Derivable affine planes and translation planes

Norbert Knarr

Abstract

It was proved by Johnson that every derivable affine plane admits a natural embedding into a 3-dimensional projective space over some skewfield. We show that under this embedding the lines of the affine plane which do not belong to the derivation set correspond to spreads of the projective space. Furthermore, we investigate the spreads associated with derivable translation planes or derivable dual translation planes more closely. Finally, we study derivable affine planes admitting a so-called affine Hughes group.

1 Derivable affine planes and associated spreads

Let $\mathcal{A} = (A, \mathcal{G})$ be an affine plane and denote the projective extension of $\mathcal{A}$ by $\bar{\mathcal{A}} = (\bar{A}, \bar{\mathcal{G}})$, where $\bar{\mathcal{G}} = \mathcal{G} \cup \{L_\infty\}$. For a subset $D \subset L_\infty$ we put $\mathcal{G}(D) = \{L \in \mathcal{G} | L \wedge L_\infty \in D\}$ and we denote by $\mathcal{B}(D)$ the set of the Baer subplanes of $\mathcal{A}$ which intersect L_∞ in D. A subset $D \subset L_\infty$ is called a derivation set of $\mathcal{A}$ if for any two distinct points $p, q \in A$ with $p \vee q \in \mathcal{G}(D)$ there exists a Baer subplane $\mathcal{A}_{p,q,D}$ of $\mathcal{A}$ which contains p, q and D. By [7], Lemma 2.4 the Baer subplane $\mathcal{A}_{p,q,D}$ is uniquely determined by p, q and D, under the assumption that D is a derivation set. If D is a derivation set of $\mathcal{A}$ we can form a new affine plane $\mathcal{A}'$ with the same set of points by replacing the lines in $\mathcal{G}(D)$ with the Baer subplanes in $\mathcal{B}(D)$. An affine plane $\mathcal{A}$ admitting a derivation set is called derivable and $\mathcal{A}'$ is called the derived plane of $\mathcal{A}$ with respect to D. The plane $\mathcal{A}'$ can be derived in such a way that $\mathcal{A}''$ is isomorphic to $\mathcal{A}$.

Extending results of Cofman [3], Johnson [8] has shown that every derivable affine plane admits a natural embedding into a 3-dimensional projective space. Johnson's theorem is most easily formulated as follows.

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Consider the geometry $\mathcal{S} = (\mathcal{G}(D), A, \mathcal{B}(D))$ with the incidence inherited from $\mathcal{A}$. Then there exists a 3-dimensional projective space $\mathrm{PG}_3F = (P, \mathcal{L}, \mathcal{E})$ over a skewfield F and a line $S \in \mathcal{L}$ such that $\mathcal{S}$ is isomorphic to the geometry obtained from PG_3F by deleting all points on S, all lines intersecting S and all planes containing S. Note that the planes through S correspond to the parallel classes of lines in $\mathcal{G}(D)$. Dually, the points on S correspond to the parallel classes of elements of $\mathcal{B}(D)$ viewed as lines of $\mathcal{A}'$.

It follows easily from the definition of a derivation set that the geometry $\mathcal{S}$ admits a diagram of the following type

$$\bigcirc \overset{Af^*}{\text{———}} \bigcirc \overset{Af}{\text{———}} \bigcirc .$$

Geometries with this type of diagram have been investigated by Cuypers [4], and Johnson's theorem on derivable affine planes, as well as the generalization of this theorem to subplane covered nets [10], can

In the sequel we identify the point set A of $\mathcal{A}$ with the set $\mathcal{L}_S = \{L \in \mathcal{L} | L \cap S = \emptyset\}$. The lines of $\mathcal{A}$ which are not in $\mathcal{G}(D)$ then become certain sets of lines of PG_3F and the question arises what kind of properties these sets of lines have. The answer is given by the following theorem, whose proof goes back to Ostrom [15], at least in the finite case.

1.1 Theorem *Let $G \in \mathcal{G} \setminus \mathcal{G}(D)$. Then the set $\mathcal{B} = \mathcal{B}(G) = \{x | x \in G\} \cup \{S\} \subset \mathcal{L}$ is a spread and a dual spread of* PG_3F.

Proof. Since G is a line of $\mathcal{A}$ whose slope is not in D, it intersects every line $M \in \mathcal{G}(D)$ in precisely one point. It follows that the point of PG_3F corresponding to M is contained in precisely one element of $\mathcal{B} \setminus \{S\}$. Of course the points of S are contained in S and hence $\mathcal{B}$ is a spread of PG_3F. Since G is also a line of the derived plane $\mathcal{A}'$, the same reasoning shows that $\mathcal{B}$ is also a dual spread. ■

The translation plane associated with the spread $\mathcal{B}(G)$ will be called $\mathcal{A}(G)$.

In the sequel we use the methods for the description of spreads of PG_3F given in [11]. We coordinatize PG_3F using the 4-dimensional right vector space $F^2 \times F^2$ over F such that $S = \{0\} \times F^2$. Then every line of PG_3F which does not intersect S is the graph of a linear mapping from F^2 to F^2, which we identify with its associated 2×2-matrix. Assume now that $\mathcal{B}$ is a spread of PG_3F which contains S. Then the linear mappings whose graphs are contained in $\mathcal{B} \setminus \{S\}$ form a spread set $\mathcal{M}$ of F^2, i.e. for any two distinct $\lambda_1, \lambda_2 \in \mathcal{M}$ the linear mapping $\lambda_1 - \lambda_2$ is bijective and for any $x \in F^2 \setminus \{0\}$ the mapping $\varrho_x : \mathcal{M} \to F^2 : \lambda \mapsto \lambda(x)$ is bijective, cp e.g. [11], Proposition 1.11. It follows from [11], 2.2. that there exists a mapping $f = (f_1, f_2) : F^2 \to F^2$ such that

$$\mathcal{M} = \left\{ \begin{pmatrix} a & f_1(a,b) \\ b & f_2(a,b) \end{pmatrix} \middle| a, b \in F \right\}$$

and f is transversal, i.e. f is bijective and there holds

(T1) For any two distinct points $x, y \in F^2$ the lines $x \vee y$ and $f(x) \vee f(y)$ are not parallel in the affine plane associated with F^2.

(T2) For each dilatation δ of the affine plane F^2 the mapping $\delta \circ f$ has a fixed point.

1.2 Lemma *A mapping $f : F^2 \to F^2$ is transversal if and only if it satisfies*

(S) For every $s \in F$ the mapping $f - \mathrm{id} \cdot s : F^2 \to F^2 : x \mapsto f(x) - xs$ is bijective.

Proof. Assume first that f is transversal. Let $s \in F$ and assume that $f - \mathrm{id} \cdot s$ is not injective. Since f is bijective we may assume $s \neq 0$. Then there are distinct $x, y \in F^2$ with $f(x) - xs = f(y) - ys$. It follows that $f(x) - f(y) = (x - y)s$ and hence the lines $x \vee y$ and $f(x) \vee f(y)$ are parallel, violating (T1).

The dilatations of F^2 are precisely the mappings $\delta : F^2 \to F^2 : x \mapsto xs^{-1} + t, s \in F^\times, t \in F^2$. From (T2) we infer that the equation $f(x)s^{-1} + t = x$ has a solution for every $s \in F^\times, t \in F^2$. It follows that $f - \mathrm{id} \cdot s$ is surjective for $s \neq 0$ and f is surjective anyhow.

The proof of the reverse direction is similar. ■

If $\mathcal{B}$ is also a dual spread then f is also $*$-transversal by[11], 2.3, i.e. besides (T1) it also satisfies

(T2*) For any two parallel lines H_1, H_2 of the affine plane F^2 we have $H_1 \cap f(H_2) \neq \emptyset$.

Note that if F is finite or if $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and f is continuous, then f is transversal if and only if it is $*$-transversal, cp. [11], 1.6.

2 Translation planes and dual translation planes

In this section we examine derivable translation planes and derivable dual translation planes and their associated spreads.

2.1 Proposition *Let $\mathcal{A}$ be a derivable affine plane, then every dilatation τ of $\mathcal{A}$ is induced by a linear mapping of F^4 whose matrix is of the form*

$$\begin{pmatrix} I & \\ C & D \end{pmatrix},$$

where I is the identity matrix and C and D are 2×2-matrices over F with D invertible.

Moreover, τ is a translation if and only if $D = I$ and if τ is a nontrivial homology then $Dy \neq y$ for all $y \in F^2 \setminus \{0\}$.

Proof. According to [9], Theorem 2.9, every collineation of $\mathcal{A}$ which fixes the derivation set D is induced by a semilinear bijection of F^4 which fixes S. Since the planes through S correspond to the parallel classes of lines in $\mathcal{G}(D)$ the dilatation τ

fixes all planes through S. Hence τ even is induced by a linear mapping $\varphi : F^4 \to F^4$ and the matrix of φ looks as follows:

$$\begin{pmatrix} aI & \\ C & D \end{pmatrix} \text{ with } a \in Z(F)^\times, C \in F^{2\times 2}, D \in \mathrm{GL}_2 F,$$

where $Z(F)$ denotes the center of F. Since we use homogeneous coordinates we may assume $a = 1$. This proves the first part of the proposition.

Let τ be a translation. Then τ is also a translation of the derived plane $\mathcal{A}'$ and hence φ fixes all points on S. It follows that $D = bI$ for some $b \in Z(F)^\times$. Let $M \in F^{2\times 2}$, then φ maps the graph of M to the graph of $DM + C = bM + C$. If $b \neq 1$ then the equation $DM + C = M$ has a solution M and hence τ cannot be a translation.

Assume now that τ is a nontrivial homology. Then the equation $DM + C = M$ has precisely one solution $M \in F^{2\times 2}$. This implies that the matrix $D - I$ is invertible and hence the equation $Dy = y$ has no solution $y \in F^2 \setminus \{0\}$. ∎

2.2 Corollary *The translation group of a derivable affine plane is isomorphic to a subgroup of the additive group of some vector space, in particular it is abelian and all nonidentity translations have the same order.*

2.3 Proposition *Let $v \in L_\infty \setminus D$ and assume that $\Gamma_{[v,L]}$ is a transitive translation group. Let G be a line of $\mathcal{A}$ which passes through v, then the translation plane $\mathcal{A}(G)$ is of Lenz type V.*

Proof. The group $\Gamma_{[v,L]}$ acts on $\mathrm{PG}_3 F$ as follows. It fixes the line S elementwise and acts regularly on $\mathcal{B}(G) \setminus \{S\}$. Hence it induces a transitive group of shears on $\mathcal{A}(G)$. ∎

2.4 Corollary *Let $\mathcal{A}$ be a derivable translation plane and let $G \notin \mathcal{G}(D)$ be a line of $\mathcal{A}$. Then the translation plane $\mathcal{A}(G)$ is of Lenz type V.*

Let $\mathcal{B}$ be a spread of a 3-dimensional pappian projective space $\mathrm{PG}_3 F$ which contains a regulus $\mathcal{R}$. Then $\mathcal{B}' = \mathcal{B} \setminus \mathcal{R} \cup \mathcal{R}'$ is also a spread of $\mathrm{PG}_3 F$, where $\mathcal{R}'$ denotes the opposite regulus of $\mathcal{R}$. The translation plane $\mathcal{A}'$ associated with $\mathcal{B}'$ is derived from the translation plane $\mathcal{A}$ associated with $\mathcal{B}$ and the derivation is said to be obtained by reversion of reguli.

2.5 Lemma *Let F be a commutative field and let $\mathcal{M} \subset F^{2\times 2}$ be a spread set. Then the translation plane associated with $\mathcal{M}$ is pappian if and only if $\mathcal{M}$ is an affine subspace of the F-vector space $F^{2\times 2}$.*

Proof. This is a well-known fact, it follows e.g. from [11], Proposition 4.8. ∎

2.6 Proposition *Let $\mathcal{A} = (A, \mathcal{G})$ be a derivable translation plane. Then the derivation is obtained by reversion of reguli if and only if for every line $G \in \mathcal{G} \setminus \mathcal{G}(D)$ the translation plane $\mathcal{A}(G)$ associated with the spread $\mathcal{B}(G)$ is pappian.*

Proof. Assume first that the derivation of $\mathcal{A}$ is obtained by reversion of reguli. Then the point set of $\mathcal{A}$ can be taken to be a vector space F^4 for some commutative field F. Furthermore, we can assume that the spread $\mathcal{B}$ of $\mathcal{A}$ contains the standard regulus consisting of the subspaces $\{(x, y, xs, ys) | x, y \in F\}$ for $s \in F$ and $S = \{(0, 0, u, v) | u, v \in F\}$. The identification of the points of $\mathcal{A}$ with the elements of $F^{2\times 2}$ can be made in such a way that (x, y, u, v) corresponds to $\begin{pmatrix} x & u \\ y & v \end{pmatrix}$. It follows that for each line $G \in \mathcal{G} \setminus \mathcal{G}(D)$ the spread set for $\mathcal{B}(G)$ is an affine subspace of $F^{2\times 2}$. By 2.5 the translation plane $\mathcal{A}(G)$ is pappian.

Assume now that for each $G \in \mathcal{G} \setminus \mathcal{G}(D)$ the translation plane $\mathcal{A}(G)$ is pappian. Since F is contained in the kernel of $\mathcal{A}(G)$ this implies that F is commutative. We identify the point set of $\mathcal{A}$ with $F^{2\times 2}$ in such a way that the elements of $\mathcal{G}(D)$ correspond to the translates of the subspaces $\left\{ \begin{pmatrix} x & xs \\ y & ys \end{pmatrix} \middle| x, y \in F \right\}$ for $s \in F$ and $\left\{ \begin{pmatrix} 0 & u \\ 0 & v \end{pmatrix} \middle| u, v \in F \right\}$. Now each line $G \in \mathcal{G} \setminus \mathcal{G}(D)$ corresponds to a spread set in $F^{2\times 2}$. Since $\mathcal{A}(G)$ is pappian each of these spread sets is an affine subspace of $F^{2\times 2}$. It follows that the spread $\mathcal{B}$ of $\mathcal{A}$ is contained in $F^{2\times 2}$ and that $\mathcal{B}$ contains the standard regulus. ∎

The following result is essentially due to Ostrom [15].

2.7 Proposition *Let $\mathcal{B}$ be a spread and a dual spread of a 4-dimensional vector space V over a skewfield F and let $\mathcal{A}$ denote the translation plane associated with $\mathcal{B}$.*

For each 3-dimensional affine subspace U of V there is precisely one component $S \in \mathcal{B}$ such that U contains a coset of S. Then the set D_U consisting of L_∞ and of all cosets of S that are contained in U is a derivation set of the dual plane $\mathcal{A}_d$ of $\mathcal{A}$. Let G be any line of $\mathcal{A}_d$ not in D_U, then the translation plane $\mathcal{A}(G)$ is isomorphic to $\mathcal{A}$.

Proof. The fact that D_U is a derivation set of $\mathcal{A}_d$ was obtained by Bose and Barlotti [2].

We may assume that the line G corresponds to the origin of V. Note that the lines in $\mathcal{G}(D_U)$ are precisely the points in U and the points of L_∞. So the derivation set D_U is already embedded in a 3-dimensional projective space over F. In order to find $\mathcal{B}(G)$ we just have to take the lines of $\mathcal{A}$ passing through the origin of V and intersect them with U. Obviously, this yields a spread which is equivalent to $\mathcal{B}$. ∎

3 Planes admitting affine Hughes groups

Let $\mathcal{A}$ be an affine plane which contains a desarguesian Baer subplane $\mathcal{Q} \cong \mathrm{AG}_2F$ such that the improper line of $\mathcal{A}$ belongs to $\mathcal{Q}$. A collineation group Θ of $\mathcal{A}$ which leaves $\mathcal{Q}$ invariant and induces an overgroup of ASL_2F on $\mathcal{Q}$ in its natural action is called an affine Hughes group of $\mathcal{A}$. This notion is taken from Löwen [13], who studies the case $F = \mathbb{R}$ and assumes that Θ even induces the group $\mathrm{AGL}_2^+\mathbb{R}$ or $\mathrm{AGL}_2\mathbb{R}$. For the general approach it seems to be convenient to allow the smaller group ASL_2F. We are mainly interested in derivable affine planes which admit an affine Hughes group such that the Baer subplane $\mathcal{Q}$ belongs to $\mathcal{B}(D)$, where we have used the notation from Section 1. According to [13], Theorem 7, this situation is realized in every 4-dimensional locally compact affine plane which admits an affine Hughes group.

We now give a description of planes admitting affine Hughes groups using the results on derivable planes obtained so far. It turns out to be more convenient to exchange the roles of $\mathcal{G}(D)$ and $\mathcal{B}(D)$, i.e. we identify the Baer subplanes in $\mathcal{B}(D)$ with the points of PG_3F not on S.

3.1 Proposition *Let $\mathcal{A}$ be a derivable affine plane admitting an affine Hughes group Θ such that the corresponding Baer subplane $\mathcal{Q}$ belongs to $\mathcal{B}(D)$. Assume moreover that $\mathcal{Q} \cong \mathrm{AG}_2F$ is pappian, i.e. that F is commutative. Let Λ be the largest subgroup of Θ which induces ASL_2F on $\mathcal{Q}$. Then Λ is the direct product of a group $\Omega \cong \mathrm{ASL}_2F$ and a group K consisting of Baer collineations with respect to $\mathcal{Q}$. The point set of $\mathcal{A}$ can be identified with $F^{2\times 2}$ in such a way that $\mathcal{Q}$ corresponds to the point $(1,0,0,0)^tF$ and Ω acts on PG_3F as the group*

$$\left\{ \begin{pmatrix} I & \\ B & A \end{pmatrix} \middle| \, A \in \mathrm{SL}_2F, B = \begin{pmatrix} 0 & s \\ 0 & t \end{pmatrix}, s,t \in F \right\}.$$

Proof. The elements of Θ leave the derivation set D invariant. Consequently they induce permutations of $\mathcal{G}(D)$ and $\mathcal{B}(D)$. It follows now from Johnson [9], Theorem 2.9, that the group induced by Θ on PG_3F is contained in the subgroup of $\mathrm{P\Gamma L}_4F$ which fixes S. Since Θ induces the natural action on $\mathcal{Q}$ this group is even contained in PGL_4F. Up to conjugation, we may assume that $\mathcal{Q} = (1,0,0,0)^tF$. Then Θ is contained in the group

$$\Phi = \left\{ \begin{pmatrix} C & \\ B & A \end{pmatrix} \middle| \, C = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, a,d \in F^\times, b \in F, B = \begin{pmatrix} 0 & s \\ 0 & t \end{pmatrix}, s,t \in F, A \in \mathrm{GL}_2F \right\}.$$

Since F is commutative and the action is projective we can assume $a = 1$. It follows that Φ is isomorphic to the direct product of the subgroups AGL_2F and AGL_1F which are obtained by setting $C = I$ or $B = 0, A = I$, respectively. The group AGL_1F is precisely the kernel of the action of Φ on the plane $\mathcal{Q}$, cp. also [3], Theorem 3. Let $\mathrm{K} \leq \mathrm{AGL}_1F$ be the kernel of the action of Λ on $\mathcal{Q}$. It follows that $A \in \mathrm{SL}_2F$ and there exists a mapping $\chi : \mathrm{ASL}_2F \to \mathrm{AGL}_1F$ such that

$$\Lambda = \left\{ \begin{pmatrix} \chi(A,B)C & \\ B & A \end{pmatrix} \middle| \, A \in \mathrm{SL}_2F, B = \begin{pmatrix} 0 & s \\ 0 & t \end{pmatrix}, s,t \in F, \, C \in \mathrm{K} \right\}.$$

Let Ξ be the group generated by K and by the elements $\chi(A, B)$, then K is normal in Ξ and the induced mapping $\bar{\chi} : \mathrm{ASL}_2F \to \Xi/\mathrm{K}$ is a surjective homomorphism. Since Ξ/K is solvable the group SL_2F is contained in the kernel of $\bar{\chi}$ and hence $\bar{\chi}$ is trivial. So we may assume that χ itself is trivial and the proof is complete. ■

Remark. In the case $F = \mathbb{R}$ and $\Theta = \mathrm{AGL}_2^+\mathbb{R}$ the action is determined by a homomorphism of the group $\mathrm{P} \cong \mathbb{R}$ of positive real homotheties to $\mathrm{AGL}_1\mathbb{R}$. These homomorphisms yield exactly the actions determined in [13], Theorem 4.

3.2 Theorem *Let the conditions of 3.1 be satisfied. Then there exists a family $\mathcal{T}$ of mappings $f : F^2 \to F^2$ with $f(0) = 0$ such that the following hold*

(P1) Every $f \in \mathcal{T}$ is transversal and $*$-transversal.

(P2) For every $A \in \mathrm{SL}_2F, f, g \in \mathcal{T}$ the mapping $A \circ f \circ A^{-1} - g$ is either bijective or identically zero.

(P3) For every $x_1, x_2, y \in F^2$ with $x_1 - x_2$ and y linearly independent there exist $f \in \mathcal{T}$ and $A \in \mathrm{SL}_2F$ such that $Ay = (f \circ A)(x_1) - (f \circ A)(x_2)$.

The point set of $\mathcal{A}$ can be identified with $F^4 = F^{2\times 2}$ in such a way that the lines are the following sets

(I) $\left\{ \begin{pmatrix} x & u \\ mx+s & mu+t \end{pmatrix} \middle| x, u \in F \right\}$ *for $m, s, t \in F$ and* $\left\{ \begin{pmatrix} c & d \\ y & v \end{pmatrix} \middle| y, v \in F \right\}$ *for $c, d \in F$.*

(II) $\left\{ A \begin{pmatrix} x & f_1(x,y) \\ y & f_2(x,y) \end{pmatrix} + \begin{pmatrix} 0 & s \\ 0 & t \end{pmatrix} \middle| x, y \in F \right\}$ *for $A \in \mathrm{SL}_2F, s, t \in F, f \in \mathcal{T}$.*

The group ASL_2F acts as follows

$$\begin{pmatrix} x & u \\ y & v \end{pmatrix} \mapsto A \begin{pmatrix} x & u \\ y & v \end{pmatrix} + \begin{pmatrix} 0 & s \\ 0 & t \end{pmatrix} \quad \textit{for } A \in \mathrm{SL}_2F, s, t \in F.$$

Conversely, every family $\mathcal{T}$ of mappings $f : F^2 \to F^2$ with $f(0) = 0$ satisfying (P1) to (P3) yields a derivable affine Hughes plane in the way just described.

Proof. We use the notation of the preceding proposition.

The line of PG_3F corresponding to the matrix $M \in F^{2\times 2}$ is the set $\{(z, Mz) \in F^2 \times F^2 | z \in F^2\}$. The matrix

$$\begin{pmatrix} I & \\ B & A \end{pmatrix} \in \Omega \cong \mathrm{ASL}_2F$$

maps this set to $\{(z, Bz + AMz) | z \in F^2\}$. This proves the assertion on the the action of ASL_2F.

Since we exchanged the roles of lines and Baer subplanes the lines in $\mathcal{G}(D)$ correspond to the planes of PG_3F not containing S. Such a plane has a basis consisting either of the vectors

$$\begin{pmatrix}1\\0\\0\\s\end{pmatrix}, \begin{pmatrix}0\\1\\0\\t\end{pmatrix}, \begin{pmatrix}0\\0\\1\\m\end{pmatrix}, \ m,s,t \in F \text{ or } \begin{pmatrix}1\\0\\c\\0\end{pmatrix}, \begin{pmatrix}0\\1\\d\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \ c,d \in F.$$

The line of PG_3F corresponding to the matrix

$$M = \begin{pmatrix} x & u \\ y & v \end{pmatrix} \in F^{2\times 2}$$

is contained in such a plane if and only if there holds $y = mx + s, v = mu + t$ or $x = c, v = d$ in the respective cases. This proves the assertion on the lines of type (I).

Every other line of $\mathcal{A}$ is a spread set and a dual spread set of $F^{2\times 2}$ and hence can be written as

$$\left\{ \begin{pmatrix} x & f_1(x,y) \\ y & f_2(x,y) \end{pmatrix} \middle| \, x,y \in F \right\}$$

for a suitable mapping $f : F^2 \to F^2$ which is transversal and $*$-transversal. Every line can be mapped onto a line through the origin by an element of the group Ω. Thus $\mathcal{A}$ is determined by a collection $\mathcal{T}$ of selfmappings of F^2 which fix the origin. Also, (P1) is satisfied.

Let $f, g \in \mathcal{T}, A \in \mathrm{SL}_2F, s,t \in F$ and put $h = A \circ f \circ A^{-1}$. Then the lines

$$\left\{ \begin{pmatrix} x & h_1(x,y)+s \\ y & h_2(x,y)+t \end{pmatrix} \middle| \, x,y \in F \right\} \text{ and } \left\{ \begin{pmatrix} x & g_1(x,y) \\ y & g_2(x,y) \end{pmatrix} \middle| \, x,y \in F \right\}$$

are either equal or parallel or they intersect in precisely one point. It follows that $h - g$ is either bijective or identically zero, which yields (P2).

Note that two points $M_1, M_2 \in F^{2\times 2}$ are on a line of type (I) if and only if $\det(M_1 - M_2) = 0$. So let $M_1, M_2 \in F^{2\times 2}$ with $\det(M_1 - M_2) \neq 0$ and denote the row vectors of M_i by x_i and y_i, $i = 1, 2$. Then $x_1 - x_2$ and $y = y_1 - y_2$ are linearly independent. Since M_1 and M_2 are on a line of type (II) there exists $f \in \mathcal{T}, A \in \mathrm{SL}_2F$ and $w \in F^2$ such that $y_i = (A \circ f \circ A^{-1})(x_i) + w$ for $i = 1, 2$. Subtracting these two equations gives us (P3).

The proof of the converse is similar. ∎

Remarks. (1) The lines of type (II) can also be written inthe form

$$\left\{ A \begin{pmatrix} x & u \\ g_1(x,u) & g_2(x,u) \end{pmatrix} + \begin{pmatrix} 0 & s \\ 0 & t \end{pmatrix} \middle| \, x,u \in F \right\}, \ A \in \mathrm{SL}_2F, s,t \in F,$$

for suitable mappings $g : F^2 \to F^2$. However, in this description there seems to be no easy way to formulate conditions on the defining functions.

(2) If F is finite or if $F \in \{\mathbb{R}, \mathbb{C}\}$ and all elements of $\mathcal{T}$ are continuous, then every transversal mapping is also $*$-transversal. It follows from 1.2 that then (P1) can be replaced by

(P1') For every $f \in \mathcal{T}$ and every $s \in F$ the mapping $f - \mathrm{id} \cdot s : F^2 \to F^2 : x \mapsto f(x) - xs$ is bijective.

In general, it is not clear how many transversal mappings one has to choose in order to determine the plane $\mathcal{A}$. If $f, g \in \mathcal{T}$ and if there exists $A \in \mathrm{SL}_2 F$ with $A \circ f \circ A^{-1} = g$, then one of them is superfluous. Because of condition (P2) one only has to check whether $(A \circ f \circ A^{-1})(1,0) = g(1,0)$.

3.3 Lemma *Let the conditions of 3.1 be satisfied and let* $f, g \in \mathcal{T}$ *with* $f_2(1,0) = g_2(1,0)$, *then there exists* $A \in \mathrm{SL}_2 F$ *with*$A \circ f \circ A^{-1} = g$.

Proof. Since f ist $*$-transversal it maps the x-axis to a set which intersects each parallel to the x-axis in precisely one point. It follows that the mapping $x \mapsto f_2(x,0)$ from F to F is a bijection. Now $f_2(0,0) = 0$ and hence $f_2(1,0) \neq 0$. Let

$$A = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \text{ with } t \in F,$$

then we have

$$(A \circ f \circ A^{-1}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} f_1(1,0) + t f_2(1,0) \\ f_2(1,0) \end{pmatrix}.$$

Since $f_2(1,0) = g_2(1,0)$ we can choose t such that $A \circ f \circ A^{-1} = g$. ■

In the particular case $F = \mathbb{R}$ this lemma can be improved as follows.

3.4 Proposition *Let the conditions of 3.1 be satisfied and assume that* $F = \mathbb{R}$ *and that all functions* $f \in \mathcal{T}$ *are continuous. Let* $f, g \in \mathcal{T}$ *with* $f_2(1,0) g_2(1,0) > 0$, *then there exists* $A \in \mathrm{SL}_2 \mathbb{R}$ *with* $A \circ f \circ A^{-1} = g$. *Hence* $\mathcal{A}$ *is determined by at most two transversal mappings.*

Proof. As was noted in the proof of the last lemma, the mapping $h : \mathbb{R} \to \mathbb{R}$ with $h(a) = f_2(a,0)$ is bijective and we have $h(0) = 0$. Since h is continuous this implies that each $b \in \mathbb{R}$ which has the same sign as $f_2(1,0)$ can be written as $a h(a)$ for a suitable $a \in \mathbb{R}$. Consider now the matrix

$$A = \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix} \text{ with } a \in \mathbb{R}^{\times},$$

then we have

$$(A \circ f \circ A^{-1}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a^{-1} f_1(a,0) \\ a f_2(a,0) \end{pmatrix}.$$

The result now follows from the considerations above and from 3.3. ■

3.5 Proposition *Let the conditions of 3.1 be satisfied. Then* $\mathcal{A}$ *is a translation plane if and only if all mappings* $f \in \mathcal{T}$ *are additive homomorphisms.*

Proof. Assume first that $\mathcal{A}$ is a translation plane. Then it follows from 2.3 that for each line G of $\mathcal{A}$ which is not contained in the derivation set the translation plane $\mathcal{A}(G)$ is of Lenz type V. Consequently, all transversal mappings $f \in \mathcal{T}$ are additive homomorphisms, cp. e.g. [11], Proposition 4.6.

Assume now that all mappings $f \in \mathcal{T}$ are additive homomorphisms of F^2. Then it is easily seen that the mappings

$$\begin{pmatrix} x & u \\ y & v \end{pmatrix} \mapsto \begin{pmatrix} x & u \\ y & v \end{pmatrix} + \begin{pmatrix} z & s \\ w & t \end{pmatrix}$$

for $z, w, s, t \in F$ form a transitive translation group of $\mathcal{A}$. It follows that $\mathcal{A}$ is a translation plane. ∎

The planes constructed in [1], Satz 5, are examples of translation planes which admit an affine Hughes group.

Remark. It is known that the finite Hughes planes are derivable. Using the description of these planes given by Ostrom [16] one can show that the translation plane obtained from a line not belonging to the derivation set is isomorphic to the plane over the nearfield from which the respective Hughes plane is constructed.

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Norbert Knarr
Institut für Analysis
Technische Universität Braunschweig
Pockelsstr. 14
D-38106 Braunschweig
E-mail: n.knarr@tu-bs.de