New partial geometries constructed from old ones

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Abstract

R. Mathon and A. Street constructed seven new partial geometries $\mathrm{pg}(7,8,4)$ partly by computer. We generalize this construction and give a computerfree proof of the fact that one can derive from the partial geometry $\mathrm{PQ}^+(7,2)$, constructed by F. De Clerck, R. H. Dye and J. A. Thas, exactly three of those partial geometries using this construction.

1 Introduction

A *partial linear space* $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order (s,t) is a (finite) incidence structure such that each point is incident with $t+1$ lines, each line is incident with $s+1$ points and two different points are incident with at most one line.

A *partial geometry* $\mathrm{pg}(s,t,\alpha)$ is a partial linear space of order (s,t) such that for each anti-flag (x,L) the incidence number $\alpha(x,L)$, being the number of points on L collinear with x, is a constant α $(\neq 0)$. The numbers s, t and α are called the *parameters* of $\mathcal{S}$. The partial geometries are introduced by Bose [1]. Note that the dual structure of a partial geometry is again a partial geometry and that

$$|\mathcal{P}| = v = (s+1)\frac{(st+\alpha)}{\alpha} \quad \text{and} \quad |\mathcal{L}| = b = (t+1)\frac{(st+\alpha)}{\alpha}.$$

The point graph $\Gamma(\mathcal{S})$ of a partial geometry $\mathcal{S}$ is an

$$\mathrm{srg}\Big((s+1)\frac{(st+\alpha)}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\Big).$$

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Each strongly regular graph Γ having parameters of this form with $t \geq 1$, $s \geq 1$, $1 \leq \alpha \leq s+1$ and $1 \leq \alpha \leq t+1$ is called a *pseudo–geometric* (s,t,α)–graph. If the graph Γ is indeed the point graph of at least one partial geometry, then Γ is called *geometric*. Translating the necessary conditions for strongly regular graphs yields conditions on the existence of partial geometries in terms of the parameters, see [6] for more details, examples and the status of the theory up to 1995.

A family of partial geometries $\mathrm{pg}(2^{2n-1}-1, 2^{2n-1}, 2^{2n-2})$ and sometimes denoted by $\mathrm{PQ}^+(4n-1,2)$ is constructed in [4]. We recall here briefly the construction. Let $\mathrm{Q}^+ = \mathrm{Q}^+(4n-1,2)$, $n \geq 2$, be a hyperbolic quadric in $\mathrm{PG}(4n-1,2)$. It is well-known that the set of maximal totally isotropic subspaces (having dimension $2n-1$) on Q^+ can be divided into two disjoint families $\mathcal{D}_1$ and $\mathcal{D}_2$. Two maximal totally isotropic subspaces belong to the same family iff their intersection has an odd dimension. A spread $\Sigma = \{\sigma_0, \ldots, \sigma_{2^{2n-1}}\}$ of Q^+, is a (maximal) set of $2^{2n-1}+1$ disjoint $(2n-1)$-dimensional spaces on Q^+. All the elements of Σ belong to the same family, without loss of generality we will assume in the sequal that they belong to $\mathcal{D}_1$. We will fix Σ and refer to it as an *orthogonal spread*. Let Ω be the set of all hyperplanes of the elements of Σ. Consider the incidence structure $\mathrm{PQ}^+(4n-1,2) = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $\mathcal{P}$ the set of points of $\mathrm{PG}(4n-1,2)$ not on the quadric, $\mathcal{L} = \Omega$ and $x \mathrm{~I~} L$, $x \in \mathcal{P}$ and $L \in \mathcal{L}$, if and only if x is contained in the polar space $L^\star$ of L with respect to Q^+. One can prove that $\mathrm{PQ}^+(4n-1,2)$ is indeed a $\mathrm{pg}(2^{2n-1}-1, 2^{2n-1}, 2^{2n-2})$.

For $q=3$ an analogous construction is given in [12], but the point set $\mathcal{P}$ is in this case restricted to the set $\{p = \langle v \rangle \| Q(v) = 1\}$. The incidence structure $\mathrm{PQ}^+(4n-1,3)$ is a partial geometry with parameters $s = 3^{2n-1}-1$, $t = 3^{2n-1}$, $\alpha = 2 \cdot 3^{2n-2}$. Up to now it is only known that $\mathrm{Q}^+(7,3)$ has a spread which yields a $\mathrm{pg}(26,27,18)$.

Cohen [3] was the first to construct a $\mathrm{pg}(8,7,4)$ using the root system E_8. In [7] Haemers and Van Lint constructed a $\mathrm{pg}(8,7,4)$ using coding theory. Kantor [8] proved that $\mathrm{PQ}^+(7,2)$ and the dual of the geometry of Haemers–Van Lint are isomorphic. Later on Tonchev [13] showed with the help of a computer that the model of Cohen and the dual of the geometry of Haemers–Van Lint are isomorphic. In [5] this isomorphism is proved without the use of a computer. It is known that the point graph of $\mathrm{PQ}^+(7,2)$ does not yield other partial geometries [5] (see also [8]). For a long time it has been conjectured that any $\mathrm{pg}(7,8,4)$ had to be isomorphic to $\mathrm{PQ}^+(7,2)$. However in [11] a construction technique for new partial geometries from other ones with the same parameters has been introduced which yield, using some computer search, seven new $\mathrm{pg}(7,8,4)$. We will generalize this construction and give a geometric proof.

Remarks

1. Four of these geometries were independently found by M. Klin and S. Reichard (private communication). They are using another construction technique, but also here some computer calculations were involved.

2. Mathon has recently proved that the Hermitian graphs $\mathcal{H}(q)$ (also called Talyor graphs) are geometric for $q = 3^{2m}$ [10], yielding a family of partial geometries

with new parameters:

$$s = 3^{2m} - 1,\ t = \frac{3^{4m}-1}{2},\ \alpha = \frac{3^{2m}-1}{2}.$$

2 New partial geometries from old ones with a replaceable spread

2.1 Replaceable spreads of a partial geometry

Let Φ be a *spread* of a pg(s,t,α) $\mathcal{S}$, i.e. a set of $st/\alpha + 1$ lines partitioning the point set, we will refer to Φ as a pg-*spread.* Assume $t > 1$ and let L be any element of $\mathcal{L} \setminus \Phi$. Let Φ_L be the set of $s+1$ lines of Φ intersecting L. We call L *regular with respect to* Φ if there exists a set of $s+1$ lines $\mathcal{L}_\Phi = \{L_0 = L, L_1, \ldots, L_s\}$ that partitions the point set $\mathcal{P}(\Phi_L)$ of Φ_L, and each element of $\mathcal{L} \setminus (\mathcal{L}_\Phi \cup \Phi)$ is intersecting Φ_L in at most s points.

Lemma 1 *If L is a regular line with respect to the* pg-*spread Φ of a* pg(s,t,α) $\mathcal{S}$, *then $t \geq s+1$. Moreover if $t = s+1$ then every line M not being an element of the spread Φ neither of $\mathcal{L}_\Phi$ intersects $\mathcal{P}(\Phi_L)$ in α points. Conversely, if every line M not being an element of the spread Φ neither of $\mathcal{L}_\Phi$ intersects $\mathcal{P}(\Phi_L)$ in α points then $t = s+1$.*

Proof. Assume that the line M_i, not being an element of the pg-spread Φ neither of $\mathcal{L}_\Phi$, $i = 1, \ldots, d = \frac{t(st+\alpha)}{\alpha} - (s+1)$ intersects $\mathcal{P}(\Phi_L)$ in a_i points. By counting the ordered pairs (p, M_i), $p \in \mathcal{P}(\Phi_L)$, $p \mathrel{\mathrm{I}} M_i$, $i = 1, \ldots, d$, in two different ways, we get

$$\sum_{i=1}^{d} a_i = (s+1)^2(t-1).$$

Counting the ordered triples (p, p', M_i), $p,\ p' \in \mathcal{P}(\Phi_L)$, $p \mathrel{\mathrm{I}} M_i \mathrel{\mathrm{I}} p'$, $i = 1, \ldots, d$, we get

$$\sum_{i=1}^{d} a_i(a_i - 1) = (s+1)^2 s(\alpha - 1).$$

Using $\sum_{i=1}^{d}(a_i - \bar{a})^2 \geq 0$ with $d\bar{a} = \sum_{i=1}^{d} a_i$, we find

$$s(\alpha - t)^2(t - s - 1) \geq 0.$$

Hence either $\alpha = t$ but then $\mathcal{P} = \mathcal{P}(\Phi_L)$, hence $a_i = s+1$, $i = 1, \ldots, d$, which is against the assumption or $t \geq s+1$. If $t = s+1$ then $a_i = \frac{\sum a_i}{d} = \alpha$, $\forall i = 1, \ldots, d$.

Conversely, assume that every line M_i, $i = 1, \ldots, d$ not being an element of the pg-spread Φ neither of $\mathcal{L}_\Phi$ intersects $\mathcal{P}(\Phi_L)$ in α points ($\alpha \neq s+1$). We count in two different ways the number of ordered pairs (p, M_i) with p a point of $\mathcal{P}(\Phi_L)$ and $p \mathrel{\mathrm{I}} M_i$. This yields

$$(s+1)^2(t-1) = \left(\frac{t(st+\alpha)}{\alpha} - (s+1)\right)\alpha.$$

This equation simplifies to $(s-t+1)(s(t-1)+(\alpha-1)) = 0$. Hence $t = s+1$. ∎

Definition

Assume that Φ is a pg-spread of a pg$(s,s+1,\alpha)$ such that every line is regular with respect to Φ. Then $\mathcal{L}\setminus\Phi$ is partitioned in $\frac{s(s+1)}{\alpha}+1$ sets $\mathcal{L}_i$ $(i=0,\ldots,\frac{s(s+1)}{\alpha})$ each containing $s+1$ mutually skew lines. The spread Φ will be called a *replaceable pg-spread* for reasons that will become clear very soon. This definition generalizes the definition given in [11]; they restrict their definition to pg-spreads of pg$(2\alpha-1,2\alpha,\alpha)$ and they call them *regular spreads*. We prefer to use another terminology as the concept of regular spreads is used in another context.

Remarks

1. If Φ is a replaceable pg-spread of a pg$(s,s+1,\alpha)$, then the incidence structure $\mathcal{D}(\Phi)$ with as points the elements of Φ and as blocks the sets Φ_L, incidence being the natural incidence, is a symmetric $2-(\frac{s(s+1)}{\alpha}+1,s+1,\alpha)$ design. This yield extra conditions on the parameters s and α. We will prove in the sequel that $\mathrm{PQ}^+(4n-1,q)$ $(q=2$ or $3)$ has replaceable spreads, which yield $2-(2^{2n}-1,2^{2n-1},2^{2n-2})$ designs in the case $q=2$ and $2-(\frac{3^{2n}-1}{2},3^{2n-1},2\cdot 3^{2n-2})$ designs in the case $q=3$. These designs have the parameters of the complement of the designs of points and hyperplanes of a $\mathrm{PG}(2n-1,q)$, $(q=2$ or $3)$.

2. If L is a regular line with respect to a pg-spread Φ of a pg(s,t,α) $\mathcal{S}$, then the $2-((st+\alpha)/\alpha,s+1,(s+1)\alpha)$ design $\mathcal{D}$ with point set Φ and block set $\mathcal{L}\setminus\Phi$, incidence being intersection; is a design with an $(s+1)$-fold block. By the well known inequality of Mann [9] $|\mathcal{L}\setminus\Phi|\geq(s+1)|\Phi|$, which yields the inequality $t\geq s+1$ of lemma 1. Hence if Φ is replaceable, then the block set of $\mathcal{D}$ is a disjoint union of $s+1$ symmetric $2-(\frac{s(s+1)}{\alpha}+1,s+1,\alpha)$ designs.

2.2 The construction

Let $\mathcal{S}$ be a pg$(s,s+1,\alpha)$ with a replaceable pg-spread Φ. Define the following incidence structure $\mathcal{S}_\Phi=(\mathcal{P}_\Phi,\mathcal{L}_\Phi,\mathrm{I}_\Phi)$. The elements of $\mathcal{P}_\Phi$ are on the one hand the points of $\mathcal{S}$ and on the other hand the sets $\mathcal{L}_i$, $i=0,\ldots,s(s+1)/\alpha$; the set of lines $\mathcal{L}_\Phi$ equals $\mathcal{L}\setminus\Phi$. Finally $p\ \mathrm{I}_\Phi\ L$ is defined by $p\ \mathrm{I}\ L$ if $p\in\mathcal{P}$ and by $L\in p$ if $p\in\{\mathcal{L}_i\|i=0,\ldots,s(s+1)/\alpha\}$.

Theorem 1 *$\mathcal{S}_\Phi$ is a* pg$(s+1,s,\alpha)$.

Proof. It is clear from the construction that $\mathcal{S}_\Phi$ is a partial linear space of order $(s+1,s)$. We only have to prove that for each anti-flag (p,L) the incidence number equals α. Let p be a point of $\mathcal{S}$ and let L_p be the line of the pg-spread Φ through p. If L_p is not intersecting L in the partial geometry $\mathcal{S}$, then the α lines of $\mathcal{S}$ through p and intersecting L are all elements of $\mathcal{S}_\Phi$ while the point of type $\mathcal{L}_i$ defined by L is not collinear with p. However if L_p is intersecting L in the partial geometry $\mathcal{S}$, then there are $\alpha-1$ lines of $\mathcal{S}$ (being also lines of $\mathcal{S}_\Phi$) through p and intersecting L. Let $\mathcal{L}_i$ be the unique set defined by L and let $L_i(p)$ be the line of $\mathcal{L}_i$ through

p, then $p \; \mathrm{I}_\Phi \; L_i(p) \; \mathrm{I}_\Phi \; \mathcal{L}_i \; \mathrm{I}_\Phi \; L$. Hence also in this case the incidence number $\alpha(p, L)$ equals α. Assume $p \in \{\mathcal{L}_i \| i = 1, \ldots, s(s+1)/\alpha + 1\}$ then as each line L of $\mathcal{S}_\Phi$ not contained in p intersects the point set of p in α points, it follows that the incidence number is again α. ∎

Remark

It has been checked by computer by Mathon and Street [11] (and also by V. Tonchev, private communication) that $\mathrm{PQ}^+(7, 2)$ has, up to isomorphism, exactly 3 replaceable spreads, yielding (after dualizing) 3 non-isomorphic partial geometries $\mathrm{pg}(7, 8, 4)$. One of these $\mathrm{pg}(7, 8, 4)$ contains replaceable spreads too which yield again partial geometries $\mathrm{pg}(7, 8, 4)$, non-isomorphic to the former ones. In total Mathon and Street have found by this technique (they call this construction *switching*) 7 partial geometries $\mathrm{pg}(7, 8, 4)$ that are not isomorphic to $\mathrm{PQ}^+(7, 2)$.

3 Replaceable spreads of $\mathrm{PQ}^+(4n-1, q)$ ($q = 2$ or 3)

Assume that $\mathcal{S}$ is a partial geometry of type $\mathrm{PQ}^+(4n-1, q)$ ($q = 2$ or 3). In the sequel we will always denote by H^*, the polar space with respect to $\mathrm{Q}^+(4n-1, q)$ of a subspace H. It is easy to check (see [5]) that two lines L and M are intersecting lines of $\mathcal{S}$ iff on the quadric $\mathrm{Q}^+(4n-1, q)$, $L \cap M^* = \emptyset$ (or equivalently $L^* \cap M = \emptyset$). Hence any subset of Ω contained in one element σ_i of the orthogonal spread Σ yields a set of mutually disjoint lines of $\mathcal{S}$. In [5], the following theorem has been proved for $q = 2$, but the proof can easily be modified for $q = 3$ (see also [8], lemma 3.5).

Theorem 2 *Suppose that p_0 is a point on $Q^+(4n-1, q)$ ($q = 2$ or 3). The set of lines V of $PQ^+(4n-1, q)$ intersecting as $(2n-1)$-dimensional subspaces of $Q^+(4n-1, q)$ in a point p_0 of the quadric is contained in exactly 2 pg-spreads.*

As before, assume that Ω is the union of all hyperplanes of the elements σ_i, $i = 0, \ldots, q^{2n-1}$ of the orthogonal spread Σ. Assume $p_0 \in \sigma_0$. One of the pg-spreads occuring in theorem 2, which we will denote by Φ_1, consists of all hyperplanes of σ_0. The other pg-spread, which we will denote by Φ_2, equals $V \cup \{p_0^* \cap \sigma_i \| i = 1, \ldots, q^{2n-1}\}$.

Assume n is even. We will construct a third type of pg-spread of $\mathrm{PQ}^+(4n-1, q)$, ($q = 2$ or 3). Let δ be an element of $\mathcal{D}_1$ such that $\delta \cap \sigma_i$ $i = 0, \ldots, q^{2n-1}$ is either empty or a $\mathrm{PG}(n-1, q)$. Without loss of generality we may assume that $\delta = \langle H_0, \ldots, H_{q^n} \rangle$ with $H_i = \delta \cap \sigma_i = \mathrm{PG}(n-1, q)$, $i = 0, \ldots, q^n$. Each subspace H_i is contained in $\frac{q^n-1}{q-1}$ hyperplanes of σ_i. The union of all these hyperplanes yields $\frac{q^{2n}-1}{q-1}$ lines of $\mathrm{PQ}^+(4n-1, q)$ forming a pg-spread which we will denote by Φ_3. We will call a spread Φ_i ($i = 1, 2, 3$) a spread of type i.

Theorem 3 *A* pg-*spread* Φ *of type* 1 *is a replaceable* pg-*spread of* $PQ^+(4n-1, q)$, $q = 2$ *or* 3, *for all* $n \geq 2$.

Proof. In order to prove the assumption, we have to prove that each line $L \in \mathcal{L} \setminus \Phi$ is regular with respect to Φ. Let L be a line of $\mathcal{S} = \mathrm{PQ}^+(4n-1, q)$ not contained in Φ, hence L is an element of Ω not contained in σ_0. Without loss of generality we may assume that L is a hyperplane of σ_1. The polar space L^* of L with respect to Q^+ intersects σ_0 in a point p_0. The q^{2n-1} elements of Φ_L are the hyperplanes of σ_0 not containing p_0. The lines $L_i = p_0^* \cap \sigma_i,\ i = 2, \ldots, q^{2n-1}$ are disjoint in $\mathcal{S}$ from L and are concurrent with all elements of Φ_L. As these lines are pairwise disjoint, they are partitioning the point set $\mathcal{P}(\Phi_L \setminus L)$, while no other line is completely contained in $\mathcal{P}(\Phi_L)$. Hence L is regular with respect to Φ; hence a pg-spread of type 1 is replaceable. ∎

Theorem 4 *The* pg*-spreads of type* 2 *and* 3 *of* $PQ^+(4n-1, q)$, $q = 2$ *or* 3, *are replaceable* pg*-spreads if and only if* $n = 2$.

Proof. Assume that Φ is a pg-spread of $\mathrm{PQ}^+(4n-1, q)$ of type 2. As in the proof of theorem 3 we assume again that $p_0 \in \sigma_0$. If $L \in \sigma_0 \setminus \Phi$ then the elements of Φ_L are the q^{2n-1} elements of Φ not contained in σ_0 while the elements of $\mathcal{L}_\Phi$ partitioning $\mathcal{P}(\Phi_L)$ are the elements of $\sigma_0 \setminus \Phi$. No other line is completely contained in $\mathcal{P}(\Phi_L)$. Hence L is regular with respect to Φ. Assume however that $L \in \sigma_i$, $L \neq L_i = p_0^* \cap \sigma_i,\ i = 1, \ldots, q^{2n-1}$. Without loss of generality we may assume that $L \in \sigma_1$. As subspaces on the quadric $L \cap L_1 = H_1 = \mathrm{PG}(2n-3, q)$. Let $L^* = \langle \sigma_1, \sigma' \rangle$, with $\sigma' \in \mathcal{D}_2$. Then $\sigma' \cap (\sigma_1 \cap \Phi) = L \cap L_1 = H_1$, while $\sigma' \cap (\sigma_i \cap \Phi)$ for $i \neq 1$ is either empty or a point. As $L \cap p_0^* \neq \emptyset$ it follows that $\sigma_0 \cap \sigma'$ is a point q_0, which is however different from p_0 as L is not contained in p_0^*. Without loss of generality we may assume that the q^{2n-2} points of $\sigma' \cap p_0^* \setminus H_1$ are the points $q_i = \sigma' \cap \sigma_i,\ i = 0, 2, \ldots, q^{2n-2}$. The elements of Φ in $\sigma_j,\ j = q^{2n-2}+1, \ldots, q^{2n-1}$ are $q^{2n-1} - q^{2n-2}$ elements of Φ_L. On the other hand the elements of Φ in σ_0 not containing q_0 are the other q^{2n-2} elements of Φ_L. The lines of the partial geometry that are not concurrent to L in the partial geometry but concurrent to all elements of Φ_L should be hyperplanes of $\sigma_1, \ldots, \sigma_{q^{2n-2}}$. The hyperplanes in σ_1 are the $q-1$ hyperplanes through H_1 and different from L and L_1. Note that for $i = 2, \ldots, q^{2n-2}$, the intersection $L_1^* \cap \sigma_i$ $(i \neq 1)$ is a point p_i and similarly $L^* \cap \sigma_i$ $(i \neq 1)$ is a point q_i, moreover the projective line $\langle p_i, q_i \rangle$ is a line of the hyperplane $L_i \in \Phi$. The $\frac{q^{2n-2}-q}{q-1}$ hyperplanes of σ_i through $\langle p_i, q_i \rangle$ and different from L_i all yield lines of the partial geometry intersecting all elements of Φ_L. Hence the total number of lines of the partial geometry and not in Φ that are completely contained in $\mathcal{P}(\Phi_L)$ equals

$$q + \frac{q^{2n-2} - q}{q-1}(q^{2n-2} - 1).$$

This number is equal to q^{2n-1} if and only if $n = 2$. Assume $n = 2$, then $H_1 = L \cap L_1$ is a line and the 3-dimensional space $\bar{\sigma} = \langle p_0, q_0, H_1 \rangle \in \mathcal{D}_1$ intersects σ_i $(i = 2, \ldots, q^2)$ in the line $\langle p_i, q_i \rangle \in L_i$ and in each of these spaces there are q planes which form together with the q planes through H_1 in σ_1 the q^3 elements of the set $\mathcal{L}_\Phi$ defined by L. Hence the spread Φ_2 is a replaceable spread of the partial geometry $\mathrm{PQ}^+(7, q)$, $(q = 2,\ 3)$. Note by the way that the union of the planes of σ_i, $(i = 0, \ldots, q^2)$ through the lines $H_i = \sigma_i \cap \bar{\sigma}$ form a pg-spread of type 3 of the partial geometry.

Finally assume that Φ is a pg-spread of type 3. The same argument as above can be used to prove that Φ is not replaceable if $n > 2$; i. e. for each line L there are more then q^{2n-1} lines covering the points of Φ_L. However assume $n = 2$, we will prove that a pg-spread of type 3 is indeed replaceable. Assume $\delta \in \mathcal{D}_1$ such that $\delta \cap \sigma_i$, $i = 0, \ldots, q^2$ is a line H_i, while $\delta \cap \sigma_i = \emptyset$ for $i > q^2$. We will denote the elements of the pg-spread in σ_i $(i = 0, \ldots, q^2)$ by L_i^j, $j = 0, \ldots q$. Assume $L \in \mathcal{L} \setminus \Phi$. Again we have to consider 2 cases, the first case being $L \in \sigma_i$, $i = 0, \ldots, q^2$. Without loss of generality we may assume $L \in \sigma_0$. The polar space L^* of L intersects each of the elements σ_i $(i > 0)$ of the orthogonal spread Σ in a point p_i. Without loss of generality we may assume that $p_i \in L_i^0$, $i = 1, \ldots, q^2$, hence $\Phi_L = \{L_i^j \| i = 1, \ldots, q^2; j = 1, \ldots q\}$. A line of the partial geometry intersecting all the elements of Φ_L should either be a plane of σ_0 or a plane of σ_i, $i > q^2$. Let p_0 be the intersecting point on the quadric of L and H_0, then the set $\mathcal{L}_\Phi$ is the union of the q^2 planes of σ_0 through p_0 (and not contained in Φ) with the $q^3 - q^2$ planes $p_0^* \cap \sigma_i$, $i = q^2+1, \ldots, q^3$. Hence L is regular with respect to Φ. The second case that we have to consider, is the case $L \in \sigma_i, i > q^2$. Without loss of generality we may assume $L \in \sigma_{q^3}$. The polar space L^* of L will intersect σ_i, $i = 0, \ldots, q^3 - 1$, in a point p_i. However one of these points, for instance p_0 will be a point of δ. Again we may assume that $p_i \in L_i^0$, $i = 1, \ldots, q^2$ hence again $\Phi_L = \{L_i^j \| i = 1, \ldots, q^2; j = 1, \ldots q\}$ and $\mathcal{L}_\Phi$ is the union of the q^2 planes of σ_0 through p_0 (and not contained in Φ) with the $q^3 - q^2$ planes $p_0^* \cap \sigma_i$, $i = q^2 + 1, \ldots, q^3$. It follows that each line L is regular with respect to Φ and hence the spread of type 3 in $\mathrm{PQ}^+(7, q)$ is replaceable. ∎

Corollary

The partial geometry $PQ^+(7, q)$, $(q = 2$ or $3)$ has up to isomorphism exactly 3 pg-*spreads, each of them being replaceable.*

Proof.

Assume that Φ is a pg-spread of $\mathrm{PQ}^+(7, q)$, $(q = 2$ or $3)$ which is not of type 1 or 2. As Φ contains $\frac{q^4-1}{q-1}$ elements, we may assume without loss of generality, that Φ contains $q+1$ planes of σ_0 intersecting in a line H_0 of $\mathrm{Q}^+(7, q)$ and of $q+1$ planes of σ_1 intersecting in a line H_1 of $\mathrm{Q}^+(7, q)$. The maximal subspace $\delta = \langle H_0, H_1 \rangle$ is an element of $\mathcal{D}_1$ intersecting exactly $q^2 - 1$ other elements of the orthogonal spread Σ in a line. Hence $\Phi \cong \Phi_3$. ∎

Remarks

1. Note that $\mathrm{PQ}^+(7, 2)$ has 9 pg-spreads of type 1, while there are 135 of type 2 and 126 of type 3. It is known that $\mathrm{Aut}(\mathrm{PQ}^+(7, 2)) = \mathrm{Alt}(9)$. This can easily be proved by regarding $\mathrm{Aut}(\mathrm{PQ}^+(7, 2))$ acting on the 9 elements of the orthogonal spread Σ. The group is transitive on the points as well as on the lines of $\mathrm{PQ}^+(7, 2)$.

2. Assume Φ is a pg-spread of $\mathrm{PQ}^+(4n - 1, q)$ $(q = 2$ or $3)$ of type 1, i. e. it is the set of hyperplanes of an element σ_0 of Σ. For every line $L \in \mathcal{L} \setminus \Phi$, the elements of Φ_L are the hyperplanes of σ_0 not containing $p_0 = L^* \cap \sigma_0$. Hence there is a canonical bijection from the sets $\mathcal{L}_i$, $i = 1, \ldots, \frac{q^{2n}-1}{q-1}$ to the points of σ_0. From this it follows that the symmetric design $\mathcal{D}(\Phi_1)$ is indeed isomorphic

to the complement of the design of points and planes of $\mathrm{PG}(2n-1,q)$. The partial geometry $\mathcal{S}_\Phi$ is the geometry with as point set the set of points of $\mathrm{PQ}^+(4n-1,q)$ union the set of points of a fixed element σ_0 of the orthogonal spread Σ. The line set is the set of hyperplanes contained in the other elements σ_i $(i=1,\ldots q^{2n-1})$ of the orthogonal spread Σ. A point p is incident with a line L, if and only if L is contained in the polar hyperplane p^* of p with respect to the quadric. Note that the point graph as well as the block graph of this geometry were known before, see [2]. If $n=q=2$, the automorphism group of the dual partial geometry (which is the geometry Γ_2 in [11]) is the alternating group Alt(8). It is transitive on the 120 points and has clearly 2 orbits of lines, one of size 15 and one of size 120.

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