus, given a plane A of Γ incident with ac (hence, not incident with ab), a point $x \in A$ is collinear with b if and only if the line ax either belongs to O_i or is coplanar with ab.

Assume that $ac \in O_2$. Then exactly five points of A are collinear with b, namely a, c and three more points c_1, c_2, c_3, with $\{a, b, c_i\}$ contained in a plane for every $i = 1, 2, 3$. Similarly, interchanging a with c, each of the triples $\{c, b, c_i\}$ is in a plane. Thus, replacing a with c_i, for $\{i, j, k\} = \{1, 2, 3\}$ exactly one of the triples $\{c_i, b, c_j\}$ and $\{c_i, b, c_k\}$ is not contained in a plane. Let the points c_1, c_2, b be non-coplanar, to fix ideas. Then, as a coplanar triple $\{c_i, b, c_j\}$ exists for $i = 1, 2$, each of $\{c_1, c_3, b\}$ and $\{c_2, c_3, b\}$ is contained in a plane. Therefore, no triple $\{c_3, b, c_j\}$ of non-coplanar points exists; contradiction.

The above forces $ac \in O_1$. That is, a point $c \in a^\perp$ is collinear with b but not coplanar with ab if and only if the 3–subsets of S corresponding to the lines ab and ac meet in a 2–subset. Consequently, given a plane A incident with a but not with ab, $A \cap b^\perp$ contains all points of A but one; furthermore, just three out of the six points of $A \cap b^\perp$ different from a are coplanar with ab. This forces the relation $\not\perp$ ('being non-collinear') to be an equivalence relation.

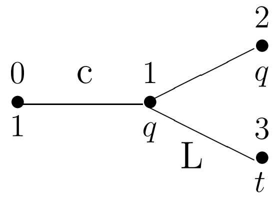
Indeed, let x, x' be distinct points non-collinear with b and assume $x \perp x'$, by contradiction. Let X be a plane incident with the line xx'. By the above, $\delta(b, X) \geq 2$. Consequently, some points of Γ have distance 2 from X. Let u be one of them and let v, w be points such that $u \perp v \perp w \in X$. According to the above, just three points of $X \setminus \{w\}$ are coplanar with the line vw. Hence, three of the planes through vw meet X in a line. Let Y be one of those planes and $\{w, w'\} = X \cap Y$. The point u, being collinear with $v \in Y$, is collinear with all but one points of Y. Therefore, $u \perp w'$, as $u \not\perp w$. However, as $w' \in X$, this contradicts the hypothesis that $\delta(u, X) = 2$.

Thus, $\not\perp$ is an equivalence relation. It also induces an equivalence relation on the set of lines through the point a, a line ax being equivalent to ab precisely when $x \not\perp b$. However, $x \not\perp b$ if and only if $ax \in O_2$. Consequently, the lines of O_2 join a with mutually non-collinear points. However, this is false: the 3–subsets of S corresponding to the lines of S mutually intersect in a 2–subset, hence they join a with mutually collinear points. We have reached a final contradiction. ∎

3.3 Adding new elements

Given a maximal clique C of the collinearity graph of Γ and a point $a \in C$, let C_a be the set of lines joining a to the points of $C \setminus \{a\}$. By property (T), C_a is a maximal set of pairwise concurrent lines of Γ^*_a. Hence either C is the set of points of some plane A of Γ incident with a or C_a is the set of lines of a plane of the projective space Γ^*_a. In the latter case we call C a 3–*element*.

Thus, we have two kinds of maximal cliques in the collinearity graph of Γ, namely the planes of Γ and the 3–elements. It is easy to see that a 3–element C and a plane A meet in 0, 1 or $q+2$ points. When the latter occurs, then we say that A and C are incident. Furthermore, we declare C to be incident with all points and lines it contains. Thus, we obtain a geometry $\overline{\Gamma}$ of rank 4, which we call the *enrichment* of Γ. It is straightforward to check that $\overline{\Gamma}$ belongs to the following diagram:



where 0, 1, 2, 3 are the types, $1, q, q, t$ are orders and $t+1 = (q^{n-1}-1)/(q-1)$. We still call *points* and *lines* the elements of $\overline{\Gamma}$ of type 0 and 1, as in Γ. Clearly, the residues of the points of $\overline{\Gamma}$ are isomorphic to the truncation of $\mathrm{PG}(n,q)$ to points, lines and planes. Hence,

Lemma 6 *The residues of the* $\{0,2\}$*–flags of* $\overline{\Gamma}$ *are* $(n-1)$*–dimensional projective spaces of order* q.

The next statement is an easy consequence of Lemma 3.

Lemma 7 *The geometry* $\overline{\Gamma}$ *is flag–transitive and the stabilizer in* $\mathrm{Aut}(\overline{\Gamma})$ *of a* $\{0,2\}$*–flag* F *of* $\overline{\Gamma}$ *induces on* $\overline{\Gamma}_F$ *a group containing* $L_n(q)$.

Clearly, $\overline{\Gamma}$ inherits (LL) from Γ. Furthermore,

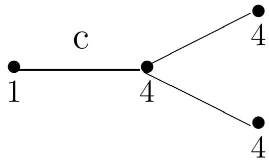
Lemma 8 *The following holds in* $\overline{\Gamma}$*:*
(T′) *every* 3*–clique of the collinearity graph of* Γ *is incident with a* (*unique*) $\{2,3\}$*–flag.*

(Easy, by (T) in Γ.) We are now ready to prove the following:

Lemma 9 *We have* $q = 2$.

Proof. As residues of 3–elements of $\overline{\Gamma}$ are extended projective planes, either $q = 2$ or $q = 4$.

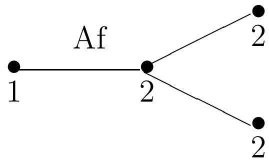
Assume $q = 4$. By Lemmas 6 and 7, the residues of the 2–elements of $\overline{\Gamma}$ are flag–transitive extensions of $(n-1)$–dimensional projective spaces of order 4 with at least $L_n(4)$ induced on point–residues. Then $n = 3$, by Delandtsheer [9] (see also [18, Theorem 9.22]). That is, $\overline{\Gamma}$ has diagram and orders as follows:



However, no flag–transitive geometry exists with diagram and orders as above and satisfying (LL) and the property (T′) of Lemma 8 (Buekenhout and Hubaut [4]). Hence $q = 2$. ■

3.4 End of the proof in the case of $n = 3$

Assume $n = 3$. By Lemma 9, $\overline{\Gamma}$ has diagram and orders as follows, where we have replaced the label c with Af, as the circular space with 4 points is the affine plane of order 2:



By [18, Theorem 7.57], $\overline{\Gamma}$ is obtained from the D_4–building over GF(2) by removing a hyperplane of its related polar space; namely, $\Gamma \cong \Gamma_3$ or Γ'_3.

3.5 The case of $n > 3$

Let $n > 3$. Let a, l, π be a point, a line and a plane of Γ forming a chamber. We know that $K_a = 1$ (Lemma 2). Henceforth we write K for K_π.

Lemma 10 *We have $G_a = L_{n+1}(2)$ and $G_{a,\pi} = K{:}L = \mathrm{ASL}_n(2)$, the group K is elementary abelian of order 2^n and $L = L_n(2)$.*

(Easy, by Lemmas 3 and 9.) Furthermore,

Lemma 11 *We have $G_\pi = K.(T{:}L)$ with $T{:}L = \mathrm{ASL}_n(2)$ and $T = 2^n$.*

Proof. By lemma 3, G acts flag-transitively on $\overline{\Gamma}$, and so G_π acts flag-transitively on $\overline{\Gamma}_\pi$, which is an extension of an $(n-1)$-dimensional projective space of order 2. The statement follows from Delandtsheer [9] and from Lemma 10. ■

With L as above, let L_l be the stabilizer of l in L. The following is obvious:

Lemma 12 *We have $G_{a,l,\pi} = K{:}L_l$ and the action of L on K is the dual of the action of L on $T \cong (KT)/K$. Furthermore, $G_\pi = (K.T){:}L$.*

Let $N = K.T$. Then,

Lemma 13 *We have $K \leq Z(N)$.*

Proof. Given $v \in N$, let $f_v \in \mathrm{Aut}(V(n,2)) = L_n(2)$ be the action of v on K. Clearly, $f_v = f_{vk}$ for every $k \in K$. Thus, given $V \in N/K$ and $v \in V$, we write f_V for f_v. Clearly, the function f sending $V \in N/K$ to f_V is a morphism from N/K to $L_n(2) = \mathrm{Aut}(K) = L$. Since N is normal in G_π and L is a subgroup of $G_{a,\pi}$ normalizing K, the image $f(N/K)$ of N/K by f is normal in L. However, $f(N/K)$ is a (possibly trivial) 2–group, as N/K is a 2–group. Hence $f(N/K) = 1$. The conclusion follows. ∎

Given $v \in N \setminus K$, we have $v^2 \in K$. Therefore, $v^4 = 1$. As $K \leq Z(N)$, the elements v and vk have the same order for any $k \in K$. Thus, and by the transitive action of L on $(N/K) \setminus \{K\}$, one of the following holds:

(i) all elements of $N \setminus \{1\}$ have order 2.
(ii) all elements of $N \setminus K$ have order 4.

Lemma 14 *Case* (ii) *is impossible.*

Proof. Assuming (ii), let $g : N/K \longrightarrow K$ be the function sending $V \in N/K$ onto v^2, with v a representative of V in N. As $g(V) \neq 1$ for some $V \in N/K$ and since L acts transitively on $(N/K) \setminus \{K\}$, g is a bijection. Clearly, g commutes with the actions of L on N/K and K. That is, if $\lambda \in L$, then $(v^\lambda)^2 = (v^2)^\lambda$. Therefore, and since g is a bijection, L acts in the same way on K and T. But this is a contradiction: indeed, by Lemma 12, the action of L on $T = N/K$ is dual to the action of L on K. ∎

As (ii) is impossible, (i) holds. Hence,

Lemma 15 *We have* $N = 2^{2n}$. *Hence* $N = K \times T$ *and* $G_\pi = (N \times T){:}L$.

We still need to describe G_l. The group $G_{a,l}$ has index 2 in G_l and, if b is the point of l other than a and $t \in G_l \setminus G_{a,l}$, then d permutes a and b. Furthermore, we can assume that t is the element of T permuting a and b. In order to determine G_l completely we only need to describe the action of t on $G_{a,l}$.

Lemma 16 *We have* $G_l = \langle t \rangle \times G_{a,l}$ *and* $G_{l,\pi} = \langle t \rangle \times G_{a,l,\pi}$.

Proof. Note that $t \in G_\pi$ $(= (K \times T){:}L)$. In G_π we see that t centralizes $G_{a,l,\pi} = K{:}L_l$. The group $G_{a,l}$ is the stabilizer of a line of $\Gamma^*_a = \mathrm{PG}(n,2)$ in $G_a = L_{n+1}(2)$. Hence $G_{a,l} = A{:}(B \times C)$ with $A = 2^{2(n-1)}$, $B = L_2(2)$ and $C = L_{n-1}(2)$. Moreover, $A = W_1 \times W_2$ with $W_1 \cong W_2 \cong V(n-1,2)$ and C stabilizes both W_1 and W_2, acting naturally on each of them. On the other hand, B acts faithfully on A. Furthermore, $G_{a,l,\pi} = A{:}C$.

The element t induces an automorphism τ on $G_{a,l}$ and an automorphism τ_A on $G_{a,l}/A$. As t centralizes $A : C$, τ_A also centralizes AC/A. Therefore, and since $n > 3$, τ_A stabilizes AB/A. The subgroups of $G_{a,l}$ isomorphic to B and acting as B on A form one conjugacy class. As t centralizes A, the automorphism τ stabilizes that conjugacy class. Therefore $B^\tau = B^v$ for some $v \in G_{a,l}$ and, as C centralizes B, we may assume that $v \in A$. Consequently, given $g \in B$, we have $g^\tau = uf$ for some

$u \in A$ and some $f \in B$. As t centralizes C, we also have $(g^x)^\tau = (g^\tau)^x$ for every $x \in C$. On the other hand, $g^x = g$ and $f^x = f$ (indeed C centralizes B). Thus, $g^\tau = (g^\tau)^x$ for every $x \in C$. That is, $uf = (uf)^x$ for every $x \in C$. So (and since $f^x = f$), $uf = u^x f$ for every $x \in B$. This forces C to centralize u. However, it is clear from the information previously given on the action of C on A that 1 is the unique element of A centralized by C. Therefore $u = 1$. That is, $g^\tau \in B$ for every $g \in B$, namely $B^\tau = B$. Furthermore, for every $g \in B$ we have $(gug^{-1})^\tau = g^\tau u g^{-\tau}$ for every $u \in A$, because τ centralizes A. Hence, $g^{1-\tau}$ acts trivially on A for every $g \in B$. However, as we have remarked above, B acts faithfully on A. Therefore $g = g^\tau$ for every $g \in B$. So, t centralizes B. ■

By Lemmas 10, 12 and 16, the structures of G_a, G_l and G_π are completely determined, as well as their intersections (note that $G_{a,\pi}$ and $G_{a,l}$ are uniquely determined inside G_a). That is,

Proposition 17 *The amalgam* $(G_a, G_l, G_\pi; G_{a,l}, G_{a,\pi}, G_{l,\pi})$ *is uniquely determined.*

End of the proof of Theorem 1. By Proposition 17 and [18, Theorem 12.28], the universal cover of Γ is uniquely determined; namely, there is a unique simply connected flag–transitive c.PG*–geometry of residual dimension n. That geometry is Γ_n, as Γ_n is indeed simply connected and flag–transitive (§2.1). Therefore, Γ is a quotient of Γ_n. On the other hand, Γ_n has no proper flag–transitive quotients (§2.1). Hence $\Gamma = \Gamma_n$.

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Cécile Huybrechts
Université Libre de Bruxelles, Département de Mathématique
Campus Plaine, B–1050 Bruxelles, Belgique.
e–mail: huyb@ulb.ac.be

Antonio Pasini
University of Siena, Department of Mathematics
Via del Capitano 15, I–53100 Siena, Italy.
e–mail: pasini@unisi.it