The classification of subplane covered nets

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Abstract

In this article, the subplane covered nets are completely classified as pseudo regulus nets.

1 Introduction.

In the sixties, T.G. Ostrom([10],[11]) conceived the notion of a derivable affine plane. These are affine planes of order q^2 which admit a set B of affine Baer subplanes which have the same set D of infinite points and which have the property that for every pair of distinct affine points whose line join belongs to a parallel class of D then there is a Baer subplane of B which contains these two points. Ostrom showed that an affine plane may be constructed by removing the lines whose parallel classes are in D and replacing these by the set B of Baer subplanes. The constructed plane is called the **derived plane**.

More generally, it is a natural question to ask of the nature of the net which contains the Baer subplanes of a derivable affine plane, and to ask if a net with such properties may always be extended to an affine plane. Futhermore, it is possible to consider infinite derivable affine planes and infinite derivable nets.

Most early attempts to determine the structure of a derivable affine plane were made by trying to show that, for every affine plane, there is a coordinate structure Q which is a right two dimensional vector space over a field F isomorphic to $GF(q)$ while the set D becomes coordinatized by $GF(q) \cup (\infty)(PG(1,q))$ (see the definition of pseudo — regulus net). These studies contrast with the ideas of Cofman [3]

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who associates an affine space with any derivable net minus a given parallel class. Recently, using Cofman's basic ideas, I was able to completely determine the structure of a derivable net (see [6], [7] and for a more complete history of the problems involved with derivation, the reader is referred to [8]).

Theorem 1.1 (Johnson [6]).

(1) Let $R = (P, L, C, B, I)$ be a derivable net. Then there exists a 3-dimensional projective space $\sum \cong PG(3, K)$ where K is a skewfield such that the points in P of R are the lines of $\sum$ which are skew to a fixed line N, the lines in L of R are the points of $\sum - N$, the parallel classes in C of R are the planes of $\sum$ which contain N and the subplanes in B of R are the planes of $\sum$ which do not contain N.

(2) Conversely, if $\sum_1 \cong PG(3, K_1)$ is a 3-dimensional projective space over the skewfield K_1 and N_1 is any fixed line, define points P_1, lines L_1, parallel classes C_1, subplanes B_1 to agree with the correspondence above with respect to $\sum_1$ and the fixed line N_1 where incidence I_1 is relative incidence in $\sum_1$. Then $R_1 = (P_1, L_1, C_1, B_1, I_1)$ is a derivable net.

To generalize these concepts further, the term "Baer subplane" may be replaced by the term "subplane". That is, a net is said to be a **subplane covered net** if and only if for each pair of distinct points which are collinear, there is a subplane which contains the two points and whose infinite points are the infinite points of the net.

When R.H. Bruck [2] proved his extension and uniqueness theorems on finite nets, the emphasis was on ideas of R.C. Bose on graph nets and more generally on partial geometries(see [1] *e.g.*). More recently, Thas and De Clerck [12] studied partial geometries which satisy the axiom of Pasch and completely determined such structures. For example, the result for finite nets is:

Theorem 1.2 (Thas and De Clerck [12])

Let S be a dual net of order $s+2$ and degree $t+1 (t+1 > s)$. If S satisfies the axion of Pasch, then S is isomorphic to $H_q^n (q-1 = s,\ t+1 = q^{n-1})$.

Here H_q^n is the set of points of the projective space $PG(n, q)$ which are not contained in a fixed subspace $PG(n-2, q) (n \geq 3)$, and lines of $PG(n, q)$ which do not have a point in common with $PG(n-2, q)$.

Very recently, De Clerck and the author combined certain of these ideas and showed that finite subplane covered nets are regulus nets:

Theorem 1.3 (De Clerck and Johnson [4]).

Let R be a finite subplane covered net. Then there is a finite projective space $\sum \cong PG(2n-1, q)$ such that the lines of the net are translates of a $(n-1)$-regulus where the net is of order q^n and degree $q+1$; a finite subplane covered net is a regulus net.

The remaining questions now involve arbitrary subplane covered nets. Since the work of Cofman and subsequent work on derivable nets by the author does not use finiteness, but the work of Thas and De Clerck and De Clerck and Johnson on partial and semi-partial geometries does use finiteness, is it possible to determine the structure of arbitrary subplane covered nets using similar combinations of methods?

Note that a $(n-1)$-regulus in $PG(2n-1,q)$ may be realized as a net of order q^n and degree $q+1$ which may be coordinatized by a field isomorphic to $GF(q)$. In the general case, given a projective space $\sum \cong PG(V,K)$ where V is a (right) vector space over a skew field K, a **pseudo-regulus net** is a net which may be coordinatized by K in a manner which will be made precise later.

Is every subplane covered net a pseudo-regulus net?

In [9], K.S. Lin and the author showed that every net whose dual may be embedded in a projective space is a pseudo-regulus net. More precisely, it is also shown that given any projective space $\sum$ of dimension ≥ 2 and any codimension 2 subspace N, the structure of "points",and "lines" as the lines of $\sum$ skew to N and points of $\sum -N$ respectively forms a pseudo-regulus net.

In this article, we are able to completely determine the structure of any subplane covered net. The arguments used involve certain ideas of Cofman and of Thas and De Clerck but do not use finiteness. Recall a Baer subplane in an arbitrary net is a subplane such that every point lies on a line of the net and every line contains a point of the subplane(in the projective setting). The main obstacle in considering the problem in the infinite case involves finding a suitable replacement for the *point/line* properties of a Baer subplane. This obstacle may be overcome once it is realized that within any subplane covered net, there is always a derivable subnet within which the subplanes are Baer(see section 2).

Our main result classifies all subplane covered nets in terms of a projective space as in Thm.(1.1) but see Thm.(3.11) for the complete statement. A corollary to this result is the generalization of the result of De Clerck and Johnson:

Theorem 1.4 If N is a subplane covered net then N is a pseudo-regulus net.

Note that a finite pseudo-regulus net is a regulus net, a derivable net is a subplane covered net, and a net whose dual satisfies the axiom of Pasch is a finite subplane covered net, so that the previously known results may be obtained as corollaries to the above theorem.

2 Derivable subnets.

In this section, it is shown that every subplane covered net contains a derivable subnet such that the subplanes contained in the subnet are Baer when restricted to this net. Most of the ideas necessary for the proofs were obtained by trying to generalize the techniques of Cofman [3], and consequently of Johnson [6], and of Thas and De Clerck [12] to the infinite case and the diligent reader can see the influence that Thas and De Clerck has had on the present work. However, since Thas and De Clerck study partial geometries satisfying the axiom of Pasch, and the duals of finite nets are the partial geometries in question, the reader who would like to read both papers must dualize our statements to find finite analogues in Thas and De Clerck. In particular, two key results might be mentioned here.

First the proof of Thas and De Clerck that dual nets satisfying the axiom of Pasch are regular uses finiteness in an essential way. The regularity condition when properly interpreted in the language of nets says that once two subplanes share two lines of a given parallel class then they share all of their lines on this parallel class. In the arbitrary case, we use a similar argument but one which does not use finiteness to prove this result(see Thm.(2.2).

Second, recall that a derivable net is a subplane covered net which is covered by Baer subplanes. Thas and De Clerck define certain substructures which when dualized become subnets of order q^2 and degree $q+1$ which are covered by subplanes of order q. Clearly, by counting, it is seen that the subplanes are Baer in the substructure and the substructure is a derivable net. In the arbitrary order case, it is still possible to prove that there are analogous structures which we show are derivable subnets wherein the subplanes are Baer (see Thm.(2.5)).

ASSUMPTIONS: Let $R = (P, L, B, C, I)$ be a subplane covered net where the sets P, L, B, C, I denotes the sets of points, lines, subplanes, parallel classes, and incidence respectively. Note it is assumed implicitly that there is more than one subplane for otherwise any affine plane would be a subplane covered net. Furthermore, occasionally we shall refer to the set of parallel classes C as the set of infinite points of the net. If P is an affine point and α is a parallel class, $P\ \alpha$ shall denote the unique line of α which is incident with P. Also, note that given a pair of distinct points P, and Q which are collinear in N then there is a subplane $\pi_{P,Q}$ which contains P and Q and which has C as its set of infinite points.

Proposition 2.1 The subplane $\pi_{P,Q}$ is the unique subplane of B which contains P and Q.

Proof: Let R be any point of the subplane which is not on the line PQ. Then RP and RQ are lines of distinct parallel classes say α and β respectively. Then $RP = P\ \alpha$ and $RQ = Q\ \beta$ and $R = P\ \alpha \cap Q\ \beta$. Hence, any point of $\pi_{P,Q}$ which is not on the line PQ may be obtained as the intersection of the lines in $\{P\ \delta | \delta \in C\}$ and in $\{Q\ \rho | \rho \in C\}$.

Similarly, any point of PQ may be obtained as the intersection of lines $R\ \alpha$ and $P\ \beta$ for a particular point $R(of$ intersection as above) for certain α, β in C.

Theorem 2.2 (The Share Two Theorem)

If π_1 and π_2 are subplanes of B that share two lines of a parallel class α in C then the subplanes share all of their lines on α.

Proof:

Existence:

First we show that the subplanes have common lines other than the given two. Let x and y be common lines to π_1 and π_2 in the parallel class α. Let z_1 and z_2 be lines of parallel classes β and δ respectively where α, β, δ are mutually distinct and lines of π_1, π_2 respectively. Let L_1, M_1 be $z_1 \cap x$, $z_1 \cap y$ respectively so that

$\pi_1 = \pi_{L_1,M_1}$. Similarly, let L_2, M_2 be $z_2 \cap x$, $z_2 \cap y$ respectively so that $\pi_2 = \pi_{L_2,M_2}$. Note that $\{L_1, M_1\}$ and $\{L_2, M_2\}$ must be disjoint in order that the subplanes π_1 and π_2 be distinct. Let $W = z_1 \cap z_2$. Note that if T is a point of a subplane π_0 then any line $T\ \delta$ for $\delta \epsilon C$ is a line of π_0; the lines thru T are lines of π_0. So, it follows that W is a point of the subplanes π_{L_1,L_2} and π_{M_1,M_2} as, for example, z_1 and z_2 are lines thru L_1 and L_2 and thus lines of the subplane π_{L_1,L_2}(such subplanes exist since L_1, L_2 are collinear with x) and as such, the intersection point W is a point of the subplane. Note that $W\ \alpha$ must be distinct from $L_1\alpha = x$ and from $M_1\alpha = y$ since π_1 and π_2 are distinct.

Choose any point U on $W\ \alpha$ distinct from W and in π_{L_1,L_2}. Hence, U and L_1 and U and L_2 are collinear. Choose any line r_1 not equal to y thru M_1 and intersect $W\ \alpha$ in R_1. Since $W\ \alpha$ and r_1 then become lines of π_{M_1,M_2}, it follows that R_1 and M_1 and R_1 and M_2 are collinear. Hence, $r_1 = R_1M_1$ and there is a line R_1M_2.

Thus, we have the lines UL_1, UL_2, R_1M_1, and R_1M_2.

Note that at this point, it is not clear that the intersections are affine; various of the lines could belong to the same parallel class. Extend the notation so that two parallel lines "intersect" in the infinite point β if and only if they belong to the parallel class β.

Form $UL_1 \cap R_1M_1 = S$ and $UL_2 \cap R_1M_2 = T$. We may choose $r_1 = R_1M_1$ to be not parallel to UL_1 but it is still possible that R_1M_2 is parallel to UL_2.

Let $UL_1 = L_1\beta_1$ and $UL_2 = L_2\beta_2$ where β_1 and $\beta_2 \epsilon\ C$. A different choice of r_1 produces a different intersection point R_1 on $W\ a$ and all of these intersection points are collinear with M_2 so the lines formed belong to different parallel classes. Hence, there is at most one line r_1 which will produce an intersection point R_1 so that R_1M_2 is parallel to UL_2.

Hence, choose r_1 different from y, different from z_1, not on β_1(*i.e.* not parallel to UL_1) and distinct from a line(at most one) which produces intersection point R_1 such that $R_1\beta_2 = R_1M_2$. Thus, assume that the degree $is \geq 5$. Then the intersection points S and T where $S = UL_1 \cap R_1M_1$ and $T = UL_2 \cap R_1M_2$ are both affine. Note that U and R_1 are collinear (there are both on $W\ \alpha$) and U and R_1 are distinct for otherwise, $R_1M_1 = UM_1$ and z_1 would be lines of π_{L_1,L_2} which intersect in M_1 so that M_1, L_1 and L_2 are points of the same subplane which cannot occur if π_1 and π_2 are distinct subplanes. So, there is a subplane π_{U,R_1}. All of the indicated lines are lines thru either U or R_1 so that the intersection points S and T are in π_{U,R_1}. Furthermore, the point S is in $\pi_{L_1,M_1} = \pi_1$ as it is the intersection of two lines of this subplane, and similarly T is a point of $\pi_{L_2,M_2} = \pi_2$. Hence, ST is a line which must be common to both subplanes. However, if the subplanes are distinct then $ST = S\ \alpha = T\ \alpha$ since otherwise, ST intersects x and y in distinct affine points which, by Prop.(2.1), forces the two subplanes to be identical.

Thus, $ST = S\ \alpha = T\ \alpha$ is a line of α which is common to both subplanes. If $ST = x$ then $S = L_1$ and $r_1 = z_1$. Similarly, $ST = y$ forces $S = M_1$ and $T = M_2$ so that $r_1 = y$. Hence, we have shown that with the exception of at most four lines thru M_1, any such line produces a line of α common to both subplanes. Moreover, two distinct lines r_1 and r_2 thru M_1 produce distinct points R_1 and R_2 on $W\ \alpha$ which produce distinct intersection points $UL_1 \cap R_1M_1 = S$ and $UL_1 \cap R_2M_1 = S_2$.

If $S\ \alpha = S_2\alpha$ then $SS_2 = S\ \alpha = S_2\alpha = UL_1$ which is a contradiction since UL_1 cannot be in the parallel class α as U is a point of $W\ \alpha \neq L_1\alpha$. Hence, each such line r_1 produces a distinct common line of π_1 and π_2. Hence, there are at least ((degree $N) - 4) + 2$ common lines all of which must be lines of the parallel class α (note, we are **not** claiming that degree N is finite as in the infinite case, degree N is an infinite cardinal number). If the degree of the net is 3 then two distinct subplanes can share at most two affine lines on α. So, we have the existence of more than 2 common lines provided the degree ≥ 5.

Completeness:

We first assume that the degree of the net is at least 5.

Now assume that π_1 and π_2 do not share all of their lines on α but share at least two. And, we assume that the degree *is* > 4. Let y_1 be a line of α of π_2 which is not a line of π_1. Let $z_1 \cap y_1 = N_1$and $z_2 \cap y_1 = N_2$. Form the subplane $\pi_{L_1,N_1} = \pi_3$ (note that L_1 and N_1 are distinct points of z_1). Furthermore, $\pi_2 = \pi_{L_2,M_2} = \pi_{L_2,N_2}$ and note that W is a point of π_{N_1,N_2} as well as a point of π_{L_1,L_2} and π_{M_1,M_2}.

Let v be a common line of π_1 and π_2 on α and distinct from x or y. Let T be a point (affine) of $v \cap \pi_2$ which is not on z_2. Since T is a point of π_2, T and N_2 are collinear. Form TN_2. Recall that $W\ \alpha$ is a line of π_{N_1,N_2} as is TN_2 so the intersection $W\ \alpha \cap TN_2 = R_2$ is a point of π_{N_1,N_2} and is affine since otherwise TN_2 would be in the parallel class α and T would be on y_1 which cannot be since y_1 is not a line of π_1.

Since T and L_2 are distinct points of π_2, form $TL_2 \cap W\ \alpha = U_1$ so that U_1 is an affine point (similarly TL_2 is not parallel to $W\ \alpha$ for otherwise, T and L_2 would be on x and $T\ \alpha = v$ would then be x). Thus, U_1 is a point of π_{L_1,L_2} and thus U_1 and L_1 are collinear.

Form R_2N_1(possible since the joining points are in the same subplane).

Now $U_1L_2 \cap R_2N_2 = T(R_1 = TN_2 \cap W\ \alpha$ and $TL_2 \cap W\ \alpha = U_1$ so that $U_1L_2 = TL_2$ and $R_2N_2 = TN_2)$*and* is, of course, in π_2. Similarly, $U_1L_1 \cap R_2N_1 = S_1$ is in $\pi_{L_1,N_1} = \pi_3$. Note that R_2 *and* U_1 are both on $W\ \alpha$ and if distinct determine a unique subplane π_{U_1,R_2}. Similar to the above argument, if $R_2 = U_1$ then R_2N_1 and z_1 are common lines of π_{L_1,L_2} so that L_1, L_2, and N_1 are in the same subplane. But, $\pi_{L_1,N_1} = \pi_3$ and $\pi_{L_2,M_2} = \pi_2$ so that π_3 and π_2 share a common point(namely L_2) and two common lines x and y_1 which forces these two subplanes to be equal. But, in this case, π_3 contains L_1 but π_2 cannot.

Thus, S_1 and T are points which are common to π_{U_1,R_2}. However, we don't know yet know that S_1 is an affine point. We know from above that there are at least ((degree $N) - 4) + 2$ lines on α which are common to π_1 and π_2. If the degree $N - 4 > 1$, let v_1 be a line on α common to π_1 and π_2 and distinct from x, y, or v. Form $TN_2 \cap v_1 = T_1$. Then T_1 is a point of π_2 distinct from T or N_2. Form $T_1L_2 \cap W\ \alpha = U_2$ and note that $T_1N_2 \cap W\ \alpha = TN_2 \cap W\alpha = R_2$ and since U_2 is a point of $\pi_{W,L_2} = \pi_{L_1,L_2}$, then we may also form the intersection $S_2 = U_2L_1 \cap R_2N_1$ and since U_1L_1 and U_2L_1 intersect in L_1 then both cannot be parallel to R_2N_1. Note $U_2 \neq U_1$ since otherwise T would be on z_2.

Now both S_1 and S_2 are points of $\pi_{L_1,N_1} = \pi_3$ and T, S_1 are points of π_{U_1,R_2} and T_1 and S_2 are points of π_{U_2,R_2} (note that U_2 is distinct from R_2 for otherwise,

$T_1N_2 = R_2N_2 = U_2\ N_2$ and $T_1L_2 = U_2L_2$ which would force U_2 to be a point of $\pi_{L_2,N_2} = \pi_2$ which would then in turn force $W\ \alpha = U_2\alpha$ to be a line of π_2 which cannot occur if π_2 and $\pi_1(\pi_3)$ are distinct). Without loss of generality, we may assume that S_1 is an affine point (note that both points S_1 and S_2 are points of R_2N_1 so are either equal or one is affine and it is direct that they cannot be equal). Since S_1 and T are collinear it follows that S_1T is a line common to π_3 and to π_2 but since π_2 and π_3 share x and y_1, it then follows that $S_1T = S_1\alpha = T\ \alpha = v$. Hence, π_3 and π_1 share a point L_1 and two common lines x and v which implies that π_1 and π_3 are identical which cannot be the case as y_1 is a line of π_3 but not π_1. Hence, we have a contradiction and the proof to our lemma provided the degree of the net is at least 6.

We now assume that the degree of the net is exactly 4. Note that we are not necessarily assuming that the net is finite for we could have a net covered by infinitely many subplanes of order 3.

With the set up as above, there are exactly four affine lines thru M_1, namely y, z_1 and say r_1 and r_2. Let $R_1 = r_1 \cap W\ \alpha$ and $R_2 = r_2 \cap W\ \alpha$. There are three affine points of π_{L_1,L_2} on $W\ \alpha$, namely W and say U_1, U_2. Note that neither R_1 nor R_2 can be in π_{L_1,L_2} since if so, for example if R_1 is a point of π_{L_1,L_2} then r_1 and z_1 are lines of this subplane which forces $r_1 \cap z_1 = M_1$ to be a point of π_{L_1,L_2} which cannot occur as we have seen previously.

Now consider U_1L_1 and U_2L_1. At least one of these two lines is not parallel to R_1M_1 and at least one is not parallel to R_2M_1. Without loss of generality, assume that U_1L_1 is not parallel to R_1M_1. Now form R_1M_2 and U_1L_2. If these latter two lines are not parallel, then we may find a common line on α of π_1 and π_2 distinct from x and y by the above argument. Hence, assume that R_1M_2 is parallel to U_1L_2.

If U_1L_1 is also not parallel to R_2M_1 then forming U_1L_2 and R_2M_2 and noting that R_1M_2 is parallel to U_1L_2 shows that U_1L_2 cannot be parallel to R_2M_2. So, we obtain a common line of π_1 and π_2v on α distinct from x and y. Hence, it must be that U_2L_1 is not parallel to R_2M_1. Forming U_2L_2 and R_2M_2, we must have these two lines parallel or we are finished.

Summarizing, we are forced into the following situation:

U_1L_2 is parallel to R_1M_2, (*so* is not parallel to R_2M_2)

U_2L_2 is parallel to R_2M_2(*so* is not parallel to R_1M_2), and

U_1L_1 is parallel to R_2M_1(since U_1L_2 is not parallel to R_2M_2),

U_2L_1 is parallel to R_1M_1(since U_2L_2 is not parallel to R_1M_2).

We have exactly four parallel classes say $\alpha, \beta, \delta, \gamma$.

U_1L_2 is parallel to R_1M_2 so these lines lie say in β(*as* they can't lie in α).

U_2L_2 is parallel to R_2M_2 but U_2L_2 cannot lie in α or β so these lines lie say in δ.

U_1L_1 is parallel to R_2M_1 but U_1L_2 cannot lie in β as U_1L_2 does and R_2M_1 cannot lie in δ as R_2M_2 does so that these two lines lie in γ.

U_2L_1 is parallel to R_1M_1 but U_2L_1 cannot lie in δ or γ as U_2L_2 lies in δ and U_1L_2 lies in γ and since R_1M_1 cannot lie in β since R_1M_2 does, U_2L_1 and R_1M_1 are forced to lie in α which is a contradiction.

Now assume the degree is 5. By the existence argument, π_1 and π_2 share lines x, y and say v on α. Let v_1 be the fourth line of π_2 on α. Form the subplane

π_3 which contains L_1 and v_1 (that is, $\pi_3 = \pi_{L_1, z_1 \cap v_1}$). Then π_3 shares x, v_1 with π_2 and by the existence result, shares either y or v also. In either case, π_3 and π_1 share L_1 and two distinct lines on α. Hence, $\pi_1 = \pi_3$. This shows that π_2 and π_1 share all four of their lines on α.

The reader might note that the argument for degree 5 originates in Thas and De Clerck who utilize this more generally in the finite case.

Hence, we have the proof to the Share Two Theorem.

THE STRUCTURES S_L^N

Let L and N be any two affine points of the net which are not collinear. Let x be any line incident with N. Form the intersection $L\ \beta \cap x$ if x does not lie in $\beta \in C$ and determine the subplane $\pi_{L, L\ \beta \cap x}$. This subplane contains all of the points $L\ \delta \cap x$ so that by Prop.(2.1) any such intersection point together with L uniquely determines the subplane. We shall use the notation $\pi_{L,x}$ for this subplane.

We define the structure S_L^N as $\cup_N \pi_{L,x}$ where x varies over the set of lines incident with N. Note that the lines of S_L^N are the lines of a subplane $\pi_{L,x}$ whereas the points of S_L^N are defined as intersections of nonparallel lines of the subplanes $\pi_{L,x}$ for various lines x.

Note also it is possible that there are other subplanes within S_L^N which are not of the type $\pi_{L,x}$. In the following lemmas, we shall describe the properties of the structures S_L^N.

Lemma 2.3 (i) Let P be an affine point of S_L^N. Then every line of the net incident with P is a line of S_L^N.

(ii) Let Q be any affine point of S_L^N which is not collinear to L. Then $\cup_Q \pi_{L,y} = S_L^Q = S_L^N$.

Proof: Note that (ii) implies (i) since if y is a line incident with P and P is incident with L then y is a line of any subplane $\pi_{L,x}$ for any line x incident with N and if P is not incident with L then y in $\pi_{L,y}$ and $S_L^P = S_L^N$ implies that y is in S_L^N.

Hence, it remains to prove (ii).

First assume that N and Q are collinear but N and Q are both noncollinear with L.

Since Q arises as an intersection of two lines of S_L^N there is a line z incident with Q such that z is in $\pi_{L,x}$ for some line x incident with N.

Case 1. z is parallel to x.

Consider x is in the parallel class α and form $L\ \alpha$. Then z, x and $L\ \alpha$ are all lines of the subplane $\pi_{L,x}$ and since Q and N are collinear, we may assume that z and x are distinct. $L\ \alpha$ is distinct from z and from x as otherwise L would be collinear to Q or N.

Since Q and N are collinear, we may form the subplane $\pi_{Q,N}$ and note that this subplane has x and z as lines. Thus, $\pi_{Q,N}$ shares x and z with $\pi_{L,x}$ and by Thm.(2.2) must share all lines with $\pi_{L,x}$ on α. Thus, $L\ \alpha$ is a line of $\pi_{Q,N}$. Now take any line x_1 incident with N and not in α and intersect $L\ \alpha$ say in P. Since L and N are not collinear then P is distinct from L. Hence, P is a point of $\pi_{Q,N}$. So, P and Q are collinear so form $PQ = z_1$. Now form the subplanes π_{L,z_1} and π_{L,x_1} and note that both subplanes contain L and P since $L\ \alpha \cap x_1 = P = L\alpha \cap z_1$ so that by Prop.(2.1), we must have $\pi_{L,z_1} = \pi_{L,x_1}$.

Hence, for each line x_1 incident with N, there is a line z_1 incident with Q such that $\pi_{L,x_1} = \pi_{L,z_1}$. Note that $\pi_{L,x} = \pi_{L,z}$.

Suppose that $z_1 = z_2$ and $\pi_{L,z_1} = \pi_{L,x_1}$ and $\pi_{L,z_2} = \pi_{L,x_2}$ where z_1 is a line incident with Q and x_1 and x_2 are lines incident with N. Then this forces x_1 and x_2 to be lines of the same subplane so that $x_1 \cap x_2 = N$(assuming x_1 and x_2 distinct) which is a contradiction as this would imply N and L are collinear.

Hence, in the case where z and x are parallel, we obtain $(\cup_N \pi_{L,x}) \subseteq (\cup_Q \pi_{L,y})$.

Conversely, the previous argument may be seen to be symmetric. Let z_1 be any line distinct from z and incident with Q and form $z_1 \cap L\ \alpha = K$ so that K is a point of $\pi_{Q,N}$ as z_1 incident with Q forces z_1 to be a line of $\pi_{Q,N}(see(2.1))$. Hence, K and N are collinear so form $KN = x_1$. Form the subplanes π_{L,z_1} and π_{L,x_1} and note that both contain K and L so are equal. This proves that $(\cup_Q \pi_{L,y}) \subseteq (\cup_N \pi_{L,x})$ so that $S_L^Q = S_L^N$ in the case that z and x are parallel and Q and N are collinear.

Now assume that Q and N are not collinear. Consider any line w incident with N and any line u incident with Q and if w and u are not parallel form the intersections $w \cap u$.

Let w lie in the parallel class β and let u and v be lines incident with Q and in parallel classes distinct from β. As the degree of the net is at least 3, we may select lines as above. Form $\pi_{w\cap u, w\cap v}$. Assume both intersection points $w \cap u$ and $w \cap u$ are collinear with L. Then we have Q and L points of the same subplane which implies that Q and L are collinear(as $Q = u \cap v$ and u and v are lines of $\pi_{w\cap u,\ w\cap v}$).

Now Q occurs as the intersection of two lines u, v of S_L^N. Take a line u incident with N and not parallel to u or u. Without loss of generality $E = u \cap v$ is not parallel to L. Hence, it follows that there is a point E of S_L^N which is collinear to both Q and N but which is not collinear to L. Case 2 below considers the case where the points Q and N are collinear but the lines z and x are not collinear in a general or generic sense. Hence, $S_L^N = S_L^E = S_L^Q$.

Case 2. z is not parallel to x.

Initially, assume that Q is collinear to N.

Let z_1 be any line incident with Q and distinct from z. Consider $z \cap x = P$. Since z and x are lines of $\pi_{L,x}$ by assumption, we have that P and L are collinear. Assuming that z_1 is not parallel to PL, let $T = z_1 \cap PL$ and note that T is distinct from L as Q and L are not collinear and z_1 is a line incident with Q. Form the subplane $\pi_{P,Q}$ and note that $N = NQ \cap (x = PN)$ and $T = PL \cap (z_1 = TQ)$ so that both N and T are points of the subplane $\pi_{P,Q}$. Hence, N and T are collinear so form $NT = x_1$. Note that the subplanes π_{L,z_1} and π_{L,x_1} both contain the points T and L so are identical by Prop.(2.1).

Now suppose z_1 is parallel to PL. Note that z_1, z, and PL are lines of $\pi_{Q,N}$(P is a point of the subplane and PL is a line incident with P). Assume that z_1 and PL belong to the parallel class δ so that $N\ \delta$ is also a line of $\pi_{Q,N}$ and Form $\pi_{L,N\ \delta}$ and note that this subplane shares two lines PL and $N\delta$ on δ with $\pi_{Q,N}$ so by Thm.(2.2) the two subplanes shares all of their lines on δ. Hence, z_1 is a line of $\pi_{L,N\ \delta}$ so that

$\pi_{L,z_1} = \pi_{L,N\ \delta}$.

Hence, for each line z_1 incident with Q there is a line x_1 incident with N such that $\pi_{L,z_1} = \pi_{L,x_1}$.

Conversely, let x_1 be a line incident with N and not parallel to PL. Let

$T = x_1 \cap PL$. Form $\pi_{N,P}$ and notice that PL and x_1 are lines of this subplane as are z and QN. Recall $Q = z \cap QN$, so that Q is in $\pi_{P,N}$. Note also that $T = x_1 \cap PL$ so that T and Q are collinear. Hence, let $TQ = z_1$ and observe that π_{L,x_1} and π_{L,z_1} both contain the points T and L so are identical.

If x_1 is parallel to PL and both lines are in the parallel class δ, note that x_1 and PL are both in $\pi_{Q,N}$(x_1 is incident with N , P is a point of $\pi_{Q,N}$ and PL is a line incident with P). Form $\pi_{L,Q\ \delta}$ and note that PL and $Q\ \delta$ are also lines of $\pi_{Q,N}$ so that, by Thm.(2.2), x_1 is also a line of $\pi_{L,Q\ \delta}$ so that it follows that $\pi_{L,x_1} = \pi_{L,Q\ \delta}$.

Hence, the previous arguments show that $\cup_Q \pi_{L,y} = S_L^Q = \cup_N \pi_{L,x} = S_L^N$ provided Q and N are collinear but Q and N are both noncollinear with L in the case where z and x are not parallel.

If Q and N are not collinear there is a point E of S_L^N which is not collinear to L but is collinear to Q and to N. Hence, $S_L^N = S_L^E = S_L^Q$. This completes the proof of Lem.(2.3) in both cases z parallel to x and z not parallel to x.

In the following, let $S_L = S_L^N = S_L^Q$ for all points Q of S_L^N which are noncollinear with L(note that N is a point of S_L^N).

Lemma 2.4 Let A, B be points of S_L where A is not collinear to B and B is not collinear to L. Then $\cup_B \pi_{A,z} = S_A^B = S_L$.

Proof: First assume that A and L are collinear and form $\pi_{A,L}$. Since A is in S_L, every line incident with A is a line of S_L and as such is in some subplane $\pi_{L,x}$ where x is a line incident with N. It then follows that $\pi_{A,L}$ is one of the basic subplanes $\pi_{L,x}$. Let B be any point of S_L which is not collinear to L. This subplane is equal to a subplane $\pi_{L,w}$ where w is a line of S_L incident with B by the previous lemma. Hence, $\pi_{A,L} = \pi_{A,w}$ for some line w incident with B. Any line z thru B is a line of S_L by the previous lemma. Any line thru L is a line of $\pi_{A,L}$. Form $\pi_{L,z}$: The initial points are determined by taking lines thru L and intersecting these with z to form points on z. If P is such a point then $P\ \delta$ for all $\delta \in C$ is a line of the subplane. Since all lines thru L are lines of $\pi_{A,L} = \pi_{A,w}$ and all lines thru B are lines $of \cup_B \pi_{A,y}$, it follows that all these initial intersection points are also points $of \cup_B \pi_{A,y}$. Since the remaining points of $\pi_{L,z}$ are generated by these initial intersection points , it follows that the points of each of the subplanes $\pi_{L,x}$ for x incident with B are points $of \cup_B \pi_{A,y}$. By applying the lemma (2.3)(i) $to \cup_B \pi_{A,y}$, it follows that on any point $Q\ of \cup_B \pi_{A,y}$, all lines on Q are also lines $of \cup_B \pi_{A,y}$. Moreover, since the lines of the subplanes $\pi_{L,x}$ for x incident with B may be obtained by taking the points P and forming $P\ \alpha$ for all $\alpha \in C$, since P is also a point $of \cup_B \pi_{A,y}$ then such lines also become lines $of \cup_B \pi_{A,y}$.

Hence, all lines of the subplanes $\pi_{L,x}$ for all lines x incident with B are also lines $of \cup_B \pi_{A,y}$ so that all subsequent points of S_L are also points $of \cup_B \pi_{A,y}$. Thus, $S_L \subset \cup_B \pi_{A,y}$. Since A and B are points of S_L, all lines incident with A and all lines incident with B are lines of S_L by Lem.(2.3)(i) and hence all subsequent points and lines generated within $\cup_B \pi_{A,y}$ are likewise in S_L (the previous argument is symmetric) so that $\cup_B \pi_{A,y} \subset S_L$.

Hence, we have shown that if A and B are points of S_L which are not collinear, A and L are collinear but B and L are not collinear then $\cup_B \pi_{A,y} = S_L$.

Now assume that there is a point C in S_L such that A is collinear with C, C is collinear with L and A, C, L are each not collinear with B. Then

$\cup_B \pi_{C,w} = \cup_B \pi_{L,x} = S_L$ and since A is then $in \cup_B \pi_{C,w}$, it follows from the above argument that $\cup_B \pi_{A,y} = \cup_B \pi_{C,w} = S_L$.

If A and L are not collinear take any two lines u and v thru A. These lines u and v are lines of S_L by Lem.(2.3). Take any line w thru L which is not parallel to either u or v.

Suppose both intersection points $u \cap w$ and $v \cap w$ are collinear with B. Then since A is collinear with both intersection points(A is $u \cap v$), it follows that A and B are points of the subplane $\pi_{u\cap w, v\cap w}$ which forces A and B to be collinear.

Since u, v and w are lines of S_L, the intersection points are also in S_L and one of these, say C, is not collinear with B but is collinear to both A and L.

Hence, it follows that $\cup_B \pi_{A,y} = \cup_B \pi_{L,x} = S_L$.

Theorem 2.5 The structures S_L are derivable subnets; the structures S_L are subnets with parallel class C and the subplanes contained within the structures are Baer subplanes of S_L.

Proof: We define a subnet as a triple of subsets of points, lines, and parallel classes. The lines of the subnet will be the the lines of the subplanes $\pi_{L,x}$ for x incident with N where L and N are not collinear. The points of the subnet shall be the intersections of lines of the subplanes indicated. The set of lines of each parallel class $\alpha \in C$ is the union of the sets of lines belonging to the subplanes $\pi_{L,x}$ which lie in α.

Note that each line on each point of S_L is a line of S_L by Lem.(2.3) so that each point P is on exactly one line of each parallel class. Hence, it easily follows that we have a subnet. It remains to show that given any pair of distinct collinear points P and Q of S_L then the subplane $\pi_{P,Q}$ is a subplane of S_L and to show that the subplane is Baer within S_L.

Each line incident with P or Q is a line of S_L by Lem.(2.3). The points of $\pi_{P,Q}$ are obtained via intersections of $P\ \alpha$ and $Q\ \beta$ for all $\alpha, \beta \in C$ so that all points are then back in S_L as are all subsequent lines by appplications of Lem.(2.3)(i). This shows that $\pi_{P,Q}$ is a subplane of S_L.

Take any subplane π_1 of the net which is within S_L and let A be any point of S_L. To show that π_1 is Baer within S_L, we must show that every line of the net contains a point of the projective extension of π_1, and that every point of the net is incident with a line of π_1. The first condition is trivial since each line projectively contains an infinite point(point of C) of π_1. To show the second condition, we first show that π_1 is of the form $\pi_{Q,x}$ where x is a line incident with a point B which is not collinear to Q and Q and B are points of S_L. Let $\pi_1 = \pi_{P,Q}$ where P and Q are any two distinct affine points of the subplane and note that P and Q must be in S_L. Take any line u of π_1 incident with P and not PQ. u must be a line of S_L. If u contains a point B in S_L which is not in π_1, then B cannot be collinear with Q for otherwise B would lie on two lines of π_1 and hence be a point of π_1. But u is in some subplane $\pi_{L,x}$ where x is a line incident with N and as such u contains at least two affine points of $\pi_{L,x}$ in S_L. If both of these points are in $\pi_{P,Q}$ then $\pi_{P,Q} = \pi_{L,x}$ by Prop.(2.1). If one of these points say B on u in $\pi_{L,x}$ is not in $\pi_{P,Q}$ then B and Q are not collinear and $\pi_{P,Q} = \pi_{Q,u}$. Now to show that there is a line of $\pi_{Q,u}$ incident with A. If A and Q are collinear, clearly AQ is a line of $\pi_{Q,u}$ incident with A.

First assume that $\pi_{Q,u}$ is a subplane of the type π_{L,x_1} for some line x_1 incident with N. We may assume that A and L are not collinear. Then, $\cup_A \pi_{L,z} = \cup_N \pi_{L,x}$ and furthermore, there is a 1-1 and onto correspondence $x \to z$ of lines x incident with N and lines z incident with A such that $\pi_{L,x} = \pi_{L,z}$. Hence, there exists a line z_1 thru A such that π_{L,x_1} contains this line; π_1 contains a line incident with A.

Now assume that $\pi_{Q,u}$ is not a subplane of the type $\pi_{L,x}$ but note that u is a line of π_{L,x_o} for some line x_o incident with N. We want to show that A is in $\cup_C \pi_{Q,w}$ where C is a point of S_L on u. We know that A is in S_L and $\cup_C \pi_{Q,w} \subset S_L$.

On any line t thru Q of $\pi_{P,Q} = \pi_1$ assume two points of t in $\pi_{P,Q}$ are incident with L. Then L must be in $\pi_{P,Q}$. Hence, if $\pi_{P,Q}$ is not of the form $\pi_{L,x}$ for some line x then at most one point of t in $\pi_{P,Q}$ is incident with L. If degree > 3, we may assume without loss of generality that neither P or Q are incident with L. Furthermore, we may assume that A and Q are not collinear for otherwise we are finished.

Let B be a point of π_{L,x_o} on u which is not in $\pi_{Q,u}$. Form the subplane $\pi_{B,P}$(note $u = BP$) and note that this subplane must be distinct from either $\pi_{Q,u}$ or π_{L,x_o} since if $\pi_{B,P}$ is π_{L,x_o} then P and L are collinear. We have established that $\pi_{B,P}$ is a subplane of S_L. Assume that the degree *is* > 3. Hence, any point C on u of $\pi_{B,P}$ distinct from B or P is not in either plane $\pi_{Q,u}$ or π_{L,x_o} (if c is in $\pi_{Q,u}$ then $\pi_{B,P} = \pi_{C,P} = \pi_{Q,u}$). Then C is not collinear to L or Q so that $\cup_C \pi_{Q,w} = \cup_C \pi_{L,y} = S_L$ (note if C is collinear to L then $\pi_{L,x_o} = \pi_{L,u}$ implies c in π_{L,x_o} so that $\pi_{B,P} = \pi_{B,C} = \pi_{L,x_o}$, a contradiction). Hence, A must be *in* $\cup_C \pi_{Q,w}$ so that we may apply the previous results to show that $\cup_A \pi_{Q,y} = \cup_C \pi_{Q,w}$. Moreover, there is a 1-1 and onto correspondence $w \to y$ of lines w incident with C and lines y incident with A such that the $\pi_{Q,w} = \pi_{Q,\,y}$. This implies that for the line u there is a line z incident with A such that $\pi_{Q,u} = \pi_{Q,z}$ so that the subplane $\pi_1 = \pi_{Q,u}$ contains a line incident with A.

Thus, it remains to show that when the degree is exactly 3, the subplanes contained in S_L are Baer.

Note that, in this case, we are not necessarily assuming that the net is finite. However, there are exactly three lines of S_L incident with N and on each line there is a unique point incident with L so there are exactly $4 \cdot 3$ lines of S_L and it follows that on each line there are exactly 4 points of S_L. That is, S_L is a subnet of degree $1 + 2 = 3$ and order 2^2. Since the subplanes contained in S_L now have order 2, it follows that such subplanes are Baer within S_L. This completes the proof of the theorem.

Corollary 2.6 Consider any of the subnets S_L of points, lines, subplanes, parallel classes, and incidence.

Then there is a 3-dimensional projective space $\sum$ and a line N of $\sum$ such that the lines of $\sum$ skew to N are the points of S_L, the points of $\sum - N$ are the lines of S_L, the planes of $\sum$ which intersect N in a point are the subplanes of S_L and the planes of $\sum$ which contain N are the parallel classes of S_L.

Proof: The main result of Johnson [6] applies to the subnets S_L.

3 The associated projective space.

The previous corollary in section 2 shows that there is a 3-dimensional projective space associated with any subnet S_L. We shall use this to show that associated with any subplane covered net is a projective space Π with a fixed codimension 2 subspace N such that the points, lines, subplanes, parallel classes of the net are(correspond to) the lines skew to N of Π, the points of $\Pi - N$, the planes of Π which intersect N in a point, and the hyperplanes of Π which contain N respectively.

3.1 The parallel classes are affine spaces

First we consider making the parallel classes into affine spaces.

Let α be any parallel class. Define the structure A_α as follows:

The points of A_α are the lines of the net on α. The lines of A_α are the sets of lines of subplanes $\pi_{P,Q}$ which lie on α. The planes of A_α are defined via the sets S_L(derivable subnets) and are denoted by $S_{L,\alpha}$. The points of $S_{L,\alpha}$ are the lines on α of the set of subplanes of S_L. A line of $S_{L,\alpha}$ is, of course, the lines on α of a subplane of S_L.

We shall define two lines of A_α to be parallel if and only if the two lines correspond to subplanes which belong to some S_L and their lines on α are disjoint or equal.

Note that it is clear that the relation of being parallel is symmetric and reflexive.

The previous result that there are derivable subnets is vital for the results in this section. Furthermore, as the structures A_α are interconnected to the net, we shall require net properties to show that the A_α are affine spaces.

We define two lines a and $b(a||b)$ of the structures A_α, A_β for $\alpha \neq \beta \epsilon C$ to be parallel if and only if these sets are the sets of lines on α, β respectively of a subplane π_o.

Again, it is clear that this relation is symmetric.

Lemma 3.1 Given any subplane π_o and any line u of the net which is not a line of π_o, there is a unique derivable subnet $< \pi_o, u >$containing π_o and u.

Proof: Take any line v in π_o which is not parallel to u. Let $N = u \cap v$ and let L be a point of π_o which is not collinear with N. Note that N cannot be a point of π_o

Form $\cup_N \pi_{L,x} = S_L^N$ and note that this derivable subnet contains π_o(simply take x to be v) and u(take x to be u). Note that any derivable net containing π_o and u must contain the intersection point N as a point and hence, must contain the set of lines incident with N. Thus, any such derivable net contains S_L^N.

Lemma 3.2 Any two distinct subplanes π_o and π_1 which share a parallel class of lines are in some unique derivable subnet $< \pi_o, \pi_1 >$.

Proof: Let u be any line of π_1 which is not a line of π_o. Form the derivable subnet $< \pi_o, u >$. Assume that the indicated subplanes share all of their lines on the parallel class $\alpha \epsilon C$. Since $< \pi_o, u > is$ a derivable net containing u, there is a subplane π_1^* of this derivable subnet which contains u and which shares the lines of π_o on α by Johnson [6]. Since π_1 and π_1^* share u and share all of their lines on the parallel class on α, it must be that π_1 and π_1^* are identical.

Lemma 3.3 Let a, b, c be lines of various of the structures A_δ for $\delta \epsilon C$. If $a||b$ and $b||c$ then $a||c$.

Proof: We consider the following cases:

Case (1): the lines a, b, c belong to the structures A_α, A_β, A_γ respectively where α, β, γ are mutually distinct.

In this case, there are subplanes π_o and π_1 such that a and b are the sets of lines of π_o on α and β respectively and b and c are the sets of lines of π_1 on β and γ respectively.

Form the derivable subnet $< \pi_o, \pi_1 > by$ lemma (3.2) and note that a, b, and c are lines of this subnet. Then, within this derivable subnet, there is a subplane π_2 such that a and c are the sets of lines of π_2 on α and γ respectively (again see Johnson [6]). Hence, a $||c$.

Case (2): a and b belong to A_α but c belongs to A_γ for $\alpha \neq \gamma$.

By assumption, there is a derivable subnet $< \pi_o, \pi_1 > such$ that a and b are the sets of lines on α of π_o, and π_1 respectively. Within this derivable subnet, there is a subplane which contains a and say d not on α or $\beta(a$ set of lines of this subplane which does not belong to either parallel class) and a subplane which contains b and d(since a and b are sets of lines of a parallel class of subplanes of the derivable net). That is, a $||d$ and $b||d$.

Hence, $c||b||d$ and all three lines are in distinct substructures A_ρ for various values $\rho \epsilon C$, it follows from case (1) that $c||d$. Hence, $c||d||$ a so that another application of case (1) shows that $c||$ a.

Case (3): a and c are in A_α and b is in A_β for $\alpha \neq \beta$.

Let π_o be a subplane whose sets of lines on α and β are c and b respectively and let π_1 be a subplane whose sets of lines on α and β are a and b respectively. Form the derivable subnet $< \pi_o, \pi_1 > . Then$ a, b and c are lines of a derivable subnet and a $||b||c$ so that a automatically becomes parallel to c.

Case (4): a, b, and c are in A_α.

Since a is parallel to b, there is a derivable subnet $< \pi_o, \pi_1 > such$ that the lines on α of π_o and π_1 are a and b respectively. Similarly, there is a derivable subnet

$<\pi_2, \pi_3>$ *such* that the lines of π_2 and π_3 are b and c respectively. Take any set of lines d of π_1 on a parallel class β distinct from α. Then $a\,||b||d$ implies $a\,||d$ from case (2) and $d||b||c$ implies $d||c(i.e.\ c||b||d)$ again from case (2). Then $a\,||d||c$ implies that $a\,||c$ from case (3).

Theorem 3.4 A_α **is an affine space for each parallel class** $\alpha \epsilon C$.

Proof: First take two distinct points a and b of A_α. Recall that a and b are lines on α. Take any line u of the net which is not in α. Then the intersections of u with a and b produce a subplane π_o such that any other subplane which shares a and b with π_o must share all of the lines on α with π_o (see Thm.(2.2)). That is, given two distinct points of A_α, there is a unique line joining them.

Note that the planes of A_a are affine planes since we may use the results of Johnson [6] as these planes are induced off of derivable subnets.

Now take three distinct points of A_α, a, b, c not all collinear. Then there is a unique plane $< a, b, c >$containing these points.

Pf: Let u be any line of the net which is not in α. Form the intersection of u with a and b and the corresponding subplane π_o. By assumption, a, b, c are not collinear so c is not a line of π_o. Form the intersection of u with b and c and construct the corresponding subplane π_1. Let $P = u \cap b$ so that P is a common point of π_o and π_1. Take any line x of π_o which is not on P and take any line z on π_1 which is not on P and not parallel to x. Let $N = x \cap z$. If P and N are collinear then PN intersects x in N so that N is a point of π_o and similarly also a point of π_1 which forces π_o to be π_1. Hence, P and N are not collinear. Form $\cup_N \pi_{P,w}$ which contains $\pi_1 = \pi_{P,z}$ and $\pi_o = \pi_{P,x}$. Hence, there is a derivable subnet containing π_o and π_1 so that there is a plane of A_α containing a, b, c. Let D be any derivable net containing a, b and c. Then the set of lines of the derivable net on a form a plane of A_a containing a, b, c by Johnson [6]. Since any plane is generated by any of its triangles, it follows that that the plane is unique.

Now assume that there are two derivable subnets that share the lines a, b.

If two planes of A_α share two distinct points a and b then they share all points on the line ab.

Pf: The two planes are defined by two derivable nets D_1 and D_2. Within D_1, there is a subplane π_o which contains the lines a and b. Any other subplane which contains the lines a and b contains as lines all of the lines of π_o on α by Thm.(2.2). Hence, any subplane on D_2 which contains a and b must contains the lines of π_o on α and thus each plane of A_α containing a and b contains all of the points on the line ab.

Lem.(3.3) shows that parallelism is an equivalence relation.

It now follows that the structures A_α are affine spaces.

This completes the proof of Thm.(3.4).

NOTATION AND ASSUMPTIONS:

By the results of Johnson [6], [7], we may assume that the net is not a derivable net. Since derivable nets induce planes in A_α**, it follows that we may assume that the structures** A_a **are affine spaces of dimension** ≥ 3.

Let D and R be derivable subnets which share three lines of the same parallel class $\alpha \epsilon C$ not all in the same subplane. Then the derivable subnets share all of their lines on α and we denote this by $D_\alpha = R_\alpha$.

The reader will need to distinguish between lines of the net or subnet and lines of the affine spaces A_α or D_α since a line of a derivable subnet D_α is the set of net lines on α of a subplane of D.

We consider the projective extensions of the affine spaces A_a. Let N_a denote the hyperplane of A_a at infinity obtained by defining **infinite points** to be parallel classes of lines of A_a and **infinite lines** to be parallel classes of planes of A_a. We want to show that $N_a = N_\beta$ for all $\alpha, \beta \epsilon C$. What this basically implies is that there is a projective space Π such that the parallel classes when properly extended become hyperplanes in Π that contain a common codimension two subspace. In order to do this, we need to define what it means for two planes of different affine spaces A_α and A_β to be parallel for possibly different parallel classes α and β. The following is similar to arguments of Thas and De Clerck in the finite case except that we make more use of the structure of derivable nets.

Let Π_α, Π_β be planes of A_α and A_β respectively. We shall say that Π_α is parallel to Π_β, written $\Pi_\alpha || \Pi_\beta$ if and only if each line of Π_α is parallel to some line of Π_β.

Before proving that the relation defined in the above definition is an equivalence relation, we provide some lemmas on derivable subnets.

Lemma 3.5 Let D be a derivable subnet and α a parallel class of the net. Let x be a line which is not in α. There there is a unique derivable subnet generated by x and D_α which we denote by $< x, D_\alpha >$.

Proof : Take any three lines u, v, w of D on α not all in the same parallel class of lines of a subplane of D. Form the intersections $u \cap x = P$, $v \cap x = Q$, and $w \cap x = R$ and form the subplanes $\pi_{P,Q}$ and $\pi_{Q,R}$. There is a unique derivable net R containing these subplanes by Lem.(3.1) and the proof to Thm.(3.4) and clearly $R_\alpha = D_\alpha$. R contains x so that $R =< x, D_\alpha >$.

We known that planes of A_α must fall into parallel classes since A_α is an affine space. What we don't know if how the derivable subnets that define these planes are related. The next two lemmas study this problem.

Lemma 3.6 Let D be a derivable subnet so that D_α is a plane of A_α. Let x be a line of α which is not in D_α. Then the unique plane of A_α incident with x and parallel to D_α may be constructed as follows: Take any line z of D not in α.

Then there exists a unique derivable net R containing x and z with the property that R_α is parallel to D_α.

Any other derivable net B so constructed from any derivable net T where $T_\alpha = D_\alpha$ and containing x has the property that $B_\alpha = R_\alpha$.

Proof: Let a be a line of D_α in A_α. Let z be a line of D in β distinct from α. Then z intersects a in a uniquely defined subplane π_o which does not contain x.

Then there is a unique derivable net containing π_o and x , $< x, \pi_o >$ by Lem.(3.1). Note that since $< x, \pi_o >_\alpha$ is an affine plane in A_α, it follows that there is a unique line $L_{a,x}$ of $< x, \pi_o >_\alpha$ parallel to a thru x. Recall that this line on A_α is the set of lines on α of some subplane. In $< x, \pi_o >$, there is a unique subplane π_1 which has $L_{a,x}$ as its lines on α and which contains z. Let $L_{a,z}$ denote the line of A_β which is the set of lines of π_1 on β. Note that a $||L_{a,x}||L_{a,z}$ so that a $||L_{a,z}$ by Lem.(3.3).

So, there is a unique subplane π_2 containing a and $L_{a,z}$ as its sets of lines on α and β respectively and since π_2 contains a and z, it follows that $\pi_2 = \pi_0$. Hence, $\cup\{L_{a,z}|$ a is a line of $D_\alpha\} = D_\beta$.

Note that $< x, D_\beta >$ is a derivable net by Lem.(3.1) and there is a unique subplane containing $L_{a,z}$ and x and this is a subplane π_1 containing $L_{a,z}$ and $L_{a,x}$ so that $\pi_1 \epsilon < x, D_\beta >$.

Hence, $\cup\{L_{a,x}|$ a is a line of $D_\alpha\} =< x, D_\beta >_\alpha$.

Hence, we have produced a derivable net R containing x such that every line of D_α is parallel to some line of R_α. Let a and b be any two lines of D_α then since A_α is an affine space, the plane generated by a and x is unique and hence the line parallel to a thru a is unique. A similar statement is valid for b and x. Hence, let B be any derivable net which contains x and contains the lines on x parallel to a and b. Then B_α is uniquely determined.

It follows that R_α and D_α are mutually parallel(since they are planes of an affine space and one is parallel to the other). Furthermore, since each line of R_α is parallel to a line thru x and parallelism on lines of the affine spaces $A'_\gamma s$ is an equivalence relation, it follows that each line of R_α is parallel to a line on z of D_β and conversely each line of D_β is parallel to a line of R_α containing x. Hence, it follows that R_α and D_β are parallel planes.

Lemma 3.7 Parallelism on planes of the affine spaces $A'_\gamma s$ is an equivalence relation.

Proof: Note that if $D_\alpha||R_\beta$ where D and R are derivable nets and α and β are distinct then if z is any line of R_β then there is a derivable *net* $< z, D_\alpha >$. Take any line a of D_α and note there is a unique subplane π_o of $< z, D_a >$ containing z and with a as its set of lines on a. Since R_β is an affine plane, every line of R_β is parallel to a line which contains z. Hence, since a is parallel to some line of R_β, it follows that a is parallel to a line b which contains z and this line b must be exactly the set of lines of π_o on β. It follows that $< z, D_\alpha >_\beta = R_\beta$. It follows that any line of R_β is parallel to some line of D_α.

To prove transitivity, simply note that if three planes $D_\alpha||R_\beta||B_\gamma$ where D, R, B are derivable subnets then every line a of D_α is parallel to some line b of R_β and every such line b is parallel to some line c of B_γ and since parallelism on lines is an equivalence relation, it follows that a is parallel to c and hence, every line of D_α is parallel to some line of B_γ and hence $D_\alpha||B_\gamma$.

This proves the lemma.

Proposition 3.8 If D and R are derivable nets and for some parallel class α, $D_\alpha||R_\alpha$ then for all parallel classes β, $D_\beta||R_\beta$.

Proof: Clearly for any derivable net B and any parallel classes γ and ρ, it follows that $B_\gamma || B_\rho$. Hence, $D_\alpha || R_\alpha || R_\beta$ implies that $D_\alpha || R_\beta$ and $D_\beta || D_\alpha || R_\beta$ implies that $D_\beta || R_\beta$.

Lemma 3.9 Let A_α be any affine space for $\alpha \epsilon C$ and let N^α denote the hyperplane at infinity of the projective extension A_α^+ of A_α. Then $N^\alpha = N^\beta = N$ for all $\alpha, \beta \epsilon C$.

Proof: In order to construct N^α, we define the points of N^α to be the equivalence classes of lines of A_α and the lines of N^α as the equivalence classes of the planes of A_α. Recall that any plane of A_α is defined by a derivable net D as D_α and any line of A_α by a subplane. Since a parallel class of lines of A_α has a representative in any A_β and any parallel class of planes of A_α has a representative in any A_β it follows that $N^\alpha = N^\beta = N$.

Theorem 3.10 Let $R = (P, L, B, C, I)$ be any subplane covered net. Then there is a projective space $\sum$ defined as follows:

Call the lines of a given parallel class of a subplane "class lines" and call the lines of a given parallel class of a derivable subnet "class subplanes". Note that there are equivalence relations on both the set of class lines and on the set of class subplanes. Call the equivalence classes of the class lines "infinite points" and the equivalence classes of the class subplanes "infinite lines". Also, note that the infinite points and infinite lines form a projective subspace N.

The points of $\sum$ are the lines L of the net and the infinite points defined above.

The lines of $\sum$ are the sets of lines on an affine point(identified with the set P), the class lines extended by the infinite point containing the class line, and the lines of the projective space N.

The planes of $\sum$ are

(1) subplanes of B extended by the infinite point on the equivalent class lines of each particular subplane where the points and lines of the subplane are now considered as above(actually the dual of the subplane extended),

(2) the affine planes whose points are the lines of a net parallel class and lines the class lines of a derivable subnet of the net parallel class extended by the infinite points and infinite line, and

(3) the projective planes of the projective space N.

The hyperplanes of $\sum$ that contain N are the parallel classes C extended by the infinite points and infinite lines of N.

Note that N becomes a codimension two subspace of $\sum$.

Proof: To complete the proof, we need only show that any three distinct points A, B, C not all collinear generate a unique projective subplane.

If the points are all infinite points then since N is a projective subspace, the result is clear.

Assume that A, B and C are all lines of the net.

If all are points of the same A_α then since A_α is an affine space, the points will generate an affine plane which then uniquely extends to a projective plane in $A_\alpha \cup N$.

If A and B are in A_α and C is in A_β where α and β are distinct then by taking intersection points of the lines, there is a unique subplane of the net containing A, B and C. By extending the subplane with the infinite point corresponding to the class points, it follows that there is a unique projective plane interpreted in the notation in the statement of the theorem generated by these points A, B and C.

Similarly if A, B and C are all in mutually distinct affine spaces A_α, A_β, A_γ, there is a unique subplane of the net containing A, B and C and the previous argument applies.

Suppose that A and B are infinite points and C is a line of the net. Let C be in the parallel class α. Since A is an infinite point, there is a unique representative class line A_1C_1 which contains $C(as$ a line of the net). Similarly, there is a unique representative class line B_1C_1 in α of B which contains C. Note that A_1C_1 and B_1C_1 extended are lines of the structure $\sum$. Now the two class lines contain C and thus there is a derivable subnet D which contains these two class lines and any other derivable subnet containing these class lines agrees on the parallel class α with D. The set D_α is a plane of A_α which when extended becomes the unique projective subplane generated by A, B, and C.

Assume that A and B are lines of the net and C is an infinite point.

If A and B are in the same parallel class α , consider the set of subplanes which contain A and B. Recall that the line of A_α, AB is uniquely determined as the set of lines of any subplane containing A and B. Now if A, B and C are not collinear then C is not an equivalence class of any subplane that contains A and B. Hence, there is a representative class line on α which contains A but not B. Take any line x not in α and intersect the lines of the class point and B. Then there is a unique derivable net D containing x and these intersection points. Furthermore, any other derivable net containing the class line and B shares the lines on α with D. Hence, there is a unique affine plane D_α of A_α which when extended is the unique projective plane generated by A, B, and C.

Finally, assume that A and B are lines in different parallel classes of the net and C is an infinite point. Let $P = A \cap B$. The set of lines of the net incident with P is a line of the structure which does not intersect the projective subspace N so that A, B and C are intrinsically noncollinear in this case. Take a representative class line on the parallel class a of the net containing A. Form the intersection points of this class line (which is a set of lines of a subplane) with B and note that there is a unique subplane of the net generated. This subplane contains A, B and when extended by C is the unique projective plane containing A, B and C when interpreted in the notation of the theorem.

This completes the proof of theorem (3.10).

4 Pseudo regulus nets.

Let R be an ordinary $(n-1)$- regulus in $PG(2n-1,q)=\sum$. This is a set of $q+1$ $(n-1)$-dimensional projective subplanes of $\sum$ which is covered by a set of transversal lines; if a line intersects at least three members of the $(n-1)$-regulus then the line intersects all members of the regulus.

Let V_{2n} denote the corresponding vector subspace over $GF(q)$ such that $\sum$ is the lattice of subspaces of V_{2n}. Then

Proposition 4.1 (Johnson [7]). In V_{2n}, every $(n-1)$-regulus R has the following canonical form:

Let $V_{2n}=W\oplus W$ for some n-dimensional vector subspace W over $GF(q)$. Then R may be represented by $x=0$, $y=\delta\ x$ for all $\delta\ \epsilon\ GF(q)$ where $x=(x_1,x_2,\ .\ .\ .\ ,x_n)$ and $y=(y_1,y_2,\ .\ .\ .\ ,\ y_n)$ are vectors in W with respect to some basis for W for x_i, y_i for $i=1,2,\ .\ .\ .,n$ are in $GF(q)$ and $\delta\ x=(\delta x_1,\delta x_2,\ .\ .\ .,\delta x_n)$.

We call the corresponding net a $(n-1)$-regulus net or simply a regulus net when there is no ambiguity.

Now we define a similar quasi-geometric structure which we only consider in its vector form.

Let W be a left vector space over a skewfield K. Let $Z(K)$ denote the center of K.

Let $V=W\oplus W$. Let R be the net defined by the following $Z(K)$ — subspaces $x=0$, $y=\delta\ x$ where $\delta\ \epsilon\ K$ and if $x=(x_i)$ for $i\ \epsilon\ \lambda$ as a tuple with respect to some K-basis for W and $y=(y_i)$ for $x_i,y_i\epsilon\ K$ for $i\ \epsilon\ \lambda$. Then we call any net which can be represented as in the form of R a pseudo regulus net.

Note that any regulus net is a pseudo regulus net and any finite pseudo regulus net is a regulus net. Also note that if K is a field then it is possible to define regulus nets over K (see. *e.g.* Johnson and Lin [9]). Also note that a pseudo regulus net is a subplane covered net by [9].

We note that the nets of section 3 in Thm(3.10) are pseudo regulus nets:

Theorem 4.2 (Johnson and Lin [9]). Let $\sum$ be any projective space of dimension at least three. Let N be any codimension two subspace. Define the structure $R=(P,L,B,C,I)$ of the sets of points P, lines L, subplanes B, parallel classes C and incidence I to be the lines of $\sum$ skew to N, points of $\sum-N$, planes of $\sum$ which intersect N in a unique point, hyperplanes of $\sum$ which contain N, incidence is the incidence inherited from $\sum$.

Then R is a pseudo regulus net.

Hence, since any subplane covered net is isomorphic to the structure $\sum$, we have the following characterization of subplane covered nets.

Theorem 4.3 Any subplane covered net is a pseudo regulus net.

Note as a finite pseudo regulus net is a regulus net that we obtain the results of De Clerck and Johnson as a corollary to Thm.(4.3).

Corollary 4.4 (De Clerck and Johnson [4]).
Any finite subplane covered net is a regulus net.

There are many translation planes whose spreads may be represented as the union of regulus nets with various intersection properties. For example, a translation plane whose spread is in PG(3,q) and which is the union of q reguli that share a line corresponds to a flock of a quadratic cone. If the spread is the union of q+1 reguli that share two lines, there is a corresponding flock of a hyperbolic quadric. Furthermore, there are many planes whose spread contains q-1 mutually disjoint reguli. Moreover, there are planes of order q^n with n not 2 with similar properties. Thus, we see that there are many open problems concerning the connections with translation planes whose spreads contain various configurations of reguli and projective spaces. We shall mention specifically only the problems associated with flocks of quadratic cones in PG(3,q).

Problem: Let F be a flock of a quadratic cone in PG(3,q) and let π_F denote the associated translation plane of order q^2 which can be represented as a set of q regulus nets that share a common line(component). There are q projective spaces each isomorphic to PG(3,q) associated with the q regulus nets . Each regulus net produces a projective space Σ and a fixed line N on the space such that the points of the net are the lines of $\Sigma - N$. Since the points of each net are the points of the translation plane, we have q different projective spaces Σ and q lines N_i such that the sets of lines of $\Sigma_i - N_i$ are identified.

The problem would be to find a combinatorial characterization of a flock of a quadratic cone in terms of these projective spaces.

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