Linear sections of $\mathrm{GL}(4,2)$

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Abstract

For $V = V(n,q)$, a *linear section* of $\mathrm{GL}(V) = \mathrm{GL}(n,q)$ is a vector subspace $\mathcal{S}$ of the n^2-dimensional vector space $\mathrm{End}(V)$ which is contained in $\mathrm{GL}(V) \cup \{0\}$. We pose the problem, for given (n,q), of classifying the different kinds of maximal linear sections of $\mathrm{GL}(n,q)$. If $\mathcal{S}$ is any linear section of $\mathrm{GL}(n,q)$ then $\dim \mathcal{S} \leq n$.

The case of $\mathrm{GL}(4,2)$ is examined fully. Up to a suitable notion of equivalence there are just two classes of 3-dimensional maximal normalized linear sections $\mathcal{M}_3, \mathcal{M}_3'$, and three classes $\mathcal{M}_4, \mathcal{M}_4', \mathcal{M}_4''$ of 4-dimensional sections. The subgroups of $\mathrm{GL}(4,2)$ generated by representatives of these five classes are respectively $\mathcal{G}_3 \cong A_7$, $\mathcal{G}_3' = \mathrm{GL}(4,2)$, $\mathcal{G}_4 \cong Z_{15}$, $\mathcal{G}_4' \cong Z_3 \times A_5$, $\mathcal{G}_4'' = \mathrm{GL}(4,2)$. On various occasions use is made of an isomorphism $T : A_8 \to \mathrm{GL}(4,2)$. In particular a representative of the class $\mathcal{M}_3$ is the image under T of a subset $\{\xi_1, \ldots, \xi_7\}$ of A_7 with the property that $\xi_i^{-1}\xi_j$ is of order 6 for all $i \neq j$.

The classes $\mathcal{M}_3, \mathcal{M}_3'$ give rise to two classes of maximal partial spreads of order 9 in $\mathrm{PG}(7,2)$, and the classes $\mathcal{M}_4', \mathcal{M}_4''$ yield the two isomorphism classes of proper semifield planes of order 16.

1 Introduction and plan

For $V = V(n,\mathbb{K})$, a *linear section* of $\mathrm{GL}(V) = \mathrm{GL}(n,\mathbb{K})$ is defined to be a vector subspace $\mathcal{S}$ of the n^2-dimensional space $\mathrm{End}(V)$ which is contained in $\mathrm{GL}(V) \cup \{0\}$.

Theorem 1.1 *For any linear section $\mathcal{S}$ of* $\mathrm{GL}(n,\mathbb{K})$, $\dim \mathcal{S} \leq n$.

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Proof. Let $H \subset V(n,\mathbb{K})$ be a subspace of V dimension $n-1$. Then $W_H = \{A \in \mathrm{End}(V) : \mathrm{Im}A \subseteq H\}$ is a subspace of $\mathrm{End}(V)$ isomorphic to $L(V,H)$, and hence of dimension $n(n-1)$. The rank of every element of W_H is at most $n-1$, and so $W_H \cap \mathcal{S} = \{0\}$. It follows that $\dim\mathcal{S}$ is at most $\dim(\mathrm{End}(V)) - n(n-1) = n$. ■

In the case $\mathbb{K} = \mathbb{R}$ of the real field, the maximal dimension m of a linear section of $\mathrm{GL}(n,\mathbb{R})$ is known for all values of n, see e.g. [13, after theorem 13.68]. Only for $n = 1,2,4,8$ does $m = n$. In fact, in this case of the real field, the ratio m/n tends to 0 with increasing n, see e.g. [15]. In the present paper we will confine ourselves to the case of a vector space $V = V(n,q)$ of dimension $n(>1)$ over the finite field $\mathrm{GF}(q)$. In contrast to the case of the real field we then find, see theorem 2.2 below, that m is always equal to n.

So we will be seeking subspaces $\mathcal{S} \subset \mathrm{End}(V) = \mathrm{End}(n,q)$ with the property that every nonzero element of $\mathcal{S}$ lies in the group $G = \mathrm{GL}(n,q)$. Now G is a $G \times G$ space, with $(A,B) \in G \times G$ acting on $X \in G$ by $X \mapsto AXB^{-1}$, and if a set $\mathcal{S}$ of linear maps is a linear section of G, then so is the left- and right-translated set $A\mathcal{S}B^{-1}(=\{AXB^{-1} : X \in \mathcal{S}\})$ for any $A,B \in G$. Consequently it seems natural to seek a classification of linear sections up to *equivalence,* where two linear sections $\mathcal{S},\mathcal{S}'$ are defined to be equivalent if and only if they lie on the same $(G \times G)$-orbit:

$$\mathcal{S}' = A\mathcal{S}B^{-1}, \quad \text{for some } A,B \in \mathrm{GL}(V). \tag{1}$$

A r-dimensional *normalized* linear section of $\mathrm{GL}(n,q)$, abbreviated $\mathrm{NLS}_r(n,q)$, is a linear section $\mathcal{S}$ which contains the identity $I \in G = \mathrm{GL}(n,q)$. Since any $G \times G$ orbit of linear sections (other than $\{0\}$) contains at least one normalized section, from now on we usually restrict our attention just to these. If $\mathcal{S}$ is a $\mathrm{NLS}_r(n,q)$, then so are $X^{-1}\mathcal{S}$ and $\mathcal{S}X^{-1}$, for each nonzero element X of $\mathcal{S}$, and we refer to such sections $X^{-1}\mathcal{S}$ and $\mathcal{S}X^{-1}, X \in \mathcal{S}$, as, respectively, left and right *mutants* of $\mathcal{S}$. Note that a left mutant of a left mutant of $\mathcal{S}$ is a left mutant of $\mathcal{S}$, and similarly, *mutatis mutandis,* for right mutants. If two normalized linear sections $\mathcal{S},\mathcal{S}'$ satisfy (1) then $AIB^{-1} \in \mathcal{S}'$, and so $A = X'B$ for some $X' \in \mathcal{S}'$. *Consequently two normalized linear sections* $\mathcal{S},\mathcal{S}'$ *of* $\mathrm{GL}(n,q)$ *are equivalent whenever* $\mathcal{S}$ *is conjugate to some left mutant of* $\mathcal{S}'$, that is whenever

$$B\mathcal{S}B^{-1} = (X')^{-1}\mathcal{S}' \qquad \text{for some } X' \in \mathcal{S}' \smallsetminus \{0\} \text{ and some } B \in \mathrm{GL}(n,q). \tag{2}$$

(We could replace "left mutant" by "right mutant" in this last statement, since $\mathcal{S}X^{-1} = X(X^{-1}\mathcal{S})X^{-1}$ is conjugate to $X^{-1}\mathcal{S}$.) Naturally we will be particularly concerned with the classification of those $\mathrm{NLS}_r(n,q)$'s which are *maximal*, that is those linear sections which are not proper subspaces of a higher-dimensional section.

It is worth noting that if $\mathcal{S}$ is a $\mathrm{NLS}_r(n,q)$, then so is its Galois conjugate $\mathcal{S}^\sigma = \{X^\sigma : X \in \mathcal{S}\}$ for any automorphism σ of the field $\mathrm{GF}(q)$, where, in matrix terms, $(X^\sigma)_{ij} = (X_{ij})^\sigma$. So, in the case of nonprime q, a broader notion of equivalence of two linear sections $\mathcal{S}$ and $\mathcal{S}'$could be appropriate, say *semilinear equivalence*, with $\mathcal{S}'$ being equivalent in the previous sense to some Galois conjugate $\mathcal{S}^\sigma$ of $\mathcal{S}$. However in the present paper we will be chiefly concerned with cases where $q = p$ is a prime.

The plan of the present paper is as follows. In section 2 we treat certain matters valid for general $\mathrm{GF}(q)$. Thereafter, in sections 3 - 8, we deal solely with the case of linear sections of $\mathrm{GL}(n,2), n \leq 4$. In the case of $\mathrm{GL}(4,2)$ we obtain a complete classification of all maximal linear sections: see section 5 for a summary. Finally, in the Appendix, we treat the connection of the present work with certain well known material concerning spreads, spread sets and those translation planes which are coordinatized by semifields. However we wish to stress that the motivation for the present work did *not* arise from this connection with spreads and translation planes: realizing how rare it is for (the nonzero vectors of) a linear subspace to lie inside a linear group, we believe that when this rare event occurs interesting mathematics is likely to ensue. See section 6 for at least one case that supports this belief, the linear group being a subgroup of GL$(4, 2)$ isomorphic to A_7.

2 Linear sections of GL(n, q)

2.1 General considerations

An element $A \in \mathrm{GL}(V) = \mathrm{GL}(n, q)$ induces a collineation, say $\mathbf{A}$, of the projective space $\mathbf{P}V = \mathrm{PG}(n-1, q)$ associated with $V = V(n, q)$. By an easy proof we obtain the following elementary, but useful, lemma in respect of 2-dimensional sections. (We use $\prec X_1, X_2, \ldots \succ$ to denote the linear span of elements X_1, X_2, ... over the agreed field, in this case GF(q).)

Lemma 2.1 *For* $A \in \mathrm{GL}(n, q)$ *the subspace* $\prec I, A \succ$ *is a* $\mathrm{NLS}_2(n, q)$ *if and only if* $\mathbf{A}$ *is fixed-point-free on* PG$(n-1, q)$. ■

Theorem 2.2 *For any prime power* q, *the group* GL(n, q), $n > 1$, *possesses a normalized linear section* $\mathcal{S}$ *of dimension* n *of the form*

$$\mathcal{S} = \prec I, A, A^2, \ldots, A^{n-1} \succ, \tag{3}$$

where A *is an element of* GL(n, q) *of order* $q^n - 1$.

Proof. Take $V(n, q)$ to be the field GF(q^n) viewed as a vector space of dimension n over the subfield GF(q), and define A by $Ax = \alpha x$, $x \in \mathrm{GF}(q^n)$, where α is a primitive element of GF(q^n). By the field properties, A generates a subgroup $\langle A \rangle \cong Z_{q^n-1}$ of GL(n, q) (called a Singer cyclic subgroup) and $\mathcal{S} = \langle A \rangle \cup \{0\}$ is a $\mathrm{NLS}_n(n, q)$. ■

Sections of the kind (3), and also their translates, will be referred to as *Singer sections.* A subspace of a Singer section will be said to be a *sub-Singer* section, or a section of *Singer type.* The next theorem demonstrates that, in general, by no means all sections are sub-Singer sections. (Also, see later, there may well exist maximal linear sections of GL(n, q) of dimension $< n$.)

Theorem 2.3 *For* $n = mk$, *where* $m > 1, k > 1$, *consider the tensor product space* $V(n, q) = V(m, q) \otimes V(k, q)$. *Set* $A = B \otimes C \in \mathrm{GL}(n, q)$, *where* $\langle B \rangle \cong Z_{q^m-1}$ *is a cyclic Singer subgroup of* GL(m, q) *and* $C \in \mathrm{GL}(k, q)$. *Put* $v = |\mathrm{PG}(m-1, q)| = (q^m - 1)/(q - 1)$. *Suppose that the order* r *of* C *is such that (i)* $v \not| r$ *(ii)* $r \not| (q^n - 1)$. *Then* $\prec I, A \succ$ *is a* $\mathrm{NLS}_2(n, q)$ *which is not of Singer type.*

Proof. Since $v \nmid r$, observe that B^r has no eigenvalues over $\mathrm{GF}(q)$, see e.g.[5], section 11.3. The same is therefore true of $A^r = B^r \otimes I$ (being the direct sum of k copies of B^r), and hence of A; so $\prec I, A \succ$ is a $\mathrm{NLS}_2(n,q)$. Since r divides the order of A, and since $r \nmid (q^n - 1)$, the order of A does not divide the order of a Singer subgroup $\cong Z_{q^n-1}$ of $\mathrm{GL}(n,q)$. ■

Example 2.4 *(i) It is always possible to choose a suitable* $C \in \mathrm{GL}(k,q)$ *for the theorem to apply. For if* $q = p^h$, *with* p *prime, take* $C = I + N$ *where* $N \neq 0$ *satisfies* $N^p = 0$. *Then* $r = p$ *and so conditions (i) and (ii) in the theorem are both satisfied.*

(ii) If $q = 2$ *we may choose* $C \in \mathrm{GL}(k,2)$ *to be any element of even order. If* $q = 2$ *and* $m = 2$, $k = 4$, *we may choose* $C \in \mathrm{GL}(4,2)$ *to be of order 7.*

Lemma 2.5 *For* $q = p^h$, *where* p *is prime, suppose that* $\mathcal{S}$ *is a* $\mathrm{NLS}_2(n,q)$. *Then so is* $(\mathcal{S})^p$, *where* $(\mathcal{S})^p = \{X^p : X \in \mathcal{S}\}$.

Proof. If $\mathcal{S} = \prec I, A \succ$ is a $\mathrm{NLS}_2(n,q)$, then $X = \lambda I + \mu A$ lies in $\mathrm{GL}(n,q) \cup \{0\}$ for all $\lambda, \mu \in \mathrm{GF}(q)$. Since $X^p = \lambda^p I + \mu^p A^p$, and $\lambda \mapsto \lambda^p$ is a field automorphism, it follows that the 2-dimensional subspace $(\mathcal{S})^p = \prec I, A^p \succ$ of $\mathrm{End}(n,q)$ is also a $\mathrm{NLS}_2(n,q)$. ■

The corresponding result for a non-normalized 2-dimensional section $\mathcal{S} = \prec A, B \succ$ holds if and only if A commutes with B. More generally, consider an arbitrary r-dimensional linear section $\mathcal{S}$ of $\mathrm{GL}(n,p^h)$ which is *abelian:* $X_1X_2 = X_2X_1$ for all $X_1, X_2 \in \mathcal{S}$; then $(\mathcal{S})^p$ will also be a linear (abelian) section of $\mathrm{GL}(n,p^h)$.

At times we will say that a particular normalized linear section $\mathcal{S}$ has *order pattern* $(n_1)^{k_1}(n_2)^{k_2}(n_3)^{k_3}\ldots$. By this we will mean that, *discounting the elements* $0, I \in \mathcal{S}$, k_i elements have order n_i, $i = 1, 2, \ldots$.

Example 2.6 *The group* $\mathrm{GL}(4,2)$ *has two classes of elements of order* 15, *the* 1344 *elements of one class* $\mathcal{C}_{15}$ *having characteristic polynomial* $t^4 + t + 1$, *and the* 1344 *elements of the other class* $\mathcal{C}'_{15}$ *having characteristic polynomial* $t^4 + t^3 + 1$, *the characteristic polynomials in each case coinciding with the minimal polynomials. If* $A \in \mathcal{C}'_{15}$, *and so satisfies* $A^4 = I + A^3$, *then* A^2, A^4, A^8 *also lie in* $\mathcal{C}'_{15}$, *while* $A^7, A^{14}, A^{13}, A^{11}$ *lie in* $\mathcal{C}_{15}$. *The following are the seven* $\mathrm{NLS}_2(4,2)$*'s which lie in the 4-dimensional Singer section* $\mathcal{S}_4 = \prec I, A, A^2, A^3 \succ$:

$$\begin{aligned} &\textit{(i)}\ \{0,I,A^3,A^4\}, \quad \{0,I,A^6,A^8\}, \quad \{0,I,A^{12},A\}, \quad \{0,I,A^9,A^2\};\\ &\textit{(ii)}\ \{0,I,A^7,A^{13}\}, \quad \{0,I,A^{14},A^{11}\}; \qquad \textit{(iii)}\ \{0,I,A^5,A^{10}\}\,. \end{aligned} \tag{4}$$

The four sections (i) are obtained one from another by successive squaring, as are the two sections (ii), while the section (iii) is its own square. Up to conjugacy we see that there are three Singer types of $\mathrm{NLS}_2(4,2)$*'s, as exemplified by representatives drawn from (i), (ii), (iii), the three conjugacy types being distinct, since a section* $\mathcal{S}_2$ *is of type (i), (ii) or (iii) according as its order pattern is* $5(15)$, $(15)^2$ *or* 3^2. *However the six sections of (i) and (ii) are all equivalent: for example,* $\{0,I,A^{14},A^{11}\}$ *is a mutant of* $\{0,I,A^3,A^4\}$, *since* $A^{-4}\{0,I,A^3,A^4\} = \{0,A^{11},A^{14},I\}$. *On the other hand the section (iii) (associated with the subfield* $\mathrm{GF}(4)$ *of* $\mathrm{GF}(16)$*) mutates only into itself. Consequently there are up to equivalence just two types of 2-dimensional sub-Singer sections of* $\mathrm{GL}(4,2)$.

Similarly one finds that the seven $\mathrm{NLS}_3(4,2)$*'s which lie in* $\mathcal{S}_4$ *form three conjugacy types, with order patterns* $3^2 5^2 (15)^2$, $3^2(15)^4$ *and* $5^2(15)^4$, *respectively. However, these three conjugacy types coalesce into just one equivalence type of Singer* $\mathrm{NLS}_3(4,2)$. *Also any two Singer* $\mathrm{NLS}_4(4,2)$*'s are conjugate, not merely equivalent.*

Theorem 2.7 *The only 2-dimensional linear sections of* $\mathrm{GL}(2,q)$ *are the Singer sections.*

Proof. If $\prec I, A \succ$ is a $\mathrm{NLS}_2(2,q)$ then A has no eigenvalues over $\mathrm{GF}(q)$ (i.e. is of elliptic type) and so, over the quadratic extension field $\mathrm{GF}(q^2)$, A is similar to the diagonal matrix $M = \mathrm{diag}(w, w^q)$ with $w \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$. So $\lambda I + \mu A$ is similar to $\mathrm{diag}(\lambda + \mu w, (\lambda + \mu w)^q)$. But, varying λ, μ over $\mathrm{GF}(q)$, $\lambda + \mu w$ yields every element of $\mathrm{GF}(q^2)$. Thus the nonzero elements of $\prec I, A \succ$ form a subgroup of $\mathrm{GL}(2,q)$ isomorphic to the multiplicative group Z_{q^2-1} of $\mathrm{GF}(q^2)$. ∎

2.2 Subgroups of $\mathrm{GL}(n,q)$ associated with $\mathrm{NLS}_r(n,q)$'s

If $\mathcal{S}$ is a $\mathrm{NLS}_r(n,q)$ we denote by $\mathcal{G}(\mathcal{S})$ the subgroup of $\mathrm{GL}(n,q)$ generated by the nonzero elements of $\mathcal{S}$, and note that $\mathcal{S}$ is a normalized linear section of the linear group $\mathcal{G}(\mathcal{S})$. Note that any mutant of a normalized linear section $\mathcal{S}$ generates the same subgroup as $\mathcal{S}$: $\mathcal{G}(X^{-1}\mathcal{S}) = \mathcal{G}(\mathcal{S}) = \mathcal{G}(\mathcal{S}X^{-1})$, for any $X \in \mathcal{S}$. (Indeed the same is true of any left mutant of any right mutant of any left mutant ... of $\mathcal{S}$.) *So subgroups* $\mathcal{G}(\mathcal{S}), \mathcal{G}(\mathcal{S}')$ *generated by equivalent normalized sections* $\mathcal{S}, \mathcal{S}'$ *are necessarily conjugates of each other within* $\mathrm{GL}(n,q)$.

Denote by $\mathcal{F} = \mathcal{F}(\mathcal{S})$ the family of left mutants of $\mathcal{S}$, and note that $\mathcal{F}(X^{-1}\mathcal{S}) = \mathcal{F}(\mathcal{S})$ for each $X \in \mathcal{S}$. We associate with such a family the subgroup $\mathcal{H}(\mathcal{F})$ of $\mathrm{GL}(n,q)$ defined by

$$\mathcal{H}(\mathcal{F}) = \{H \in \mathrm{GL}(n,q) : H\mathcal{S}H^{-1} \in \mathcal{F} \text{ for each } \mathcal{S} \in \mathcal{F}\}. \tag{5}$$

It is easy to see that *an element* H *of* $\mathrm{GL}(n,q)$ *belongs to* $\mathcal{H}(\mathcal{F})$ *provided merely that* $H\mathcal{S}H^{-1} \in \mathcal{F}$ *for one (any) choice of* $\mathcal{S} \in \mathcal{F}$. So we also write $\mathcal{H}(\mathcal{F})$ as $\mathcal{H}(\mathcal{S})$. Let $\mathcal{H}_0(\mathcal{S})$ denote the set-stabilizer of $\mathcal{S}$ under the action by conjugacy of $\mathrm{GL}(n,q)$:

$$\mathcal{H}_0(\mathcal{S}) = \{H \in \mathrm{GL}(n,q) : H\mathcal{S}H^{-1} = \mathcal{S}\}. \tag{6}$$

Then each of the groups $\mathcal{H}_0(\mathcal{S}), \mathcal{S} \in \mathcal{F}$, is a subgroup of $\mathcal{H}(\mathcal{F})$. Define further

$$\mathcal{G}_0(S) = \{G \in S \mid GS = S\}, \qquad \mathcal{G}_0'(S) = \{G \in S \mid SG = S\}. \tag{7}$$

Clearly $\mathcal{G}_0, \mathcal{G}_0'$ are both subgroups of $\mathcal{G}$. But more is the case, for since S is closed under the formation of linear combinations over $\mathrm{GF}(q)$, we have the result:

Lemma 2.8 *If* $F_0 = \mathcal{G}_0 \cup \{0\}$ *and* $F_0' = \mathcal{G}_0' \cup \{0\}$, *then each of* F_0 *and* F_0' *is a field which contain* $\mathrm{GF}(q)$ *as a subfield.* ∎

Note also that $G \in \mathcal{G}_0(S)$ if and only if $X^{-1}GX \in \mathcal{G}_0(X^{-1}S)$. So the subgroups $\mathcal{G}_0(S), \mathcal{S} \in \mathcal{F}$, are conjugates of each other; similarly for the $\mathcal{G}_0'(S), \mathcal{S} \in \mathcal{F}$.

3 Linear sections of $\mathrm{GL}(3,2)$

From now on we restrict our attention to the case of the field $\mathrm{GF}(2)$. Before dealing with dimension $n=3$ it may be worth mentioning the baby case $n=2$. Observe that the group $\mathrm{GL}(2,2)\cong S_3\cong Z_3\rtimes Z_2$ has a unique $\mathrm{NLS}_2(2,2)$, namely the Singer section $\{0\}\cup Z_3$. If we view $\mathrm{End}(2,2)\smallsetminus\{0\}$ as the projective space $\mathrm{PG}(3,2)$, then the left action $X\mapsto AX, A\in Z_3$, of Z_3 on $\mathrm{PG}(3,2)$ has for its orbits a spread of 5 lines; similarly for the right action $X\mapsto XA^{-1}$, the two spreads sharing two lines, namely the two cosets of Z_3 in $\mathrm{GL}(2,2)$. Of course the remaining 9 points of $\mathrm{PG}(3,2)$, that is the elements of $\mathrm{End}(2,2)$ having rank 1, comprise a hyperbolic quadric $\mathcal{H}_3$ with equation $\det X=0$, and the orbits for the left and right actions of Z_3 on $\mathcal{H}_3$ are the two systems of generators of $\mathcal{H}_3$, a regulus and the opposite regulus.

Theorem 3.1 *All linear sections of* $\mathrm{GL}(3,2)$ *are of Singer type.*

Proof. Let us call an element $A\in\mathrm{GL}(n,2)$ fixed-point-free if it induces a fixed-point-free collineation of $\mathrm{PG}(n-1,2)$. Since the order of an element A belonging to $\mathrm{GL}(3,2)$ is 1, 2, 3, 4 or 7, and since $|\mathrm{PG}(2,2)|=7$, a fixed-point-free element $A\in\mathrm{GL}(3,2)$ necessarily has order 7. Consequently, by lemma 2.1, any 2-dimensional linear section $\mathcal{S}_2=\prec I,A\succ$ is of Singer type, since it lies inside the 3-dimensional Singer section $\prec I,A,A^2\succ$ of the form (3). Let $\mathcal{S}_3=\prec I,A,B\succ$ be any extension of $\mathcal{S}_2$ to a $\mathrm{NLS}_3(3,2)$ of non-Singer type, that is with $BA\neq AB$. It follows that both B and $A^{-1}B$ are fixed-point free. So, for given A, we seek solutions $B\in\mathrm{GL}(3,2)$ of

$$A^7=B^7=(A^{-1}B)^7=I,\quad AB\neq BA\,. \tag{8}$$

For fixed nonzero $v\in V(3,2)$, let the 7 elements $\{v,Av,A^2v,\dots,A^6v\}$ of the Fano 7-point plane $\mathrm{PG}(2,2)=V(3,2)\smallsetminus\{0\}$ be labelled $\{0,1,2,\dots,6\}$. In the case when $A^3=I+A$ the 7 lines of the Fano plane are thus $\{013,\ 124,\ 235,\ 346,\ 450,\ 561,\ 601\}$. A particular solution of (8) is given, in terms of the permutation representation of $\mathrm{GL}(3,2)$ on $\mathrm{PG}(2,2)$, by

$$A=(0123456),\quad B_0=(0631524),\quad A^{-1}B_0=(0514623). \tag{9}$$

Moreover, cf. [17, lemma 4.2], any other solution B of the conditions (8) is of the form $B_r=A^rB_0A^{-r}$, for some $r=0,1,\dots,6$. But note that the element $C_0=I+A+B_0$ sends A^4v to $(A^4+A^5+I)v=0$; similarly $C_rA^{4+r}v=0$. So $I+A+B$ does not lie in $\mathrm{GL}(3,2)$ for any solution B of (8), and so $\mathcal{S}_3$ is not a $\mathrm{NLS}_3(3,2)$. Of course a similar proof of the impossibility of constructing a 3-dimensional section $\mathcal{S}_3=\prec I,A,B\succ$ of $\mathrm{GL}(3,2)$ of non-Singer type goes through in the case when A, lying in the other conjugacy class of elements of order 7, satisfies $A^3=I+A^2$. $\blacksquare$

Remark 3.2 *A pair of elements* $A,B\in\mathrm{GL}(3,2)$ *satisfying (8) gives rise to a maximal partial spread of size* 5 *in the space* $V(6,2)=X\oplus Y$, *where* X,Y *are two copies of* $V(3,2)$. *In terms of* $(x,y)\in X\oplus Y$ *the* 5 *components of the partial spread have equations* $x=0,\ y=0,\ y=x,\ y=Ax,\ y=Bx$. *In projective terms these are the equations of* 5 *mutually skew planes in* $\mathrm{PG}(5,2)$. *While far from obvious from the point of view of these equations, it turns out that the underlying 35-set of these* 5 *planes supports an "opposite" maximal partial spread, giving rise to the* double-five *configuration of planes in* $\mathrm{PG}(5,2)$ *considered in [17], [16], [14].*

The 511 elements of $\mathrm{PG}(8,2) = \mathrm{End}(3,2) \setminus \{0\}$ comprise 168 of rank 3, 294 of rank 2 and 49 of rank 1. A given choice of Singer group $\langle A \rangle \cong Z_7$ gives rise to two partitions of each of these subsets into families of mutually skew 7-point planes. For the first we use the orbits under the left action $X \mapsto AX, A \in Z_7$, of the Singer group, and for the second we use the right action. In the case of the 49 elements of rank 1, the two partitions into 7 skew planes form "double-seven" configurations, cf. [14]; in fact we have a Segre variety $\mathcal{S}_{2,2}$, cf. [7].

Concerning the intersections of the two families of 24 mutually skew planes for the 168 elements of $\mathrm{GL}(3,2)$, these are best considered in terms of corresponding properties of the left and right cosets of the normalizer $F_{21} \cong Z_7 \rtimes Z_3$ of Z_7, each coset of F_{21} consisting of three mutually skew planes. One finds that the 7 left cosets $\{L_1, \ldots, L_7\}$ of F_{21}, other than F_{21}, intersect the 7 right cosets $\{R_1, \ldots, R_7\}$ of F_{21}, other than F_{21}, uniformly in the manner $|L_i \cap R_j| = 3$, $i, j \in \{1, 2, \ldots, 7\}$. Moreover, for any particular F_{21} subgroup, there exists a natural bijection $L_i \leftrightarrow R_i$, $i = 1, 2, \ldots, 7$, which arises as follows. The group F_{21} possesses seven subgroups $Z_3^a \cong Z_3$, where Z_3^a keeps fixed the point $a \in \mathrm{PG}(2,2)$. The normalizer of Z_3^a in $\mathrm{GL}(3,2)$ is a subgroup $S_3^a \cong S_3 \cong Z_3 \rtimes Z_2$. Now $S_3^a \setminus Z_3^a$ consists of three involutions, say J_a, J'_a, J''_a, and the product of any two of these involutions lies in Z_3^a, and hence in F_{21}. So J_a, J'_a, J''_a lie in the same left coset of F_{21}, say L_a, and also in the same right coset, say R_a. So our bijection is $L_a \leftrightarrow R_a$, with the seven "diagonal" intersections $L_a \cap R_a$ accounting for the entire class of 21 involutions in $\mathrm{GL}(3,2)$. Moreover one find that all 42 off-diagonal intersections $L_a \cap R_b$, $a \neq b$, uniformly have the same order pattern $3, 4, 7$.

4 Linear sections of $\mathrm{GL}(4,2)$: preliminaries

4.1 The 2-dimensional sections of $\mathrm{GL}(4,2)$

By lemma 2.1 we need only those classes of $\mathrm{GL}(4,2)$ which are fixed-point-free (f.p.f.) on $\mathrm{PG}(3,2)$. So of relevance are the five classes listed in table 1:

Table 1. The f.p.f. classes of $\mathrm{GL}(4,2)$

Class	Length	Minimal polynomial	cycle type
$\mathcal{C}_3$	112	t^2+t+1	3^5
$\mathcal{C}_5$	1344	$t^4+t^3+t^2+t+1$	5^3
$\mathcal{C}_6$	1680	t^4+t^2+1	$6^2 3$
$\mathcal{C}_{15}$	1344	t^4+t+1	15
$\mathcal{C}'_{15}$	1344	t^4+t^3+1	15

The Singer elements, of order 15, see classes $\mathcal{C}_{15}, \mathcal{C}'_{15}$, were noted in example 2.6, each $A \in \mathcal{C}_{15} \cup \mathcal{C}'_{15}$ permuting the 15 points of $\mathrm{PG}(3,2)$ in a single cycle. So A^3, of order 5, permutes the 15 points in three cycles of length 5, and A^5, of order 3, permutes the 15 points in five cycles of length 3: see classes $\mathcal{C}_5, \mathcal{C}_3$. Finally there is a class $\mathcal{C}_6$ of length 1680 consisting of those elements of order 6 which permute the 15 points in two cycles of length 6 and one of length 3.

Lemma 4.1 *Up to equivalence there are just three types of* $\mathrm{NLS}_2(4,2)$*'s, with a section* $\mathcal{S}_2$ *belonging to type (i), (ii) or (iii) according as its group* $\mathcal{G}(\mathcal{S}_2)$ *is isomorphic to*

$$\textit{(i)}\ Z_{15} \quad \textit{(ii)}\ Z_3 \quad \textit{(iii)}\ (Z_2)^2 \times Z_3.$$

Type (i) splits into two conjugacy classes, with order patterns $5(15)$ *and* $(15)^2$*, while each of the types (ii) and (iii) consists of a single conjugacy class, of order pattern (ii)* 3^2 *and (iii)* 6^2*, respectively.*

If $\mathcal{S}_2$ *is a section of type (iii), with group* $\mathcal{G} \cong (Z_2)^2 \times Z_3$*, then* $\mathcal{G} \cup \{0\}$ *contains three* $\mathrm{NLS}_2(4,2)$*'s of type (iii) and also a* $\mathrm{NLS}_2(4,2)$ *of type (ii). Moreover the linear span* $\prec \mathcal{G} \succ$ *has dimension* 4*, the* 15 *points of the associated projective space* $\mathrm{PG}(3,2) = \prec \mathcal{G} \succ \setminus \{0\}$ *being the* 12 *elements of* $\mathcal{G}$ *along with a line of* 3 *singular elements* $\{I + J_i : i = 1,2,3\}$*, where* $\{I, J_1, J_2, J_3\}$ *is the* $(Z_2)^2$ *subgroup of* $\mathcal{G}$*.*

Proof. If A lies in $\mathcal{C}_{15} \cup \mathcal{C}'_{15} \cup \mathcal{C}_5 \cup \mathcal{C}_3$ then $\mathcal{S}_2 = \prec I, A \succ$ is a $\mathrm{NLS}_2(4,2)$ of sub-Singer kind, as considered in example 2.6, thus giving rise to the equivalence types (i) and (ii) of the lemma. By the preamble to the lemma, the only other possibility is for A to lie in $\mathcal{C}_6$. It then *must* be the case that $B = I + A$ lies in $\mathcal{C}_6$, since there are no further classes consisting of fixed-point-free elements. In fact a direct proof that B satisfies $B^4 = B^2 + I$ and is of order 6 is easily given: $B^4 = (I + A)^4 = I + A^4 = A^2 = (I + B)^2 = I + B^2$, and so $B^6 = B^2 + B^4 = I$. On setting $J_1 = B^3$, $J_2 = A^3$ and $W = B^2 = A^4$, the abelian group $\mathcal{G} = \langle A, B\rangle$, generated by the commuting elements A, B of order 6, is seen to have the structure

$$\mathcal{G} = \langle J_1 \rangle \times \langle J_2 \rangle \times \langle W \rangle \cong Z_2 \times Z_2 \times Z_3. \tag{10}$$

Consider now the second half of the lemma, which spells out how close the subspace $\prec \mathcal{G} \succ \subset \mathrm{End}(4,2)$ spanned by $\mathcal{G}$ is to being a $\mathrm{NLS}_4(4,2)$. On setting $J_3 = J_1 J_2$, we have an abelian group $\mathcal{G} = \{I, J_1, J_2, J_3\} \times \langle W \rangle$ where the J_i are involutions, where W, of order 3, satisfies $W^2 + W + I = 0$ and so $\prec I, W \succ$ is a $\mathrm{NLS}_2(4,2)$ of type (ii), and where $J_2 W + J_1 W^2 = I$. The last relation (a re-write of $A + B = I$) gives two further relations after mutation, so that we have three $\mathrm{NLS}_2(4,2)$'s of type (iii), given by the three relations

$$J_3 W + J_2 W^2 = I, \qquad J_1 W + J_3 W^2 = I, \qquad J_2 W + J_1 W^2 = I. \tag{11}$$

Moreover we also have the relation $I + J_1 + J_2 + J_3 = 0$. For from $J_2 W + J_1 W^2 = I$ and $W^2 = W + I$ we obtain $I + J_1 = (J_1 + J_2) W = (I + J_3) J_1 W$. But $(I + J_3)^2 = I + (J_3)^2 = I + I = 0$, and so $(I + J_3)(I + J_1) = 0$, that is $I + J_1 + J_2 + J_3 = 0$. Fortified by these relations we quickly check that the projective space $\prec \mathcal{G} \succ \setminus \{0\}$ is a $\mathrm{PG}(3,2)$, since it comprises just 15 points, namely the 12 elements of $\mathcal{G}$ and the 3 elements $I + J_i$, $i = 1,2,3$, which form a projective line. (Note the rather subtle fact that the foregoing relations are symmetric only under an *even* permutation of J_1, J_2, J_3.) ∎

4.2 Aspects of the isomorphism $T : A_8 \to \mathrm{GL}(4,2)$

As is very well-known, $\mathrm{GL}(4,2)$ is isomorphic to the alternating group A_8, consisting of the even permutations of the symbols $\{1, 2, \ldots, 8\}$. Table 2 lists the five classes of A_8 which correspond to the $\mathrm{GL}(4,2)$ classes of table 1.

Table 2. Relevant classes of A_8

Class	Length	Representative
$\overline{\mathcal{C}_3}$	112	(123)
$\overline{\mathcal{C}_5}$	1344	(12345)
$\overline{\mathcal{C}_6}$	1680	(123)(45)(67)
$\overline{\mathcal{C}_{15}}$	1344	(123)(45678)
$\overline{\mathcal{C}'_{15}}$	1344	(132)(45678)

Any isomorphism $T : A_8 \to \mathrm{GL}(4,2)$ necessarily maps $\overline{\mathcal{C}}_5$ onto $\mathcal{C}_5$, $\overline{\mathcal{C}}_3$ onto $\mathcal{C}_3$ and $\overline{\mathcal{C}}_6$ onto $\mathcal{C}_6$. In the following we *choose* to deal with an isomorphism with effect

$$T : \overline{\mathcal{C}}_{15} \mapsto \mathcal{C}_{15}, \quad \overline{\mathcal{C}}'_{15} \mapsto \mathcal{C}'_{15} \tag{12}$$

on the elements of order 15. Of course if θ is the outer automorphism of A_8 defined by $\sigma \mapsto \rho\sigma\rho^{-1}$ where $\rho \in S_8 \setminus A_8$ is any odd permutation of 12345678, then $T' = T \circ \theta : A_8 \to \mathrm{GL}(4,2)$ will be an isomorphism with the opposite effect

$$T' : \overline{\mathcal{C}}_{15} \mapsto \mathcal{C}'_{15},\ \overline{\mathcal{C}}'_{15} \mapsto \mathcal{C}_{15}. \tag{13}$$

Lemma 4.2 *Let* $T : A_8 \to \mathrm{GL}(4,2)$ *be as in (12), let* $ijklmnrs$ *denote an arbitrary even permutation of* 12345678 *and put* $\omega = (ijk)$ *and* $\phi = (lmnrs)$*. Then the following three relations hold in* $\mathrm{End}(4,2)$*:*

$$T_{\omega\phi} + T_{\omega\phi^{-1}} = I, \quad T_{\omega^2\phi} + T_{\phi^2} = I, \quad T_\omega + T_{\omega^{-1}} = I. \tag{14}$$

The first two relations yield the two mutant versions of a $\mathrm{NLS}_2(4,2)$ *of equivalence type (i), see lemma 4.1, either version generating the group* $\langle T_{\omega\phi}\rangle \cong Z_{15}$*, and the third relation yields a* $\mathrm{NLS}_2(4,2)$ *of type (ii), generating the group* $\langle T_\omega\rangle \cong Z_3$*.*

The following relations also hold:

$$\begin{aligned} T_{(ijk)(lm)(nr)} + T_{(ikj)(nl)(mr)} &= I \\ T_{(ijk)(mn)(lr)} + T_{(ikj)(lm)(nr)} &= I \\ T_{(ijk)(nl)(mr)} + T_{(ikj)(mn)(lr)} &= I. \end{aligned} \tag{15}$$

They yield three mutant versions of a $\mathrm{NLS}_2(4,2)$ *of type (iii), see lemma 4.1 and Equation (11), with each version generating the same group*

$$\langle T_\omega\rangle \times \{I, T_{(mn)(lr)}, T_{(nl)(mr)}, T_{(lm)(nr)}\} \cong Z_3 \times (Z_2)^2.$$

Proof. By (12), $T_{\omega\phi}$ has minimal polynomial $t^4 + t + 1$, and so we have the first of the relations (14). Multiplying this by $T_{\omega^2\phi}$ yields the second relation. The third of the relations (14) holds, since T_ω lies in $\mathcal{C}_3$ and so has minimal polynomial $t^2 + t + 1$.

Now the second relation in (14) reads $T_{(ikj)(lmnrs)} + T_{(lnsmr)} = I$, and on multiplying this from the left by $T_{(snl)}$ we obtain the first of the following two relations

$$T_{(ikj)(lm)(nr)} + T_{(smr)} = T_{(snl)}, \quad T_{(ijk)(mn)(lr)} + T_{(srm)} = T_{(snl)}, \tag{16}$$

with the second following from the first upon conjugating with $T_{(jk)(mr)}$. Addition of the last two relations yields the second of the relations (15), and the other two follow upon conjugating twice with $T_{(lmn)}$. The tie-in with lemma 4.1, and in particular of Equation (15) with Equation (11), is clear: set $W = T_\omega$ and let J_1, J_2, J_3 be any *even* permutation of $T_{\kappa_1}, T_{\kappa_2}, T_{\kappa_3}$, where

$$\kappa_1 = (mn)(lr), \quad \kappa_2 = (nl)(mr), \quad \kappa_3 = (lm)(nr). \qquad \blacksquare \tag{17}$$

5 Linear sections of $\mathrm{GL}(4,2)$: summary of results

The following theorem summarizes our main results.

Theorem 5.1 *There are just two equivalence classes of 3-dimensional maximal normalized linear sections of* $\mathrm{GL}(4,2)$*, say* $\mathcal{M}_3$ *and* $\mathcal{M}'_3$*, and three equivalence classes of 4-dimensional sections, say* $\mathcal{M}_4, \mathcal{M}'_4$ *and* $\mathcal{M}''_4$*. Information concerning these equivalence classes is displayed in* Table 3 *below. In the table the second column indicates the structure of the group* $\mathcal{G}(\mathcal{S})$ *generated by a(ny) representative* $\mathcal{S}$ *of a class, and the third column indicates, for* $\mathcal{F} \ni \mathcal{S}$*, the structure of the group* $\mathcal{H}(\mathcal{F})$*, see equation (5). The fourth column lists the associated order patterns, one for each conjugacy type of section, and the final column indicates the structure of the group* $\mathcal{H}_0(\mathcal{S})$*, see equation (6), for these conjugacy types. In the case of* $\mathcal{S} \in \mathcal{M}'_4$ *each of the groups* $\mathcal{G}_0(\mathcal{S}), \mathcal{G}'_0(\mathcal{S})$*, see equation (7), is isomorphic to* Z_3*; in all other cases the groups* $\mathcal{G}_0, \mathcal{G}'_0$ *are trivial.*

Proof. Concerning existence, in succeeding sections we give explicit constructions of linear sections belonging to the four non-Singer classes $\mathcal{M}_3, \mathcal{M}'_3, \mathcal{M}'_4, \mathcal{M}''_4$. We also provide information there concerning the $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{H}_0$ groups. See equation (30) for the $\mathcal{G}_0, \mathcal{G}'_0$ groups in the case of $\mathcal{S} \in \mathcal{M}'_4$. However we made repeated use of the computer algebra system MAGMA, see [1], in order to check maximality in respect of the classes $\mathcal{M}_3, \mathcal{M}'_3$, and especially to prove that our list of five classes was complete. (We also found MAGMA helpful as a back-up to check the accuracy of our statements concerning the $\mathcal{H}$ and $\mathcal{H}_0$ groups.) $\blacksquare$

Table 3. Equivalence classes of maximal $\mathrm{NLS}_r(4,2)$'s

Class	$\mathcal{G}$	$\mathcal{H}$	Order patterns	$\mathcal{H}_0$
$\mathcal{M}_3$	A_7	$\mathrm{GL}(3,2)$	6^6	S_4
$\mathcal{M}'_3$	$\mathrm{GL}(4,2)$	Z_7	$5^2(15)^4$	$\{I\}$
$\mathcal{M}_4$	Z_{15}	$Z_{15} \rtimes Z_4$	$3^2 5^4 (15)^8$	$Z_{15} \rtimes Z_4$
$\mathcal{M}'_4$	$\mathrm{GL}(2,4)$	$S_3 \times S_3$	$3^4 5^4 6^2 (15)^4$	$S_3 \times Z_2$
			$3^2 6^6 (15)^6$	$Z_3 \times S_3$
$\mathcal{M}''_4$	$\mathrm{GL}(4,2)$	$(Z_3)^2 \rtimes Z_2$	$3^2 6^6 (15)^6$	S_3
			$5^4 6^6 (15)^4$	Z_2
			$6^8 (15)^6$	S_3

Corollary 5.2 *(i) There are (at least) two inequivalent classes of maximal partial spreads of order* 9 *in* PG(7, 2).

(ii) There are precisely two non-isotopic proper semifields of order 16.

Proof. (i) As explained in section A.1 of the Appendix, if $\mathcal{S}$ is a $\mathrm{NLS}_3(4,2)$ it gives rise to a partial spread Σ in PG(7, 2) of order 9. If $\mathcal{S} \in \mathcal{M}_3$, or if $\mathcal{S} \in \mathcal{M}'_3$, then Σ is a *maximal* partial spread, for *any* extension of $\mathcal{S}$ *qua* a spread set would imply (a result peculiar to GF(2)!) a *linear* extension, contradicting the maximality of $\mathcal{S}$ as a linear section.

(ii) This is a known result: see [9], and the independent computer check in [10]. However, see section A.3, it also follows from our present results, since, leaving aside the class $\mathcal{M}_4$ associated with the *field* of order 16, we have shown that there exist precisely two other equivalence classes of $\mathrm{NLS}_4(4,2)$'s. (See also examples A.7 and A.8 for more details linking our results with those in [10].) ■

Knowing representatives for the conjugacy classes of 4-dimensional linear sections it is a relatively straightforward matter to look at their 3-dimensional subspaces. It turns out that there are five equivalence classes $\mathcal{N}_3, \mathcal{N}'_3, \mathcal{N}''_3, \mathcal{N}'''_3, \mathcal{N}^{iv}_3$ of non-maximal $\mathrm{NLS}_3(4,2)$'s. Information concerning these is given in table 4; in particular each equivalence class contains three conjugacy classes with order patterns as listed in column 4. (In the $\mathcal{N}''_3$ entry, we use the ATLAS [2] abbreviation $(2^2 \times 3){:}2$ for the structure $((Z_2)^2 \times Z_3) \rtimes Z_2$.)

Table 4. Equivalence classes of non-maximal $\mathrm{NLS}_3(4,2)$'s

Class	$\mathcal{G}$	$\mathcal{H}$	Order patterns	$\mathcal{H}_0$
$\mathcal{N}_3$	Z_{15}	$Z_{15} \rtimes Z_4$	$3^2 5^2 (15)^2$	$Z_{15} \rtimes Z_2$
			$3^2 (15)^4$	$Z_{15} \rtimes Z_4$
			$5^2 (15)^4$	Z_{15}
$\mathcal{N}'_3$	GL(2, 4)	$Z_3 \times S_3$	$3^2 6^2 (15)^2$	$Z_3 \times Z_2$
			$5^2 6^2 (15)^2$	$Z_3 \times Z_2$
			6^6	$Z_3 \times S_3$
$\mathcal{N}''_3$	GL(2, 4)	$(2^2 \times 3){:}2$	$3^2 5^2 (15)^2$	S_3
			$3^4 6^2$	$(2^2 \times 3){:}2$
			$6^2 (15)^4$	$Z_3 \times (Z_2)^2$
$\mathcal{N}'''_3$	GL(4, 2)	Z_4	$5^2 6^2 (15)^2$	$\{I\}$
			6^6	Z_4
			$6^2 (15)^4$	Z_2
$\mathcal{N}^{iv}_3$	GL(4, 2)	S_3	$5^2 6^2 (15)^2$	Z_2
			6^6	S_3
			$6^4 (15)^2$	Z_2

Remark 5.3 *Within an equivalence class it turns out that, for* GL(4, 2), *the order pattern suffices to distinguish the conjugacy classes. However linear sections with the same order pattern may be non-conjugate. For example tables 3 and 4 show that there are four distinct conjugacy classes of* $\mathrm{NLS}_3(4,2)$ *'s which share the same order pattern* 6^6.

See [4] for more information concerning $\mathrm{NLS}(n,2)$'s for $n = 4$, and for a preliminary look at the $n = 5$ case.

6 Linear sections generating an A_7 subgroup of $\mathrm{GL}(4,2)$

Concerning linear sections $\mathcal{S}$ of $\mathrm{GL}(4,2)$ belonging to the class $\mathcal{M}_3$, a recipe for their construction can be given based upon the existence in the alternating group A_7 of *7-clusters.* We now outline this recipe, but refer to [15] for further details.

Definition 6.1 *A 7-cluster is a subset* $\Pi = \{\xi_1, \xi_2, \dots, \xi_7\}$ *of* A_7 *which satisfies*

$$(\xi_i)^{-1}\xi_j \text{ is of order } 6 \text{ for all } i \neq j \in \{1, 2, \dots, 7\}. \tag{18}$$

We will restrict our attention to *normalized* 7-clusters Π, for which $\xi_i = Id$ for some i, the other six elements of Π therefore having order 6. Note that elements of A_7 of order 6 form a single class (of length 210) with representative (123)(45)(67). Since we will not deal in this note with any other kind of cluster, we will refer to a normalized 7-cluster simply as a cluster.

If Π is a cluster then so are all its *left and right mutants* $\xi^{-1}\Pi$ and $\Pi\xi^{-1}$, $\xi \in \Pi$. Each cluster Π determines a family $\Phi(\Pi) = \{\xi^{-1}\Pi : \xi \in \Pi\}$ of left mutants, and also a family $\Phi'(\Pi) = \{\Pi\xi^{-1} : \xi \in \Pi\}$ of right mutants, of Π. These families are in fact democracies since, for any $\xi \in \Pi$, we have $\Phi(\xi^{-1}\Pi) = \Phi(\Pi)$, and $\Phi'(\Pi\xi^{-1}) = \Phi'(\Pi)$.

We use some well-known facts concerning 2-(7,3,1) designs ($\mathrm{PG}(2,2)$'s, STS(7)'s) based on the point-set $\{1,2,3,4,5,6,7\}$. Under the action of S_7 such designs form a single orbit of length $7!/168 = 30$, which splits into two A_7-orbits of length 15, say Ω, represented by the design $\mathcal{D}_0$ whose triples are $124, 235, 346, 457, 561, 672, 713$, and Ω', represented by $\tau\mathcal{D}_0$ for any $\tau \in S_7 \setminus A_7$. Two designs lying on the same A_7-orbit share precisely one triple, while if on different A_7-orbits either they share three triples, or they are disjoint.

To each $\mathcal{D} \in \Omega \cup \Omega'$ we associate the 42 (distinct) elements $\xi_{ij}(\mathcal{D})$, $i \neq j \in \{1,2,3,4,5,6,7\}$, of A_7 of order 6 defined by

$$\xi_{ij}(\mathcal{D}) = (ijk)(ln)(mr), \qquad (i \neq j), \tag{19}$$

where k is such that ijk is a triple of the design $\mathcal{D}$, and where kln and kmr are the other two triples of $\mathcal{D}$ which contain k. Observe that $\xi_{ij}(\mathcal{D})^{-1} = \xi_{ji}(\mathcal{D})$. Acting by conjugation, the group $\mathrm{Aut}\,\mathcal{D} \cong \mathrm{GL}(3,2)$ is transitive on the 42 elements (19), with point-stabilizer $\cong (Z_2)^2$. We define also $\xi_{ii}(\mathcal{D}) = Id$, for each i, and so associate with each $\mathcal{D} \in \Omega \cup \Omega'$ a 7×7 array

$$\Xi(\mathcal{D}) = (\xi_{ij}(\mathcal{D}))_{i,j\in\{1,2,3,4,5,6,7\}}, \tag{20}$$

whose diagonal elements are all equal to Id and whose off-diagonal elements are the 42 permutations of equation (19), each of order 6.

Theorem 6.2 *([15]) (i) For all* $i, j, k \in \{1, 2, \dots, 7\}$, $\xi_{ij}(\mathcal{D})\xi_{jk}(\mathcal{D}) = \xi_{ik}(\mathcal{D})$.

(ii) Each row $\Pi_i(\mathcal{D}) = \{\xi_{ij}(\mathcal{D}) : j \in \{1,2,3,4,5,6,7\}\}$ *of* $\Xi(\mathcal{D})$ *is a cluster, the* 7 *rows forming a family* $\Phi(\mathcal{D})$ *of* 7 *(distinct) left mutants. Also each column* $\Pi'_i(\mathcal{D}) = \{\xi_{ji}(\mathcal{D}) : j \in \{1,2,3,4,5,6,7\}\}$ *of* $\Xi(\mathcal{D})$ *is a cluster, the* 7 *columns forming a family* $\Phi'(\mathcal{D})$ *of* 7 *(distinct) right mutants.*

(iii) Under the action of A_7 *by conjugation, the clusters in* A_7 *form two orbits* $\mathcal{O}$, $\mathcal{O}'$*, each of length* 105*. Every cluster* $\Pi \in \mathcal{O}$ *is of the form* $\Pi_i(\mathcal{D})$ *for some* $\mathcal{D} \in \Omega$ *and some* i*, and also of the form* $\Pi'_i(\mathcal{D}')$ *for some* $\mathcal{D}' \in \Omega'$ *and some* i*. Every cluster* $\Pi' \in \mathcal{O}'$ *is of the form* $\Pi_i(\mathcal{D}')$ *for some* $\mathcal{D}' \in \Omega'$ *and some* i*, and also of the form* $\Pi'_i(\mathcal{D})$ *for some* $\mathcal{D}' \in \Omega'$ *and some* i*.* ■

The duplication in part (iii) of the theorem arises from the fact that $\Pi'_i(\mathcal{D}) = \Pi_i(\mathcal{D}^i)$ where $\mathcal{D}^i$ is that design which shares with $\mathcal{D}$ precisely those three triples of $\mathcal{D}$ which contain i.

Let A_7 be that (maximal) subgroup of A_8 which fixes the symbol 8 and denote by $\mathcal{A}_7$ its image in GL(4, 2) under T of (12). For $\mathcal{D} \in \Omega \cup \Omega'$ we set

$$X_{ij}(\mathcal{D}) = T_{\xi_{ij}(\mathcal{D})}, \quad i, j \in \{1, 2, 3, 4, 5, 6, 7\}. \tag{21}$$

So each $X_{ij}(\mathcal{D})$ lies in $\mathcal{A}_7$, and $X_{ii}(\mathcal{D}) = I$ for each i. We associate with each $\mathcal{D} \in \Omega \cup \Omega'$ the 7×7 array

$$\mathbf{X}(\mathcal{D}) = (X_{ij}(\mathcal{D}))_{i,j \in \{1,2,3,4,5,6,7\}}, \tag{22}$$

the image under T of the array (20), whose 42 off-diagonal elements lie in $\mathcal{A}_7 \cap \mathcal{C}_6$. From the ith row, and the ith column, of this array, we form the 7-subsets of End(4, 2)

$$\begin{aligned} P_i(\mathcal{D}) &= \{X_{ij}(\mathcal{D}) : j \in \{1, 2, 3, 4, 5, 6, 7\}\}, & (23)\\ P'_i(\mathcal{D}) &= \{X_{ji}(\mathcal{D}) : j \in \{1, 2, 3, 4, 5, 6, 7\}\}, & (24) \end{aligned}$$

which are the images under T of the clusters $\Pi_i(\mathcal{D})$, $\Pi'_i(\mathcal{D})$.

Theorem 6.3 *([15]) (i) Given $\mathcal{D} \in \Omega$ let ijk be any of its triples and let $l \in \{1, 2, 3, 4, 5, 6, 7\} \setminus \{i, j, k\}$. Then (with T in* (21) *having effect* (12)*) the following linear relations hold in* End(4, 2)*:*

$$X_{ki}(\mathcal{D}) + X_{kj}(\mathcal{D}) + I = 0, \tag{25}$$

$$X_{li}(\mathcal{D}) + X_{lj}(\mathcal{D}) + X_{lk}(\mathcal{D}) = 0. \tag{26}$$

(ii) Put $\mathcal{S}_i(\mathcal{D}) = P_i(\mathcal{D}) \cup \{0\}$ and $\mathcal{S}'_i(\mathcal{D}) = P'_i(\mathcal{D}) \cup \{0\}$. Then, for $\mathcal{D} \in \Omega$, each $\mathcal{S}_i(\mathcal{D})$ is a NLS$_3$(4, 2). Moreover $\mathcal{H}(\mathcal{S}_i(\mathcal{D})) \cong$ GL(3, 2) and $\mathcal{H}_0(\mathcal{S}_i(\mathcal{D})) \cong S_4$. (Also, for $\mathcal{D}' \in \Omega'$, each $\mathcal{S}'_i(\mathcal{D}')$ is a NLS$_3$(4, 2).)

Proof. Equation (15ii) yields (25), and on left-multiplying (25) by $X_{lk}(\mathcal{D})$ we obtain the relation (26). It follows that each $P_i(\mathcal{D})$ is a PG(2, 2), and so each $\mathcal{S}_i(\mathcal{D})$ is a NLS$_3$(4, 2). The group $\mathcal{H} \subset$ GL(4, 2) which, acting by conjugation, preserves the family $\{\mathcal{S}_i(\mathcal{D}); i = 1, \dots, 7\}$ of left mutants is isomorphic to Aut$\mathcal{D} \cong$ GL(3, 2), and the subgroup $\mathcal{H}_0$ which preserves the ith member $\mathcal{S}_i(\mathcal{D})$ is isomorphic to Aut$\mathcal{D} \cap$ Stab$(i) \cong S_4$. ■

Remark 6.4 *If $\mathcal{D} \in \Omega$ it is should be stressed that $\mathcal{S}'_i(\mathcal{D}) = P'_i(\mathcal{D}) \cup \{0\}$ is* not *a NLS$_3$(4, 2); indeed the 7 elements of $P'_i(\mathcal{D})$, $\mathcal{D} \in \Omega$, are linearly independent. (Moreover the only 3-term linear dependencies amongst the* 43 *distinct elements of the array* (22) *are those of the kind* (25), (26), *involving* 3 *elements of a single* row *of the array* **X**.*) However if we had chosen to use an isomorphism T' with effect* (13), *rather than* (12), *then, for $\mathcal{D} \in \Omega$, the $\mathcal{S}'_i(\mathcal{D})$, and not the $\mathcal{S}_i(\mathcal{D})$, would have been the NLS$_3$(4, 2)'s.*

Under the action by conjugacy of $\mathcal{A}_7 \subset \mathrm{GL}(4,2)$ the 105 normalized linear sections $\mathcal{S}_i(\mathcal{D})$, $\mathcal{D} \in \Omega$, $i = 1, \dots, 7$, form a single conjugacy class, which is also an equivalence class. Let $A_7(s)$ be the subgroup of A_8 which fixes the symbol s, and let $\mathcal{A}_7(s) \subset \mathrm{GL}(4,2)$ denote its image under T. Our foregoing considerations dealt with the case $s = 8$, but apply equally to any $s = 1, \dots, 8$. The 2-(7,3,1) designs based on seven out of the eight points $\{1, 2, \dots, 8\}$ form two A_8 orbits, say Δ, Δ', each of length $8 \times 15 = 120$, with $\Delta = \cup_{s=1}^{8} \Omega(s)$ and $\Delta' = \cup_{s=1}^{8} \Omega'(s)$, where $\Omega(8) = \Omega$ and $\Omega'(8) = \Omega'$, and where $\Omega(s)$, $\Omega'(s)$ are the two $A_7(s)$ orbits of 2-(7,3,1) designs based on $\{1, 2, \dots, 8\} \setminus \{s\}$. Under the isomorphism T, each of the 120 designs $\mathcal{D} \in \Delta$ gives rise to seven $\mathrm{NLS}_3(4,2)$'s $\mathcal{S}_i(\mathcal{D})$, $i = 1, \dots, 7$.

Theorem 6.5 *([15]) The* 840 *elements of* $\mathcal{M}_3 = \{\mathcal{S}_i(\mathcal{D}) : \mathcal{D} \in \Delta, i = 1, \dots, 7\}$ *form a single conjugacy class, and equivalence class, of maximal* $NLS_3(4,2)$*'s, with each of the* 8 *subgroups of* $GL(4,2)$ *isomorphic to* A_7 *contributing* 105 $NLS_3(4,2)$*'s. For any* $\mathcal{S} \in \mathcal{M}_3$ *the following group isomorphisms hold.*

$$\mathcal{G}(\mathcal{S}) \cong A_7, \qquad \mathcal{H}(\mathcal{S}) \cong GL(3,2), \qquad \mathcal{H}_0(\mathcal{S}) \cong S_4. \quad \blacksquare$$

7 Linear sections generating a $\mathrm{GL}(2,4)$ subgroup of $\mathrm{GL}(4,2)$

7.1 Use of 2×2 matrices $\in \mathrm{GL}(2,4)$

In this section we will be dealing with 2×2 matrices $X \in \mathrm{GL}(2,4) \cup \{0\}$. Since elements of $\mathrm{GF}(4) = \{0, 1, w, w^2\}$ may be viewed as 2×2 matrices over $\mathrm{GF}(2)$, for example by interpreting w as the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, we may view X as a 4×4 matrix $\in \mathrm{GL}(4,2) \cup \{0\}$. The following readily proven lemma is of help in determining the orders of the various matrices X encountered below.

Lemma 7.1 *(i) If* $X \in \mathrm{GL}(2,4)$ *then* X *satisfies the equation*

$$X^2 = \tau X + \delta I, \quad \textit{where } \tau = \mathrm{tr} X \textit{ and } \delta = \det X, \tag{27}$$

this being its minimal equation except when $X \in \{I, wI, w^2 I\}$.

(ii) If $X \in \mathrm{GL}(2,4) \setminus \{I, wI, w^2 I\}$ *then*

$$X \textit{ has order} \begin{cases} 2 \\ 3 \\ 5 \\ 6 \\ 15 \end{cases} \iff (\tau, \delta) = \begin{cases} (0,1) \\ (1,1), (w, w^2) \textit{ or } (w^2, w) \\ (w, 1) \textit{ or } (w^2, 1) \\ (0, w) \textit{ or } (0, w^2) \\ (1, w), (1, w^2), (w, w) \textit{ or } (w^2, w^2) \end{cases} \tag{28}$$

(iii) If $X \in \mathrm{GL}(2,4)$ *then* $I + X \in \mathrm{GL}(2,4)$ *if and only if one of the following holds: (a)* X *has order* 3 *and either* $X \in \{wI, w^2 I\}$ *or* $(\tau, \delta) = (1,1)$ *(b)* X *has order* $5, 6$ *or* 15. $\blacksquare$

Consider the three sets $\mathcal{S} = \{X_u \mid u = (a,b) \in \mathrm{GF}(4)^2\}$ of 2×2 matrices over GF(4) defined by

$$\text{(i) } X^{\mathrm{i}}_{a,b} = \begin{pmatrix} a & wb \\ b & a+b \end{pmatrix}, \quad \text{(ii) } X^{\mathrm{ii}}_{a,b} = \begin{pmatrix} a & b^2 \\ b & a^2+b^2 \end{pmatrix}, \quad \text{(iii) } X^{\mathrm{iii}}_{a,b} = \begin{pmatrix} a & wb^2 \\ b & a^2 \end{pmatrix}. \tag{29}$$

Each set consists of 16 matrices, and for each set $\det X_u$ is nonzero for $u \neq (0,0)$. Moreover $X_u + X_v = X_{u+v}$, and $X_{1,0} = I$. Viewing elements of GF(4) as 2×2 matrices over GF(2), it follows that each set is a $\mathrm{NLS}_4(4,2)$.

In the case of the first set $\mathcal{S}^{\mathrm{i}}$, the 15 nonzero elements form an abelian group $\cong Z_{15}$, with generator $X^{\mathrm{i}}_{0,1}$. So $\mathcal{S}^{\mathrm{i}}$ is a Singer section. The other two sections are non-abelian, and generate the whole of GL(2, 4). Using lemma 7.1, we see that $\mathcal{S}^{\mathrm{ii}}$ has order type $3^4 5^4 6^2 (15)^4$ and $\mathcal{S}^{\mathrm{iii}}$ has order type $3^2 6^6 (15)^6$. In fact $\mathcal{S}^{\mathrm{ii}}$ is a left mutant of $\mathcal{S}^{\mathrm{iii}}$, since

$$(X^{\mathrm{iii}}_{1,w^2})^{-1} X^{\mathrm{iii}}_{a,b} = X^{\mathrm{ii}}_{wa+b,a+wb}.$$

Each of the sections $\mathcal{S}^{\mathrm{ii}}, \mathcal{S}^{\mathrm{iii}}$ has groups $\mathcal{G}_0, \mathcal{G}'_0$, see (7), of order 3:

$$\mathcal{G}_0(\mathcal{S}^{\mathrm{ii}}) = \left\langle X^{\mathrm{ii}}_{0,1} \right\rangle, \quad \mathcal{G}'_0(\mathcal{S}^{\mathrm{ii}}) = \left\langle X^{\mathrm{ii}}_{w,0} \right\rangle, \qquad \mathcal{G}_0(\mathcal{S}^{\mathrm{iii}}) = \left\langle X^{\mathrm{iii}}_{w,0} \right\rangle = \mathcal{G}'_0(\mathcal{S}^{\mathrm{iii}}). \tag{30}$$

Indeed note that $X^{\mathrm{ii}}_{0,1} X^{\mathrm{ii}}_{a,b} = X^{\mathrm{ii}}_{b,a+b}$, $X^{\mathrm{ii}}_{a,b} X^{\mathrm{ii}}_{w,0} = X^{\mathrm{ii}}_{wa,wb}$, $X^{\mathrm{iii}}_{w,0} X^{\mathrm{iii}}_{a,b} = X^{\mathrm{iii}}_{wa,w^2b}$ and $X^{\mathrm{iii}}_{a,b} X^{\mathrm{iii}}_{w,0} = X^{\mathrm{iii}}_{wa,wb}$. Now if $A^{-1}\mathcal{S}$ is a left mutant of $\mathcal{S}$ then so is $(GA)^{-1}\mathcal{S}$ for each $G \in \mathcal{G}_0(\mathcal{S})$. So the family $\mathcal{F}$ of left mutants of $\mathcal{S}^{\mathrm{ii}}$ (or of $\mathcal{S}^{\mathrm{iii}}$) has $15/3 = 5$ distinct members. It is easy to see that $\mathcal{F} = \{\mathcal{S}^{\mathrm{ii}}, \mathcal{S}^{\mathrm{iii}}, \mathcal{S}', \mathcal{S}'', \mathcal{S}'''\}$, where $\mathcal{S}'$ and $\mathcal{S}''$ are conjugates of $\mathcal{S}^{\mathrm{ii}}$ which consist of matrices of the form (29ii) except that the entry $a^2 + b^2$ is replaced by $a^2 + wb^2$ and $a^2 + w^2b^2$, respectively, and where $\mathcal{S}'''$ is a conjugate of $\mathcal{S}^{\mathrm{iii}}$ which consists of matrices of the form (29iii) except that the entry wb^2 is replaced by w^2b^2.

We now provide details of the $\mathcal{H}(\mathcal{S})$ and $\mathcal{H}_0(\mathcal{S})$ groups for $\mathcal{S} = \mathcal{S}^{\mathrm{ii}}$ and $\mathcal{S} = \mathcal{S}^{\mathrm{iii}}$. Let $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $(= (X^{\mathrm{iii}}_{0,w^2})^3)$, and let $K \in \mathrm{GL}(4,2)$ be defined by $(a,b) \mapsto (a^2, b^2)$. Set $Z_2 = \langle K \rangle$, $Z'_2 = \langle J \rangle$, $Z_3 = \langle wI \rangle$, $Z'_3 = \left\langle X^{\mathrm{ii}}_{w,0} \right\rangle = \left\langle X^{\mathrm{iii}}_{w,0} \right\rangle$, $S_3 = Z_3 \rtimes Z_2$ and $S'_3 = Z'_3 \rtimes Z'_2$. Of course Z_3 centralizes both $\mathcal{S}^{\mathrm{ii}}$ and $\mathcal{S}^{\mathrm{iii}}$. Then

$$\begin{aligned} \mathcal{H}_0(\mathcal{S}^{\mathrm{ii}}) &= S_3 \times Z'_2, \qquad \mathcal{H}_0(\mathcal{S}^{\mathrm{iii}}) = Z_3 \times S'_3, \\ \mathcal{H}(\mathcal{S}^{\mathrm{ii}}) &= S_3 \times S'_3 \;=\; \mathcal{H}(\mathcal{S}^{\mathrm{iii}}). \end{aligned} \tag{31}$$

Noting that conjugation by K fixes $\mathcal{S}^{\mathrm{ii}}$ (mapping $X^{\mathrm{ii}}_{a,b}$ to $X^{\mathrm{ii}}_{a^2,b^2}$) and effects the interchanges $\mathcal{S}' \leftrightarrow \mathcal{S}''$ and $\mathcal{S}^{\mathrm{iii}} \leftrightarrow \mathcal{S}'''$, it is easy to check that the $\mathcal{H}_0$ and $\mathcal{H}$ groups are at least as big as indicated. That they are no larger was confirmed using MAGMA.

The order patterns of subspaces of $\mathcal{S}^{\mathrm{ii}}$ and $\mathcal{S}^{\mathrm{iii}}$ are easily listed. In the case of $\mathcal{S}^{\mathrm{ii}}$, 3-dimensional subspaces of order patterns (a) $3^2 5^2 (15)^2$ (b) $3^4 6^2$ (c) $5^2 6^2 (15)^2$ are obtained by the restrictions (a) $a \in \mathrm{GF}(2)$ (b) $b \in \mathrm{GF}(2)$ (equivalently $X^t = X$) (c) $wa + b \in \{0, w\}$, respectively. In the case of $\mathcal{S}^{\mathrm{iii}}$, 3-dimensional subspaces of order patterns (d) 6^6 (e) $3^2 6^2 (15)^2$ (f) $6^2 (15)^4$ are obtained by the restrictions (d) $a \in \mathrm{GF}(2)$ (equivalently $\mathrm{tr} X = 0$) (e) $b \in \{0, w^2\}$ (equivalently $X^t = X$) (f) $a + b \in \mathrm{GF}(2)$, respectively. Every 3-dimensional subspace of $\mathcal{S}^{\mathrm{ii}}$ and $\mathcal{S}^{\mathrm{iii}}$ has one of the indicated order patterns. These six order patterns correspond to six conjugacy classes of $\mathrm{NLS}_3(4,2)$'s, and sort themselves out into two equivalence classes as indicated in table 4.

Concerning the $\mathcal{H}_0$ groups of these six conjugacy classes of $\mathrm{NLS}_3(4,2)$'s, we content ourselves with providing details of the structure $((Z_2)^2 \times Z_3) \rtimes Z_2$ of the largest one, namely $\mathcal{H}_0(\mathcal{S}_3)$ where $\mathcal{S}_3$ is as in case (b) above: $\mathcal{S}_3 = \{X \in \mathcal{S}^{\mathrm{ii}} \mid X^t = X\}$. This section $\mathcal{S}_3$ has order type $3^4 6^2$, with the two elements of order 6 being $A = X^{\mathrm{ii}}_{w^2,1}$ and $B = X^{\mathrm{ii}}_{w,1}$. Let $\mathcal{G}_{12}$ denote the group $\langle A, B\rangle$ generated by A and B. Since $A + B = X^{\mathrm{ii}}_{1,0} = I$, it follows, as in the proof of lemma 4.1, that $\mathcal{G}_{12} = \{I, A^3, B^3, AB\} \times Z_3 \cong (Z_2)^2 \times Z_3$, where $Z_3 = \langle B^2\rangle = \langle A^4\rangle = \langle wI\rangle$ is as in equation (31), and where $AB = J$. Now Z_3 centralizes $\mathcal{S}_3$, and one checks that, acting by conjugation, the four-group $\{I, A^3, B^3, AB\}$ is regular on the four elements of $\mathcal{S}_3$ of order 3, and of course fixes A and B. So $\mathcal{G}_{12} \subset \mathcal{H}_0(\mathcal{S}_3)$. But the condition $X^t = X$ is preserved under conjugation by K, and so $\langle K\rangle \subset \mathcal{H}_0(\mathcal{S}_3)$. Thus we arrive at the result that $\mathcal{H}_0(\mathcal{S}_3) = \mathcal{G}_{12} \rtimes \langle K\rangle$, after using MAGMA to check that $\mathcal{H}_0(\mathcal{S}_3)$ is no larger. (Note incidentally that $\mathcal{H}_0(\mathcal{S}_3)$ has centre $\langle AB\rangle = \langle J\rangle$.)

7.2 Use of permutations $\in Z_3 \times A_5$

Since $\mathrm{SL}(2,4)$ is isomorphic to A_5, and since $\mathrm{Aut}(\mathrm{GF}(4)) \cong Z_2$, note that $\mathrm{GL}(2,4) \cong Z_3 \times A_5$, and $\Gamma\mathrm{L}(2,4) \cong (Z_3 \times A_5) \rtimes Z_2$. We now give a $Z_3 \times A_5$ version of the preceding linear section $\mathcal{S}^{\mathrm{iii}}$ of order pattern $3^2 6^6 (15)^6$. Thus in A_8 terms we will be dealing with one of the $\binom{8}{3} = 56$ subgroups which respect a particular $3+5$ partition of the 8 symbols, and moreover, as far as $\mathrm{GL}(2,4)$, rather than $\Gamma\mathrm{L}(2,4)$, is concerned, one that permutes the 3 symbols cyclically.

As in lemma 4.2 let $ijklmnrs$ denote an arbitrary even permutation of 12345678. Starting out from the relation $I + T_{(lmn)} + T_{(lnm)} = 0$, we obtain, on left multiplication by $T_{(mn)(rs)}$, a relation $L_1 + L_2 + L_3 = 0$, where

$$L_1 = T_{(mn)(rs)}, \quad L_2 = T_{(nl)(rs)}, \quad L_3 = T_{(lm)(rs)}\,. \tag{32}$$

Observe that the $L_a,\ a = 1,2,3$, satisfy relations $L_2L_3 = L_3L_1 = L_1L_2 = M$, and $L_3L_2 = L_1L_3 = L_2L_1 = M^2$, where $M = T_{(lmn)}$. Setting $A_a = T_{(ijk)}L_a,\ a = 1,2,3$, then $\sum_a A_a = 0$, with each $A_a \in \mathcal{C}_6$. Moreover the A_a satisfy relations

$$A_2A_3^{-1} = A_3A_1^{-1} = A_1A_2^{-1} = M, \qquad A_3A_2^{-1} = A_1A_3^{-1} = A_2A_1^{-1} = M^2. \tag{33}$$

Upon joining to I we obtain a $\mathrm{NLS}_3(4,2)$

$$\mathcal{S}_3 = \{0, I, A_1, B_1, A_2, B_2, A_3, B_3\} \tag{34}$$

where $B_a = I + A_a,\ a = 1,2,3$; explicitly, after using Equations (15),

$$\begin{aligned} A_1 &= T_{(ijk)(mn)(rs)}, \quad A_2 = T_{(ijk)(nl)(rs)}, \quad A_3 = T_{(ijk)(lm)(rs)}, \\ B_1 &= T_{(ikj)(rm)(ns)}, \quad B_2 = T_{(ikj)(rn)(ls)}, \quad B_3 = T_{(ikj)(rl)(ms)}. \end{aligned} \tag{35}$$

Thus $\mathcal{S}_3$ has order pattern 6^6 and lies inside the image, under T, of the subgroup $\langle (ijk)\rangle \times A_5$ of A_8. We now find that $\mathcal{S}_3$ extends to a $\mathcal{S}_4 \subset T(\langle (ijk)\rangle \times A_5)$ by adjoining the elements

$$\mathcal{S}_4 \setminus \mathcal{S}_3 = \{M, M^2, MB_1, MB_2, MB_3, M^2B_1, M^2B_2, M^2B_3\}. \tag{36}$$

The six elements $MB_a,\ M^2B_a$ are explicitly

$$\begin{aligned} MB_1 &= T_{(ikj)(lmrns)}, \quad MB_2 = T_{(ikj)(lsmnr)}, \quad MB_3 = T_{(ikj)(lrmsn)}, \\ M^2B_1 &= T_{(ikj)(lnsmr)}, \ M^2B_2 = T_{(ikj)(lsnrm)}, \ M^2B_3 = T_{(ikj)(lrnms)}; \end{aligned} \tag{37}$$

they all have order 15, and so $\mathcal{S}_4$ has order pattern $3^2 6^6 (15)^6$. Observe that $\mathcal{G}_0(\mathcal{S}_4) = \langle M \rangle = \mathcal{G}_0'(\mathcal{S}_4)$, in agreement with equation(30).

8 Maximal linear sections which generate $\mathrm{GL}(4,2)$

We now give constructions, in A_8 terms, of maximal linear sections belonging to the classes $\mathcal{M}_3'$ and $\mathcal{M}_4''$. By using the relations in lemma 4.2 we may search for a $\mathrm{NLS}_3(4,2)$ as the span of two suitable $\mathrm{NLS}_2(4,2)$'s. One example, which yields a linear section $\mathcal{S}_3' \in \mathcal{M}_3'$, arises from the permutations

$$\begin{aligned} \sigma_1 &= (45678),\ \sigma_2 = (14325),\ \sigma_3 = (168)(25743), \\ \sigma_4 &= (123)(47586),\ \sigma_5 = (678)(12453),\ \sigma_6 = (168)(23475). \end{aligned} \tag{38}$$

By Equation (14ii) we have $T_{\sigma_1} + T_{\sigma_4} = I = T_{\sigma_2} + T_{\sigma_5}$. (Also, by Equation (14i) we have $T_{\sigma_3} + T_{\sigma_6} = I$.) But we also have $T_{\sigma_1} = T_{\sigma_2} + T_{\sigma_3}$, since $I = T_{\sigma_1^{-1}\sigma_2} + T_{\sigma_1^{-1}\sigma_3}$ is seen to hold as another instance of Equation (14ii). Hence

$$\mathcal{S}_3' = \{0, I, T_{\sigma_1}, \ldots, T_{\sigma_6}\} \tag{39}$$

is a $\mathrm{NLS}_3(4,2)$ of order pattern $5^2(15)^4$. Each mutant of $\mathcal{S}_3'$ also has order pattern $5^2(15)^4$ and is conjugate to $\mathcal{S}_3'$. By use of MAGMA we checked that $\mathcal{S}_3'$ has no extensions to a 4-dimensional section, and that any other non-abelian $\mathrm{NLS}_3(4,2)$ of order pattern $5^2(15)^4$ is conjugate to $\mathcal{S}_3'$. (*Abelian* sections of order pattern $5^2(15)^4$ exist, see example 2.6.) Moreover the permutations (38) generate A_8 (indeed σ_1 and σ_2 generate A_8), and so $\mathcal{G}(\mathcal{S}_3') = \mathrm{GL}(4,2)$. Concerning the $\mathcal{H}_0(\mathcal{S}_3')$ and $\mathcal{H}(\mathcal{S}_3')$ groups, note that if the set of six permutations (38) is stable under conjugation by $\rho \in A_8$, then so is the subset $\{\sigma_1, \sigma_2\}$ of permutations of order 5. But only the identity simultaneously centralizes both σ_1 and σ_2; it is also easy to check that $\rho = Id$ is the only element of A_8 which satisfies $\rho\sigma_1 = \sigma_2\rho$ and $\rho\sigma_2 = \sigma_1\rho$. So $\mathcal{H}_0(\mathcal{S}_3') = \{I\}$. Since the 7 left mutants of $\mathcal{S}_3'$ are distinct, and are conjugate to $\mathcal{S}_3'$, it follows that $\mathcal{H}(\mathcal{S}_3') \cong Z_7$.

Finally we show that there exist $\mathrm{NLS}_4(4,2)$'s which generate the full group $\mathrm{GL}(4,2)$. To this end consider the six permutations

$$\begin{aligned} \alpha_1 &= (123)(56)(78), \quad \alpha_2 = (123)(64)(78), \quad \alpha_3 = (123)(45)(78), \\ \beta_1 &= (132)(75)(68), \quad \beta_2 = (132)(76)(48), \quad \beta_3 = (132)(74)(58), \end{aligned} \tag{40}$$

whose images under T, see equation (35), are the elements $\neq 0, I$ of a $\mathrm{NLS}_3(4,2)$, say $\mathcal{S}_3$, of order pattern 6^6 and group $\mathcal{G}(\mathcal{S}_3) \cong Z_3 \times A_5$. To obtain a $\mathrm{NLS}_4(4,2)$, consider the extension of $\mathcal{S}_3$ by the images of the eight permutations

$$\begin{aligned} \sigma &= (27)(38)(465), \quad \rho = (28)(37)(456), \\ \sigma_1 &= (18657)(243), \quad \rho_1 = (17568)(243), \\ \sigma_2 &= (18467)(253), \quad \rho_2 = (17648)(253), \\ \sigma_3 &= (18547)(263), \quad \rho_3 = (17458)(263)\,. \end{aligned} \tag{41}$$

Observe that conjugation by (456) fixes σ and ρ, and effects $\gamma_1 \mapsto \gamma_2 \mapsto \gamma_3 \mapsto \gamma_1$ for $\gamma = \alpha, \beta, \sigma, \rho$.

Theorem 8.1 *There exists an equivalence class* $\mathcal{M}_4''$ *of* $\mathrm{NLS}_4(4,2)$*'s consisting of three conjugacy classes* $\mathcal{K}_1, \mathcal{K}_2$ *and* $\mathcal{K}_3$*, with respective order patterns* $3^2 6^6 (15)^6$, $5^4 6^6 (15)^4$ *and* $6^8 (15)^6$*. Moreover, for any* $\mathcal{S}_4 \in \mathcal{M}_4''$,

$$\mathcal{G}(\mathcal{S}_4) = \mathrm{GL}(4,2), \qquad \mathcal{H}(\mathcal{S}_4) \cong (Z_3 \times Z_3) \rtimes Z_2 \,. \tag{42}$$

Also $\mathcal{H}_0(\mathcal{S}_4)$ *is isomorphic to* S_3*,* Z_2 *or* S_3*, according as* $\mathcal{S}_4 \in \mathcal{K}_1, \mathcal{K}_2$ *or* $\mathcal{K}_3$*.*

Proof. By lemma 4.2, and recalling the preceding observation concerning conjugation by (456), we have $T_\sigma + T_\rho = I$, and $T_{\sigma_a} + T_{\rho_a} = I$, $a = 1,2,3$; we also have $T_\sigma + T_{\alpha_a} = T_{\sigma_a}, a = 1,2,3$, as can be checked upon multiplication on the left by $(T_\sigma)^{-1}$. These relations (more than) suffice to show that the considered extension of $\mathcal{S}_3$ by the images of the eight permutations (41) is indeed a 4-dimensional section $\mathcal{S}_4$. Note that this $\mathcal{S}_4$ has order pattern $6^8(15)^6$, and so belonging to $\mathcal{K}_3$. One finds that the family $\mathcal{F} = \mathcal{F}(\mathcal{S}_4)$ of left mutants of $\mathcal{S}_4$ comprises 15 distinct sections, with the order patterns $3^2 6^6 (15)^6$, $5^4 6^6 (15)^4$ and $6^8 (15)^6$ occurring $3, 9$ and 3 times, respectively, and that members of $\mathcal{F}$ having the same order pattern are conjugate. The index in $\mathcal{H}(\mathcal{S}_4) = \mathcal{H}(\mathcal{F})$ of the three $\mathcal{H}_0$ subgroups corresponding to the three classes $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}_3$ is thus $3, 9, 3$, respectively. In the case of the preceding section $\mathcal{S}_4 \in \mathcal{K}_3$ it is easy to check that $T_{(456)}$ and $T_{(45)(78)}$ belong to $\mathcal{H}_0(\mathcal{S}_4)$, and generate a subgroup $\cong S_3$ which in fact is the whole of $\mathcal{H}_0(\mathcal{S}_4)$. Moreover $T_{(278)}$ is seen to belong to $\mathcal{H}(\mathcal{S}_4)$, since $T_{(278)} \mathcal{S}_4 T_{(278)}^{-1} = T_\sigma^{-1} \mathcal{S}_4$. Consequently

$$\mathcal{H}(\mathcal{S}_4) = (\langle T_{(456)} \rangle \times \langle T_{(278)} \rangle) \rtimes \langle T_{(45)(78)} \rangle \cong (Z_3 \times Z_3) \rtimes Z_2 \,, \tag{43}$$

and the three $\mathcal{H}_0$ subgroups are as stated in the last assertion in the theorem. ■

We conclude by giving an example in matrix form of an $\mathcal{S}_4 \in \mathcal{K}_3$. Consider the 4×4 matrices $X_{\boldsymbol{\lambda}}$ and $Y_{\boldsymbol{\lambda}}$, $\boldsymbol{\lambda} = (\alpha, \beta, \gamma, \delta) \in \mathrm{GF}(2)^4$, defined by

$$X_{\boldsymbol{\lambda}} = \begin{pmatrix} \alpha & \beta & \delta & \gamma \\ \beta & \alpha+\beta & \gamma & \gamma+\delta \\ \gamma & \delta & \alpha+\beta & \beta \\ \delta & \gamma+\delta & \beta & \alpha \end{pmatrix}, \quad Y_{\boldsymbol{\lambda}} = \begin{pmatrix} \alpha & \beta & \beta+\delta & \gamma \\ \beta & \alpha+\beta & \beta+\gamma & \gamma+\delta \\ \gamma & \beta+\delta & \alpha+\beta & \beta \\ \delta & \gamma+\delta & \beta & \alpha \end{pmatrix}. \tag{44}$$

The set $\{X_{\boldsymbol{\lambda}}\}$ of 16 matrices is as in equation (29iii): $X_{(\alpha,\beta,\gamma,\delta)} = X^{\mathrm{iii}}_{\lambda_1+\beta w, \gamma+\delta w}$, and so is a $\mathcal{S}_4' \in \mathcal{M}_4'$ with order pattern $3^2 6^6 (15)^6$. Consider the 3-dimensional subspace $\mathcal{S}_3 = \{X_{\boldsymbol{\lambda}} \mid \beta = 0\}$ of $\mathcal{S}_4'$, of order pattern 6^6, and let us seek extensions of $\mathcal{S}_3$ to a $\mathrm{NLS}_4(4,2)$. We find that there is a unique extension of $\mathcal{S}_3$ to a $\mathrm{NLS}_4(4,2) \in \mathcal{M}_4'$, namely to $\mathcal{S}_4' = \{X_{\boldsymbol{\lambda}}\}$, and that there are precisely three other extensions, each being a $\mathcal{S}_4 \in \mathcal{M}_4''$ of order pattern $6^8(15)^6$, one of these being the set $\{Y_{\boldsymbol{\lambda}}\}$ of 16 matrices. As a check that the set $\{Y_{\boldsymbol{\lambda}}\}$ is indeed a linear section we may take advantage of the fact, peculiar to $\mathrm{GF}(2)$, that there is a *unique* function $I_1(\boldsymbol{\lambda})$ such that $I_1(\mathbf{0}) = 0$, and $I_1(\boldsymbol{\lambda}) \neq 0$ for $\boldsymbol{\lambda} \neq \mathbf{0}$, namely $I_1(\boldsymbol{\lambda}) = 1 + \Pi_i(1+\lambda_i)$, where now $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. A simple computation shows that $\det Y_{\boldsymbol{\lambda}} = I_1(\boldsymbol{\lambda})$, and so indeed $Y_{\boldsymbol{\lambda}} \in \mathrm{GL}(4,2)$ for $\boldsymbol{\lambda} \neq \mathbf{0}$.

A Appendix: Spreads, linear sections and semifield planes

In this appendix we assume a familiarity with certain well known results concerning spreads, spread sets and translation planes, as can be found in the books [3], [12] and in the Handbook chapter [8]; references may be found in these works to original sources, such as the papers of J. André and of R.H. Bruck & R.C. Bose.

A.1 Spreads, spread sets and reguli

For a vector space $V_d = V(d, q)$, we define a *(normalized) partial spread set* to be a subset $\mathcal{S}$ of $\mathrm{End}(V_d) = \mathrm{End}(d, q)$ such that

i) $0 \in \mathcal{S}$, $I \in \mathcal{S}$ (normalization condition)

ii) for all $X, Y \in \mathcal{S}$, $X \neq Y$, the linear mappings $X - Y$ are nonsingular.

If $\mathcal{S}$ satisfies the further condition

iii) $|\mathcal{S}| = q^d$,

then $\mathcal{S}$ is a *(normalized) spread set.* Because of the normalization condition i), note that $\mathcal{S} - \{0\} \subset \mathrm{GL}(V_d) = \mathrm{GL}(d, q)$.

Consider the vector space $V_{2d} = V_d \oplus V_d$, of dimension $2d$ over $\mathrm{GF}(q)$. Each partial spread set $\mathcal{S} \subset \mathrm{End}(V_d)$ gives rise to a corresponding partial spread Σ for V_{2d} as follows. The d-dimensional subspaces of V_{2d} which comprise the *components* of Σ are the subspaces U_X, $X \in \mathcal{S}$, and U_∞, where

$$\begin{aligned} U_X &= \{(v, Xv) \mid v \in V_d\}, \quad X \in \mathcal{S}, \\ U_\infty &= \{(0, v) \mid v \in V_d\}. \end{aligned} \tag{45}$$

If $\mathcal{S}$ is of order N, note that the associated partial spread Σ is of order $N + 1$; in particular, if $\mathcal{S}$ is a spread set then Σ has $q^d + 1$ components, and so is a spread for V_{2d} (each nonzero vector of V_{2d} lying in precisely one of the components (45)). In the other direction, the components of any (partial) spread Σ for V_{2d} can without loss of generality be assumed to be as in (45) for some (partial) spread set S.

If $\mathcal{S}$ is a $\mathrm{NLS}_r(d, q)$ then it is a partial spread set of order q^r, and the corresponding partial spread Σ, of order $q^r + 1$, is a spread if $r = d$. Note that a $\mathrm{NLS}_r(d, q)$ is a partial spread set $\mathcal{S}$ *of a special kind*, namely one that is closed under the formation of arbitrary linear combinations:

$$\lambda, \mu \in \mathrm{GF}(q) \text{ and } X, Y \in \mathcal{S} \Longrightarrow \lambda X + \mu Y \in \mathcal{S}.$$

One consequence of this is best described in projective terms. So let us view a partial spread Σ of d-dimensional subspaces of $V(2d, q)$ also as a partial spread of $(d-1)$-dimensional projective subspaces of the associated projective space $\mathrm{PG}(2d-1, q)$. Recall that a *regulus* of $\mathrm{PG}(2d-1, q)$ is a partial spread $\mathfrak{R}$ of order $q+1$ with the following property: if a line l meets three distinct components of $\mathfrak{R}$, then l intersects all components of $\mathfrak{R}$. Such a line l is called a transversal of $\mathfrak{R}$. If A, B and C are three mutually disjoint $(d-1)-$dimensional subspaces of $\mathrm{PG}(2d-1, q)$ there is a unique regulus $\mathfrak{R} = \mathfrak{R}(A, B, C)$ containing A, B and C (see [3, Sec. 5.1]). We say that a spread Σ of $\mathrm{PG}(2d-1, q)$ is $A - \mathit{regular}$ for an element A of Σ if the regulus $\mathfrak{R}(A, B, C)$ is contained in Σ for all B and C in $\Sigma \setminus \{A\}$. If Σ is $A-$regular for all A in Σ, we say that Σ is *regular.* (The case $q = 2$ is exceptional: if $q = 2$

then $\mathfrak{R}(A, B, C)$ has just the three components A, B, C, and so every spread of $\mathrm{PG}(2d-1, 2)$ is regular.)

The $A-$regular spreads have been studied in [11] and in [6]. If Σ is the partial spread of $\mathrm{PG}(2d-1, q)$ arising from a partial spread set $\mathcal{S} \subset \mathrm{End}(d, q)$, with $q > 2$, then the regulus $\mathfrak{R}(U_\infty, U_X, U_Y)$ belongs to Σ if and only if $(1-\lambda)X + \lambda Y$ belongs to S for all $\lambda \in \mathrm{GF}(q)$, see for example [11, proof of Teorema 5]. Consequently:

Theorem A.1 *If S is a normalized linear section of $\mathrm{GL}(d, q)$, and Σ is the associated partial spread for $\mathrm{PG}(2d-1, q)$, then the regulus $\mathfrak{R}(U_\infty, U_X, U_Y)$ is contained in Σ for all X and Y in S.*

If $dim S = d$, the spread Σ is U_∞-regular. ■

A.2 Linear sections and semifield planes

A spread Σ for V_{2d} gives rise to an associated translation plane $\mathcal{T}$ whose points are the vectors of V_{2d} and whose lines are the components of Σ together with their translates in V_{2d}. Moreover, [3, p. 221], if $q > 2$ the plane $\mathcal{T}$ is desarguesian if and only if the spread Σ is regular. For the spread Σ with components given as in (45) by a spread set $\mathcal{S} \subset \mathrm{End}(V_d) = \mathrm{End}(d, q)$, the affine plane $\mathcal{T}$ is coordinatized by a quasifield $\mathcal{D}$ defined, relative to a choice of nonzero vector $e \in V_d$, as follows. The additive group of $\mathcal{D}$ is that of V_d and the product xy is defined by

$$x(Ye) = Yx, \qquad Y \in S, \tag{46}$$

(every $y \in V_d$ being of the form Ye for a unique $Y \in S$). The quasifield $\mathcal{D}$ has the chosen nonzero vector e as identity, and its kernel

$$K(\mathcal{D}) = \{k \in \mathcal{D} \mid k(x+y) = x+y, \quad k(xy) = (kx)y, \quad \text{for all } x, y \in \mathcal{D}\} \tag{47}$$

contains $\mathrm{GF}(q)$ as a subfield. In the other direction, a finite quasifield $\mathcal{D}$, of dimension d' over its kernel $K \cong \mathrm{GF}(q')$, yields a spread set $\mathcal{R} = \{R_y \mid y \in \mathcal{D}\} \subset \mathrm{End}(d', q')$ consisting of the right multiplication operators $R_y : x \mapsto xy$.

Recall that a *semifield* is a distributive quasifield. So for a semifield $\mathcal{D}$ the right multiplication operators satisfy

$$R_{x+y} = R_x + R_y, \quad \text{for all } x, y \in \mathcal{D}. \tag{48}$$

In the other direction, if a spread set $\mathcal{S}$ is closed under addition it is easy to see that it yields, via (46), a quasifield $\mathcal{D}$ which satisfies (48), and hence is a semifield. Thus ([3, p. 220]): *a quasifield $\mathcal{D}$ described by a spread set $\mathcal{S}$ is a semifield if and only if $\mathcal{S}$ is closed under addition.* The next lemma is an immediate consequence.

Lemma A.2 *Each $\mathrm{NLS}_d(d, q)$ gives rise to a translation plane coordinatized by a semifield of order q^d.* ■

If $\mathcal{D}$ is a finite semifield of characteristic p and order p^n its additive group is a vector space V_n of dimension n over $\mathrm{GF}(p)$ (and $\mathcal{D}$ is a division algebra over $\mathrm{GF}(p)$). For $y \in \mathcal{D}$, as well as the right multiplication operator R_y, we will, when discussing isotopy, make use also of the left multiplication operator $L_y : \mathcal{D} \to \mathcal{D} : x \mapsto yx$.

Lemma A.3 *(i) For a semifield* $\mathcal{D}$ *of order* p^n *each of* L_y, R_y, $y \neq 0$, *lies in* $\mathrm{GL}(V_n)$ *and each of the mappings* $V_n \to \mathrm{End}(V_n)$ *defined by* $y \mapsto L_y$, $y \mapsto R_y$ *is a linear injection.*

(ii) Each of the images $\mathcal{L} = \mathrm{Im}L$ *and* $\mathcal{R} = \mathrm{Im}R$ *is a* $\mathrm{NLS}_n(n,p)$.

Proof. (i) Straightforward: the main point is that *linear combinations over the prime field* $\mathrm{GF}(p)$ *are merely sums of vectors.* Thus the additive property (48) implies that R is linear over $\mathrm{GF}(p)$:

$$R_{\lambda x + \mu y} = \lambda R_x + \mu R_y, \quad \text{for all } x, y \in \mathcal{D} \text{ and all } \lambda, \mu \in \mathrm{GF}(p), \tag{49}$$

with a corresponding result stemming from the additive property $L_x + L_y = L_{x+y}$ of the left multiplications. Part (ii) follows from (i). ■

It is true that for q nonprime a spread set $\mathcal{S} \subset \mathrm{End}(d,q)$ arising from a semifield may not be a $\mathrm{NLS}_d(d,q)$, since $\mathcal{S}$ may not be closed under multiplication $v \mapsto \lambda v$ by scalars $\lambda \in \mathrm{GF}(q)$. Nevertheless, in consequence of the preceding lemmas, note that *all finite semifield planes can be constructed from* $\mathrm{NLS}_n(n,p)$*'s, with p prime.*

A.3 Isotopy, equivalence, subgroups

Let $V_n = V(n,p)$, with p prime, and let $\mathcal{R}, \mathcal{R}^\circ \subset \mathrm{End}(n,p)$ be two normalized spread sets which are closed under addition (and so they are $\mathrm{NLS}_n(n,p)$'s). Let Σ, Σ° be the associated, see (45), spreads for $V_{2n} = V_n \oplus V_n$. Note that the two spreads Σ, Σ° share the three components U_∞, U_0, U_I. Let $\mathcal{T}, \mathcal{T}^\circ$ be the associated translation planes coordinatized by semifields $\mathcal{D}, \mathcal{D}^\circ$ whose identities are $e, e_0 \in V_n$ and whose multiplications, see (46), are written xy, $x \circ y$, where $x \circ (Y_0 e_0) = Y_0 x$, with $Y_0 \in \mathcal{R}^\circ$. The spread sets $\mathcal{R}, \mathcal{R}^\circ$ can be viewed as the sets $\{R_y\}, \{R_y^\circ\}$ of right multiplication operators in the two semifields.

Theorem A.4 *The following statements are equivalent:*

(i) The normalized linear sections $\mathcal{R}, \mathcal{R}^\circ$ *of* $\mathrm{GL}(n,p)$ *are equivalent.*

(ii) There exists $D \in \mathrm{GL}(2n,p)$ *which maps* Σ *onto* Σ° *and which fixes both* U_∞ *and* U_0.

(iii) There is an isotopy (P,Q,S) *of* $\mathcal{D}$ *onto* $\mathcal{D}^\circ$ *: that is*

$$S(xy) = Px \circ Qy, \qquad \textit{for all } x, y \in V_n\,, \tag{50}$$

holds for the triple (P,Q,S) *of elements of* $\mathrm{GL}(n,p)$.

(iv) The semifield planes $\mathcal{T}, \mathcal{T}^\circ$ *are isomorphic.*

Proof. For $P, S \in \mathrm{GL}(n,p)$ define $D_{P,S} \in \mathrm{GL}(2n,p)$ by $D_{P,S}(x,y) = (Px, Sy)$ and note that $D_{P,S}$ fixes U_∞ and U_0 and maps U_Y onto U_{Y_0}, where $Y_0 = SYP^{-1}$. Now if $\mathcal{R}, \mathcal{R}^\circ$are equivalent then

$$\text{there exist } P, S \in \mathrm{GL}(n,p) \text{ such that } Y \in \mathcal{R} \Rightarrow Y_0 \equiv SYP^{-1} \in \mathcal{R}^\circ. \tag{51}$$

So $D_{P,S}$ maps $U_Y \in \Sigma$ onto $U_{Y_0} \in \Sigma^\circ$ and fixes U_∞. Hence (i) implies (ii). In the other direction, if $D \in \mathrm{GL}(2n,p)$ fixes both U_∞ and U_0, i.e. respects the direct sum decomposition $V_{2n} = U_\infty \oplus U_0$, then $D = D_{P,S}$ for some $P, S \in \mathrm{GL}(n,p)$, and the reverse implication (ii) ⇒ (i) is seen to follow.

Note that in (51) Y_0 depends linearly upon Y. So, for $Y = R_y,\ y = Ye$ and $Y_0 = R^\circ_{y_0},\ y_0 = Y_0e_0$, we have $y_0 = Qy$ for some $Q \in \mathrm{GL}(n,p)$. Thus the equivalence of $\mathcal{R}, \mathcal{R}^\circ$ as in (51) entails the existence of a triple (P, Q, S) of elements of $\mathrm{GL}(n,p)$ which satisfy

$$SR_y = R^\circ_{Qy}P, \qquad \text{for all } y \in V_n\,. \tag{52}$$

Hence P, Q, S satisfy (50). In the other direction, given (50), and hence (52), we obtain (51). Hence (i) is equivalent to (iii).

The equivalence of (iii) and (iv) is well-known: ([3, p. 135], [8, p. 153]). ■

Remark A.5 *If instead we dealt with spread sets $\mathcal{R}, \mathcal{R}^\circ \subset \mathrm{End}(d,q)$, closed under addition, then in part (i) we would need to allow* semilinear *equivalence, and in part (ii) have $D \in \Gamma\mathrm{L}(2d,q)$.*

The isotopies of the semifield $\mathcal{D}$ onto itself form a group $\mathcal{A}(\mathcal{D})$, the *autotopy group* of $\mathcal{D}$. An *automorphism* of $\mathcal{D}$ is an autotopy with $P = Q = S$, and all such P, satisfying therefore

$$P(xy) = (Px)(Py), \qquad \text{for all } x, y \in V_n, \tag{53}$$

form the *automorphism group* $\mathcal{A}_0(\mathcal{D})$ of $\mathcal{D}$. The left, middle and right nuclei N_l, N_m, N_r of $\mathcal{D}$ are fields, [3, p. 134], say $\mathrm{GF}(q_l), \mathrm{GF}(q_m), \mathrm{GF}(q_r)$, which contain $\mathrm{GF}(p)$, and their nonzero elements $N_l^\times, N_m^\times, N_r^\times$ are thus, under multiplication, cyclic groups of orders $q_l - 1$, $q_m - 1$, $q_r - 1$. Let us relate these groups to ones defined in terms of the $\mathrm{NLS}_n(n,p)$ given by the right multiplications $\mathcal{R} = \{R_y \mid y \in \mathcal{D}\}$ of the semifield.

In section 2.2, we noted that a $\mathrm{NLS}(n,p)$ gave rise to various subgroups $\mathcal{G}$, $\mathcal{G}_0$, $\mathcal{G}'_0$, $\mathcal{H}$, $\mathcal{H}_0$ of $\mathrm{GL}(n,p)$. Recall, lemma 2.8, that both $F_0 = \mathcal{G}_0 \cup \{0\}$ and $F'_0 = \mathcal{G}'_0 \cup \{0\}$ are fields (even for a $\mathrm{NLS}_r(n,q)$ with $r < n$). Define also the subgroup $\mathcal{C}(\mathcal{R}) \subseteq \mathcal{H}_0(\mathcal{R})$ to be the centralizer of $\mathcal{R} \setminus \{0\}$ in $\mathrm{GL}(n,p)$. Because $\mathcal{R}$ acts irreducibly it follows (via Schur's lemma and Wedderburn's theorem) that $\mathcal{C}(\mathcal{R}) \cup \{0\}$(the *commutant* $[\mathcal{R}]$ of $\mathcal{R}$) is also a field.

Theorem A.6 *(i) $\mathcal{C}(\mathcal{R}) \cong N_l^\times$; $\quad \mathcal{G}_0(\mathcal{R}) \cong N_r^\times$, $\quad \mathcal{G}'_0(\mathcal{R}) \cong N_m^\times$.*

(ii) The mapping $(P, Q, S) \mapsto P$ is a homomorphism of $\mathcal{A}(\mathcal{D})$ onto $\mathcal{H}(\mathcal{R})$ whose kernel is isomorphic to $\mathcal{G}_0(\mathcal{R})$. In particular $|\mathcal{A}(\mathcal{D})| = (q_r - 1)\,|\mathcal{H}(\mathcal{R})|$.

(iii) The automorphism group $\mathcal{A}_0(\mathcal{D})$ is that subgroup $\mathrm{Fix}\,e$ *of $\mathcal{H}_0(\mathcal{R})$ which fixes the identity e of the semifield $\mathcal{D}$.*

Proof. (i) For $C \in \mathrm{GL}(n,p)$ it is easy to see that

(a) $C \in \mathcal{C}(\mathcal{R})$ if and only if $C = L_c$ for $c \in N_l^\times$;
(b) $C \in \mathcal{G}_0(\mathcal{R})$ if and only if $C = R_c$ for $c \in N_r^\times$;
(c) $C \in \mathcal{G}'_0(\mathcal{R})$ if and only if $C = R_c$ for $c \in N_m^\times$.

For example, $C \in \mathcal{G}'_0(\mathcal{R})$ if and only if for each $y \in V_n$ there exists $y' \in V_n$ such that $R_yC = R_{y'}$, i.e. such that $(Cx)y = xy'$. But from $(Cx)y = xy'$ it follows on setting $x = e$ that $y' = cy$, where $c = Ce \neq 0$, and on setting $y = e$ (and so $y' = c$) that $C = R_c$. So $(Cx)y = xy'$ reads $(xc)y = x(cy)$, which last is the condition for $c(\neq 0)$ to belong to $N_m^\times$.

(ii) If $(P, Q, S) \in \mathcal{A}(\mathcal{D})$ then, see (52),

$$PR_yP^{-1} = X^{-1}R_{Qy}, \tag{54}$$

where $X = SP^{-1}$. So $P \in \mathcal{H}(\mathcal{R})$. Conversely, if $P \in \mathcal{H}(\mathcal{R})$ then (54) holds for some $Q \in \mathrm{GL}(n, p)$ (and with $X = R_q$, where $q = Qe$). So $(P, Q, S) \in \mathcal{A}(\mathcal{D})$, where $S = R_qP$. Hence $\phi : (P, Q, S) \mapsto P$ is an epimorphism $\mathcal{A}(\mathcal{D}) \to \mathcal{H}(\mathcal{R})$. Taking $x = e$ in (50) observe that $(R_qP =)S = L_{Pe}Q$, and so $(P, Q, S) = (P,\ L_{Pe}^{-1}R_qP,\ R_qP)$. In order to determine the kernel of ϕ, set $P = I$ and observe that if (I, Q, S) is an autotopy then $Q = S$ and, from (50), $SR_y = R_{Sy}$, whence $S \in \mathcal{G}_0(\mathcal{R})$.

(iii) Equation (53) asserts that $PR_yP^{-1} = R_{Py}$, and hence that $P \in \mathcal{H}_0(\mathcal{R})$. Also $(P, L_{Pe}^{-1}R_qP, R_qP) = (P, P, P)$ if and only if $R_q = I$ and $Pe = e$. ∎

Example A.7 *For* $\mathcal{S} = \mathcal{S}^{ii}$, *or* $\mathcal{S} = \mathcal{S}^{iii}$, *and so* $\mathcal{S} \in \mathcal{M}_4'$, *we found, see Equation (31), that* $\mathcal{H}(\mathcal{S}) \cong S_3 \times S_3$ *is of order* 36. *Also, Equation (30),* $\mathcal{G}_0(\mathcal{S}) \cong Z_3$. *Hence, by theorem A.6(ii), the number* $|\mathcal{A}(\mathcal{D})|$ *of autotopies is* $3 \times 36 = 108$ — *in agreement with the number given by Knuth for the semifield W in [10, p. 209].*

For $\mathcal{S} = \mathcal{S}^{ii}$ *we find from Equation (31) that* $|\mathrm{Fix}\, e| = 4$, *for* 3 *choices of* e, *and* $= 2$ *for* 12 *choices, while for* $\mathcal{S} = \mathcal{S}^{iii}$, $|\mathrm{Fix}\, e| = 3$, *for* 6 *choices of* e, *and* $= 2$ *for* 9 *choices. So, by theorem A.6(iii), the semifields arising from class* $\mathcal{M}_4'$ *have at most four automorphisms.*

Example A.8 *For* $\mathcal{S} \in \mathcal{M}_4''$ *we have* $|\mathcal{H}(\mathcal{S})| = 18$, $|\mathcal{G}_0(\mathcal{S})| = 1$, *see theorems 8.1, 5.1. Hence the number* $|\mathcal{A}(\mathcal{D})|$ *of autotopies is* 18 — *in agreement with the number given by Knuth for the semifield V in [10, p. 209]. Let* $\mathcal{S}$ *be the left mutant* $(T_{\alpha_1})^{-1}\mathcal{S}_4$, *where* $\mathcal{S}_4$ *is the section arising from the permutations (40), (41). Then* $\mathcal{S}$ *has order pattern* $3^2 6^6 (15)^6$ *and its group* $\mathcal{H}_0(\mathcal{S})$ *is a subgroup* $\cong S_3$ *of the group (43). In fact we find that* $\mathcal{H}_0(\mathcal{S}) = T(K_0)$ *where* $K_0 = \langle (456)(287) \rangle \rtimes \langle (45)(28) \rangle$. *Now it is easy to see that there exist two 3-(8,4,1) designs* $\mathbf{D}$ *and* $\mathbf{D}'$ *each of which is preserved by* K_0; *moreover the designs lie on different* A_8*-orbits, since* $\mathbf{D}' = (13)\mathbf{D}$. *Consequently, cf. [2, p. 22],* $\mathcal{H}_0(\mathcal{S})$ *stabilizes both a point* $e \in \mathrm{PG}(3, 2)$ *and a plane* $\pi \subset \mathrm{PG}(3, 2)$. *(In fact* $e \in \pi$, *since* $\mathbf{D}$ *and* $\mathbf{D}'$ *share the three blocks* 1324, 1385, 1376, *together with the complements of these blocks.) So* $|\mathrm{Fix}\, e| = 6$, *and the semifield using e as identity has six automorphisms, again in agreement with [10, p. 209].*

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