Asymptotic properties of Abelian integrals arising in quadratic systems*

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Abstract

We consider quadratic perturbations of the vector field $(-y+ax^2+by^2)\partial_x+x(1+cy)\partial_y$ and study its limit cycles via Abelian integrals. The asymptotic analysis suggests that such systems have no more than 4 limit cycles.

1 Introduction

The 16-th Hilbert problem is to find a bound $N(n)$ for the number of limit cycles of planar vector fields of degree n. Even for quadratic systems the answer is unknown. There are examples [2], [9] of quadratic systems with 4 limit cycles. In the present paper the author examines the possibility of finding quadratic systems with >4 limit cycles in one specific situation.

We consider the vector field

$$\dot{x} = -y + ax^2 + by^2,\ \dot{y} = x(1+cy),\ c \leq 0 \tag{1}$$

which is time–reversible, (invariant under $(x, y, t) \to (-x, y - t)$), and has two centers: $x = y = 0$ and $x = 0, y = 1/b$, (see below). One can check that the center $(0, 0)$ has cyclicity 2 for $3a + 5b \neq c$ and 3 for $3a + 5b = c$, (see Section 5 below). The other center also has cyclicity 2 or 3. It seems that the configuration with 3 limit cycles around one focus and 2 cycles around the other focus for a perturbation of (1) is possible.

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Because the problem of limit cycles for systems, which are close to systems with Darboux first integral $\prod f_i^{\alpha_i}$, reduces to the problem of zeroes of certain Abelian integral, we study this integral. We do not investigate the Abelian integral completely, we study only its behaviour near the ends of its domain of definition. The results show that our expectations of finding 5 limit cycles were premature. In the author's opinion the situation considered here is the last one where using relatively simple methods one could show existence of >4 limit cycles. Now the author believes that $N(2) = 4$.

There are some works devoted to study the Abelian integrals arising from the system (1). Chicone and Jacobs [3] investigated Abelian integrals for the cases of isochronous center: $(a, b, c) = (a, b, -2)$ with (a, b) equal to $(-1, 1)$ (S_1), to $(-2, 0)$ (S_2), to $(-1/2, 0)$ (S_3) and to $(-4, 1)$ (S_4) (see Fig. 3). They have shown that the maximal number of zeroes of Abelian integral corresponding to cycles around $(0, 0)$ is 1 for S_1 and 2 for other S_j.

Another result was obtained by Gavrilov and Khorozov [4]. They studied the Abelian integrals in the case when the system (1) is Hamiltonian ($2a + c = 0$) and $c = -2,\ 0 < b$ (see Fig. 1). They estimated the number of zeroes of the Abelian integral by 3 for $0 < b < 2$ and by 1 for $2 < b$. Li Chengzhi, Llibre and Zhang Zhi-fen [6] have shown that this bound is 1 for all $0 < b$.

Shafer and Zegeling [8] have proven that, if $c = -2$ and the point (a, b) belongs to the four convex domains bounded by: the line $3a+5b+2 = 0$, the half-line $a+b = 0,\ a > 0$, the half-line $a + b + 2 = 0,\ a < -2$ and the interval $-2 < a < 0,\ b = 0$, then there exist perturbations with three limit cycles.

In the paper [7], an asymptotic analysis of limit cycles, which are close to the boundary of half of the Poicaré disc, was performed. Two cases of the system (1) were considered: $c = -2,\ a = 0,\ 0 < b < 2$ (the graphic H_9^1) and $c = a = -2,\ 0 < b < 2$ (the graphic H_{11}^1). The cyclicity of the graphic H_9^1 is 2 and the ciclicity of H_{11}^1 is 3. There the authors used the method of asymptotic expansion of the return map.

Świrszcz has investigated the cyclicity of (infinite) contours consisting of a hyperbola and an arc at infinity for $c = -2,\ a < -2,\ 0 < b < -a$. This cyclicity is generally 2 but is 3 along an analytic curve in the (a, b)–plane; this curve begins at $a = -2, b = 0$, passes through the point $a = -4, b = 2$ and is asymptotic to the half-line $b = -a, a < -2$. His method is a combination of asymptotic analysis of the return map and of Abelian integrals.

All the above results concern only limit cycles around one focus. Li Chengzhi with Dumortier [4] studied Abelian integrals corresponding to limit cycles around both centers for generic perturbations of the system (1) with $c = -2, a = -3, b = 1/2$. The resulted complete bifurcational diagram is very complicated, (with several cones in the parameter space corresponding to systems with 4 limit cycles around both foci). This shows that namely the stratum Q_3^R (of reversible systems) of the center variety is the most rich and most difficult in analysis.

In [7] there is a short discussion about large limit cycles around two singular points in the case H_{11}^1. It turns out that there can be at most 4 such large cycles.

Our asymptotic analysis covers the open domain $c = -2,\ -2 < a < 0,\ 0 < b < 2$ of parameters and we investigate limit cycles around both foci. We study asymptotics of the cycles which are close to the boundary of half of the Poincaré disc and which lie near the foci.

2 The phase portraits

If $c \neq 0$, then after some rescaling we can put

$$c = -2.$$

Performing the change $u = 1/2 - y$, we get the system

$$\dot{x} = ax^2 + bu^2 - (b-1)u + (b-2)/4, \ \dot{u} = -2xu$$

with the first integral

$$H = |u|^a (x^2 + \alpha u^2 - \beta u + \gamma) \tag{2}$$

and the integrating factor $M = |u|^a/u$. Here $\alpha = \frac{b}{a+2}$, $\beta = \frac{b-1}{a+1}$, $\gamma = \frac{b-2}{4a}$ for $a \neq 0, -1, -2$.

The analysis of bifurcations of singular points of (1) and of invariant algebraic curves defined by (2) gives the following diagram of phase portraits of (1), (see also [11]).

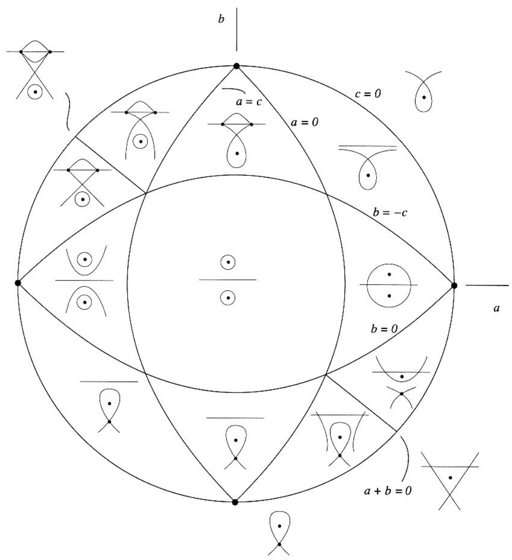

Figure 1.

To be precise we take the intersection of the bifurcational diagram with the semi-sphere $a^2 + b^2 + c^2 = 1, c \leq 0$ and project it onto (a, b)-plane. We do not treat the surface $a = c/2 (= -1)$ as bifurcational (see the first integral (2)) because the topological picture does not change there. We see that the region $0 < b < -c$ consists of systems with 2 centers.

3 The perturbation

We choose the following 3-parameter perturbation of the system (1)

$$\dot{x} = ax^2 + by^2 - y + \epsilon_1 x + \epsilon_2 xy, \ \dot{y} = x(1-2y) + \epsilon_1 y + \epsilon_3 x^2 \tag{3}$$

or

$$\begin{aligned}\dot{x} &= ax^2 + bu^2 - (b-1)u + (b-2)/4 + (\epsilon_1 + \epsilon_2/2)x - \epsilon_2 xu \\ \dot{u} &= -2xu - \epsilon_1/2 + \epsilon_1 u - \epsilon_3 x^2\end{aligned} \tag{4}$$

ϵ_i - small.

Remark 1. On the 7–dimensional space of systems

$$\dot{z} = (i+\lambda)z + Az^2 + Bz\bar{z} + C\bar{z}^2,\ z = x + iy, \tag{5}$$

the group S^1 of changes $z \to e^{i\psi}z$ acts. The systems with center form an algebraic set, called the center manifold, consisting of 4 components: $Q_3^H:\ \lambda = 2A + \bar{B} = 0$ (the Hamiltonian systems), $Q_3^{LV}:\ \lambda = B = 0$ (Lotka–Volterra systems with 3 invariant lines), $Q_3^R:\ \lambda = \mathrm{Im}(AB) = \mathrm{Im}(A^3C) = \mathrm{Im}(\bar{B}^3C) = 0$ (reversible systems) and $Q_4:\ \lambda = A - 2\bar{B} = |B| - |C| = 0$ (with quadratic and cubic invariant curves), (see [11]). The system (1) belongs to Q_3^R (A, B, C - real). The perturbations of the system (1) are three-fold: 3–dimensional (non-essential) changes of the parameters a, b, c, 1–dimensional (non-essential) perturbation along the orbit of S^1 - action and 3–dimensional essential perturbations. The non-essential perturbations are within Q_3^R. The perturbation (3) is just the essential perturbation, (except for $b = 2$ where the essential part of the perturbation (3) is 2–dimensional, see Remark 6 below).

Remark 2. We ought consider the case $b = 2$ separately. In what follows we shall have many limit cases where general formulas and properties fail. We shall omit a separate analysis of them. The reason is two-fold. Firstly, they usually are not difficult but make the work more complicated. Secondly, we are interested in examples of systems with many limit cycles and the limit cases cannot add new examples. Let us state these restrictions explicitely

$$c = -2,\ (a-1)a(a+1)(a+2)b(b-1)(b-2)(a+b)(b-a+2) \neq 0.$$

Limit cycles of the system (4) surrounding the point $x = 0, u = 1/2$ are represented by fixed points of the Poincaré map $\mathcal{P}$. If we expand $\mathcal{P} - id$ into powers of ϵ, then the linear part of this expansion forms the following linear Poincaré–Pontriagin–Melnikov integral along the component $\gamma_h \subset \{u > 0\}$ of the curve $H(x,u) = h$ oriented by the vector field (4), (see [3], [4], [5], [6], [11]),

$$\begin{aligned}&I(h) = \\ &\textstyle\int_{\gamma_h} M[(-(\epsilon_1 + \epsilon_2/2)x + \epsilon_2 xu)du + (-\epsilon_1/2 + \epsilon_1 u - \epsilon_3 x^2)dx] \\ &= \epsilon_1 I_1 + \epsilon_2 I_2 + \epsilon_3 I_3\end{aligned} \tag{6}$$

where

$$\begin{aligned}I_1 &= \tfrac{a-1}{2}J_0 - (a+1)J_1 \\ I_2 &= -J_1/2 + J_2 \\ I_3 &= \tfrac{a-1}{3}\textstyle\int_{\gamma_h} |u|^{a-2}x^3 du\end{aligned}$$

and

$$J_i = J_i(h;\alpha,\beta,\gamma) = \int_{\gamma_h} |u|^{a-2}u^i x du = -2\int_{u_1}^{u_2} u^{a-2+i}x(u)du.$$

Here $x^2(u) = hu^{-a} - \alpha u^2 + \beta u - \gamma$ and $0 < u_1 < u_2$ are the roots of the function $x^2(u)$ (see Fig. 2). The number of zeroes of the function I is equal to the number of limit cycles of the perturbation (4).

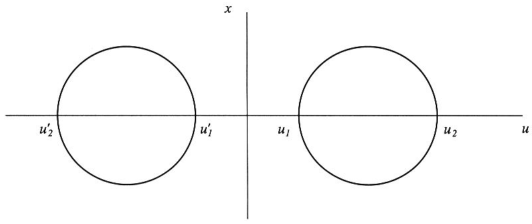

Figure 2.

Remark 3. The above statement is generally true but there are some exceptions. If two or more components of the center manifold meet at some point, then the number of zeroes of the linear Poincaré–Pontriagin–Melnikov integral is less or equal to the number of limit cycles of the perturbation. We ought to take into account higher order terms of the expansion of the Poincaré map. The intersections of the space of systems (1) with other components of the center manifold are: $a = 1$ (Q_3^H), $a + b = 0$ (Q_3^{LV}) and $a = -2/3, b = 0$ and $a = -4, b = 2$ (Q_4).

The integral I_3 is nonzero for $a \neq 1$. Therefore, we can introduce the functions $P_j = I_j/I_3, j = 1,2$ and the equation $I(h) = 0$ is equivalent to

$$L: \quad \epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3 = 0. \tag{7}$$

If we denote by $\Omega = \Omega(a,b)$ the curve $\{(P_1,P_2)(h)\} \subset \mathbb{R}^2$ then then the number of zeroes of the integral I is equal to the number of intersection points of the line L (defined by (7)) with the curve Ω.

If $0 < b < 2$, then the unperturbed system (1) possesses another center $x = 0, y = 1/b$. Limit cycles surrounding this point correspond to zeroes of the Poincaré–Pontriagin–Melnikov integral $I' = I'(h; a, b)$ along the component $\delta_h \subset \{u < 0\}$ of the curve $H(x,u) = h$. The subintegral function is the same as in (5) and $I' = \sum \epsilon_j I'_j$, where

$$\begin{aligned} I'_1 &= \tfrac{a-1}{2} J'_0 - (a+1) J'_1 \\ I'_2 &= -J'_1/2 + J'_2 \\ I'_3 &= -2\tfrac{a-1}{3} \int_{u'_1}^{u'_2} |u|^{a-2} x^3(u) du \end{aligned} \tag{8}$$

$$J'_i = J'_i(h; , \alpha, \beta, \gamma) = -2 \int_{u'_1}^{u'_2} |u|^{a-2} u^i x(u) du, \ u'_2 < u'_1.$$

Introducing the functions $P'_j = I'_j/I'_3, j = 1,2$ we can define the curve $\Omega' = \Omega'(a,b) = \{(P'_1, P'_2)(h)\} \subset \mathbb{R}^2$. We have the following statement.

Lemma 1. *Let (a,b) does not belong to the intersection of Q_3^R with other strata of the center manifold. If $b < 0$ or $b > 2$, then the number of limit cycles of the system (3) is equal to the number of intersections of the line (7) with the curve Ω. If $0 < b < 2$, then the number of limit cycles is equal to the number of intersections of the line (7) with the set $\Omega \cup \Omega'$.*

The remaining part of the work is wholly devoted to studying the Poincaré–Pontriagin–Melnikov integrals. Let us mention some of their properties.

Lemma 2. *We have* $J_i'(h;\alpha,\beta,\gamma) = (-1)^{i+1}J_i(h;\alpha,-\beta,\gamma)$ *and* $I_3'(h;\alpha,\beta,\gamma) = -I_3(h;\alpha,-\beta,\gamma)$. *(Here we forget that* α,β,γ *depend on* a,b*).*

Lemma 3. *We have*

$$(a+2)I_3 = (1-a)[((b-2)/4)J_0 - (b-1)J_1 + bJ_2]$$

$$(a+2)I_3' = (1-a)[((b-2)/4)J_0' - (b-1)J_1' + bJ_2']$$

Proof. We have $(ax^2 + bu^2 - (b-1)u + (b-2)/4)du + 2xudx = 0$ along γ_h. Multiplying this identity by $u^{a-2}x$ and integrating along γ_h we get

$$\begin{aligned} &\tfrac{a+2}{a-1}I_3 \\ &= \tfrac{a+2}{3}\int u^{a-2}x^3du \\ &= a\int u^{a-2}x^3du + 2\int u^{a-1}x^2dx \\ &= -bJ_2 + (b-1)J_1 - \tfrac{b-2}{4}J_0 \end{aligned}$$

In the same way we prove the second identity. ∎

Lemma 4. *The curve* $\Omega'(a,b)$ *is equal to the the image of the curve* $\Omega(a,2-b)$ *under the following transformation of* $\mathbb{R}^2$ *(with* $b' = 2-b$*)*

$$\begin{aligned} P_1' &= \tfrac{b-a-1}{b'}P_1 + \tfrac{2b(a^2-1)}{b'^2}P_2 + \tfrac{2(a+1)(a+2)}{b'^2} \\ P_2' &= \tfrac{-1}{2b}P_1 + \tfrac{ab+b^2-5b+4}{bb'}P_2 + \tfrac{a+2}{bb'} \end{aligned}$$

Proof. From Remark 1 it follows that the space of linear Poincaré–Pontriagin–Melnikov integrals is 3–dimensional. Moreover, from Lemma 6 below it follows that for $b \neq 2$ the integrals I_1, I_2, I_3 are independent and can be chosen as the basis of the space of linear Poincaré–Pontriagin–Melnikov integrals along γ_h. The transition from the singular point $x = y = 0$ to the singular point $x = 0, y = 1/b$ means the transformation $(a,b) \to (a,b')$ of the (a,b) - diagram of the stratum Q_3^R of the center manifold. Indeed, the change $y' = 1/b - y$ gives $\dot{x} = ax^2 + y'(by'-1)$, $\dot{y}' = x(b'/b - 2y')$. The next transformation $x \to (b'/b)^{1/2}x$, $y' \to (b'/b)y'$, $t \to (b/b')^{1/2}t$ leads to the initial system with b replaced with b'.

Now we apply the above transformation to the perturbed system (3). We obtain

$$\begin{aligned} \dot{x} &= ax^2 + y'(b'y'-1) + (k\epsilon_1 + l\epsilon_2)x - m\epsilon_2 xy' \\ \dot{y}' &= x(1-2y') - n\epsilon_1 + k\epsilon_1 y' - k\epsilon_3 x^2, \end{aligned} \tag{9}$$

where $k = (b/b')^{1/2}, l = (bb')^{-1/2}, m = (b'/b)^{1/2}, n = b^{1/2}(b')^{-3/2}$. So, the Poincaré–Pontriagin–Melnikov integral $I'(h';a,b)$ along $\delta_{h'} = \delta_{h'}(a,b)$, connected with the perturbation (3), is equal to an integral along $\gamma_h = \gamma_h(a,b')$, $h = \text{const}\cdot h'$, corresponding to the perturbation (9). But we know that the space of Poincaré–Pontriagin–Melnikov integrals is generated by I_1, I_2, I_3. Hence, $I_j'(h';a,b)$ are equal to some linear combination of $I_i(h;a,b')$. Moreover, $I_1' = (k/2-n)(a-1)J_0 - k(a+1)J_1$, $I_2' = (m/2-l)J_1 - mJ_2$,

$$I_3'(h';a,b) = -kI_3(h;a,b'),\ k > 0. \tag{10}$$

Now Lemma 4 follows from straightforward calculations using (6), (8) and Lemma 3. ∎

4 Behaviour of integrals near critical values

The first integral H has the critical point $x = y = 0$ with the critical value h_c and $H(x,y) - h_c = D\rho^2 + \dots,\ \rho^2 = x^2 + y^2,\ D > 0$. The analysis of behaviour of the integral $I(h)$ can be reduced to study of the Poincaré map $\mathcal{P} : \{x = \rho > 0, y = 0\} \to \{x = \rho > 0, y = 0\}$. Bautin [1] has proved that $\mathcal{P}(\rho) - \rho = \sum_{i=0}^{3} v_{2i+1}\rho^{2i+1}(1 + \dots)$, where $v_1 = e^{2\pi\lambda} - 1$, $v_3 = -2\pi \mathrm{Im}(AB)$, $v_5 = \frac{-2\pi}{3}\mathrm{Im}[(2A + \bar{B})(A - 2\bar{B})\bar{B}C]$, $v_7 = \frac{-5\pi}{4}(|B|^2 - |C|^2)\mathrm{Im}[(2A + \bar{B})\bar{B}^2C]$ for the system (5).

Remark 4. The above form of the expansion of $\mathcal{P}(\rho)$ is not unique. One can add to v_j a combination of $v_i, i < j$.

Remark 5. The Lyapunov quantities v_j are different from those given by Bautin and are taken from the author's work [11].

Lemma 5. *We have* $\lambda = \epsilon_1$, $A = \frac{1}{4}[a - b - 2 + i(\epsilon_3 - \epsilon_2)]$, $B = \frac{1}{2}(a + b + i\epsilon_3)$, $C = \frac{1}{4}[a - b + 2 + i(\epsilon_2 + \epsilon_3)]$ *and therefore*

$$
\begin{aligned}
&v_1 = 2\pi\epsilon_1 + \dots \\
&v_3 = \tfrac{-\pi}{4}[-(a + b)\epsilon_2 + 2(a - 1)\epsilon_3] + \dots \\
&v_5 \,|_{v_3=0} = \tfrac{\pi}{12}(a + b)(3a + 5b + 2)(b - 2)\epsilon_2 + \dots \\
&v_7 \,|_{3a+5b+2=0} = -\pi \cdot 3 \cdot 10^{-4}(a + 4)^2(3a + 2)(a - 1)^2\epsilon_2 + \dots
\end{aligned}
$$

The proof consists of straightforward calculations.

Remark 6. It seems that we should consider also the case $b = 2$. But if $b = 2$ then the perturbation of the system (5) arising from the change $z \to ze^{i\psi} : A \to A(1 + i\psi)$, $B \to B(1 - i\psi)$, $C \to C(1 - 3i\psi)$ lies in the space $\{\frac{1}{4}[(\epsilon_3 - \epsilon_2)z^2 + 2\epsilon_3 z\bar{z} + (\epsilon_2 + \epsilon_3)\bar{z}^2]\frac{\partial}{\partial z}\}$ defined in Lemma 5, (and generated by the perturbation (3)). Therefore the perturbation (3) is not correct for $b = 2$, its essential part is 2–dimensional. We ought take into account also the term $y^2\frac{\partial}{\partial y}$, but we shall not do it in this paper.

By the definition of the linear Poincaré–Pontriagin–Melnikov integral, we have $\Delta H = H(\mathcal{P}(\rho)) - H(\rho) = \sum \epsilon_j I_j(h) + O(|\epsilon|^2)$, where $h = H(\rho, 0) = h_c + D\rho^2 + \dots$. Next, $\Delta H \approx (2D\rho)\Delta\rho \approx 2D \sum v_{2i+1}\rho^{2i+2}$. From Lemma 5 we get that

$$
\begin{aligned}
&\epsilon_1 I_1 \sim \tfrac{dH}{d\rho} v_1\rho \sim 2\pi(2D\rho^2)\epsilon_1 \\
&I_2 \sim \pi(a + b)(2D\rho^4)/4 \\
&I_3 \sim -\pi(a - 1)(2D\rho^4)/2 \\
&I_2 + \tfrac{a+b}{2(a-1)} I_3 \sim \begin{cases} \frac{\pi}{12}(a + b)(3a + 5b + 2)(b - 2)(2D\rho^6), & 3a + 5b + 2 \neq 0 \\ \frac{-3\pi}{10^4}(a + 4)^2(3a + 2)(a - 1)^2(2D\rho^8), & 3a + 5b + 2 = 0 \end{cases}
\end{aligned}
$$

Moreover, it is also known that the asymptotic analysis of the map $\mathcal{P}$, near the critical point by means of the focus quantities, is the same as the asymptotic analysis of the Abelian integrals, provided that the integrals behave correctly (i.e. are independent).

Therefore we have the following result.

Lemma 6. *For $h \to h_c$ we have*

$$P_1 \to \infty, \quad (P_1 \sim \frac{-4}{(a-1)\rho^2})$$

$$P_2 = K_1(1 + K_2 P_1^{\nu}) + \dots$$

where $K_1 = -\frac{a+b}{2(a-1)}$ *and* $\nu = -1$, $K_2 = -\frac{4(3a+5b+2)(b-2)}{3(a-1)}$ *if* $3a+5b+2 \neq 0$ *and* $\nu = -2$, $K_2 = -\frac{3(a+4)^2(3a+2)}{4 \cdot 125(a-1)}$ *otherwise.*

Using Lemma 4 and Lemma 6 we obtain the asymptotic behaviour of the integrals I_j' and of the curve Ω' as $h' \to h_c'$.

Lemma 7. *Near the critical point h_c' we have*

$$P_2' \to -\infty$$
$$P_1'/P_2' \to 2b(a-b+1)/b'$$

5 Behaviour of integrals near infinity

It this section we assume that

$$-2 < a < 0 < b < 2.$$

Lemma 8. *If $h \to \infty$, then*
(a) $J_0 \sim J_0' \sim K_0 h^{1-1/a}$, $K_0 < 0$;
(b)

$$J_1 \sim K_1 h + \begin{cases} -\beta\gamma^{\tau} L h^{1+1/a} & \text{if } -2 < a < -1 \\ -\frac{2(a+2)}{a}\alpha^{1+\sigma} M h^{1-1/(a+2)} & \text{if } -1 < a < 0 \end{cases}$$

$$J_1' \sim -K_1 h + \begin{cases} -\beta\gamma^{\tau} L h^{1+1/a} \\ -\frac{2(a+2)}{a}\alpha^{1+\sigma} M h^{1-1/(a+2)} \end{cases}$$

where $K_1 = -2\pi[a(b-2)]^{-1/2}$, $\tau = \frac{3a+2}{-2a}$, $\sigma = -\frac{3a+4}{2a+4}$, $L, M > 0$ *and* α, β, γ *are defined in Section 2;*
(c)

$$J_2 \sim K_2 h + \begin{cases} \frac{-2a}{a+2}\gamma^{1+\tau} L h^{1+1/a} & \text{if } -2 < a < -1 \\ -\beta\alpha^{\sigma} M h^{1-1/(a+2)} & \text{if } -1 < a < 0 \end{cases}$$

$$J_2' \sim K_2 h + \begin{cases} \frac{-2a}{a+2}\gamma^{1+\tau} L h^{1+1/a} \\ \beta\alpha^{\sigma} M h^{1-1/(a+2)} \end{cases}$$

where $K_2 = -\pi[(a+2)b]^{-1/2}$.

Proof. We have $J_i = \int_{\gamma_h} u^{a+i-2} x du = -2\int_{u_1}^{u_2} u^{a+i-2} x(u) du$. If $h \to \infty$, then γ_h tends to the polycycle composed of the line $u = 0$ and a half of the equator at infinity. The dominating contributions arise either from integration near $u = 0$ or near $u = \infty$. We have

$$x^2(u) = hu^{-a} - \alpha u^2 + \beta u - \gamma \sim hu^{-a} - \gamma \text{ near } u = 0 \tag{11}$$

$$x^2(u) \sim hu^{-a} - \alpha u^2 \ \textit{near}\ u = \infty \tag{12}$$

Let us estimate the orders of the integrals J_i in these two domains.

(i) $u = 0$: then $u \sim h^{1/a}$, $x \sim 1$ and $\int du$ gives $h^{1/a}$. Hence $J_i \sim h^{1+(i-1)/a}$.

(ii) $u = \infty$: then $u \sim h^{1/(a+2)}$, $x \sim h^{\frac{1}{2}-\frac{a}{2a+4}}$ and $\int du \sim h^{1/(a+2)}$. Hence $J_i \sim h^{(a+i)/(a+2)}$.

Therefore $J_0 \sim h^{1-1/a}$. For J_1, the region near $u = 0$ is dominating. Putting $x(u)$ in the form (11) and introducing the new variable $z = \gamma u^a h^{-1}$ we obtain $J_1 \sim K_1 h$. For J_2, we take the approximation (12) and introduce the variable $v = \alpha u^{a+2} h^{-1}$. This gives $J_2 \sim K_2 h$.

In order to find the next terms in the asymptotic expansions of J_1 and J_2, we notice that they arise from taking into account the terms in $x^2(u)$ which were neglected in the approximations (11) and (12). We have

$$\begin{array}{l} \frac{\partial J_1}{\partial \beta} = -\frac{\partial J_2}{\partial \gamma} = -\int u^a x^{-1}(u) du \\ \frac{\partial J_1}{\partial \alpha} = -\frac{\partial J_2}{\partial \beta} = \int u^{a+1} x^{-1} du. \end{array} \tag{13}$$

The integrals $\int u^{a+i} x^{-1}$ are estimated as before. Near $u = 0$ they are of the order $h^{1+(1+i)/a}$ and the terms with $i = 0$ are dominating. Near $u = \infty$ the growth order is $h^{1+(i-2)/(a+2)}$ and we have to choose $i = 1$. Of course, $h^{1+1/a} > h^{1-1/(a+2)}$ for $-2 < a < -1$ and $h^{1+1/a} < h^{1-1/(a+2)}$ for $-1 < a < 0$.

Thus, if $-2 < a < -1$, then introduction of the variable $z = \gamma u^a h^{-1}$, gives $\int u^a x^{-1} \sim h^{1+1/a} \gamma^\tau B(\frac{3a+2}{2a}, \frac{1}{2})$ and if $-1 < a < 0$, then the change $u \to v = \alpha u^{a+2} h^{-1}$ shows that $\int u^{a+1} x^{-1} \sim h^{1-1/(a+2)} \alpha^\sigma B(\frac{2a+4}{2a+4}, \frac{1}{2})$. (Here $B(\cdot,\cdot)$ denotes the Euler Beta–function). Integrating the formulas (13), we obtain the second terms of the expansions of J_1 and J_2.

The asymptotes of J_i' follow from Lemma 2. ∎

Using the above lemma we can describe the asymptotic behaviour of the curves Ω and Ω'.

Lemma 9. *If $h \to \infty$, then*

(a) $P_1, P_1' \to \frac{2(a+2)}{2-b}$;

(b) $P_2 \sim C_1(a+b)h^{1/a}$, $P_1' \sim C_2 h^{1/a}$, $C_{1,2} > 0$;

(c)

$$\begin{array}{l} P_2 = \lambda(P_1 - \frac{2(a+2)}{2-b}) + \mu(P_1 - \frac{2(a+2)}{2-b})^\theta + \dots \\ P_2' = \lambda'(P_1' - \frac{2(a+2)}{2-b}) + \mu'(P_1' - \frac{2(a+2)}{2-b})^\theta + \dots \end{array}$$

where $\theta = 2$ for $-2 < a < -1$ and $\theta = 1 - \frac{a}{a+2} > 1$ for $-1 < a < 0$;

(d) $\lambda \neq \lambda'$, (they may be equal to $\pm\infty$);

(e) $\mu \neq 0$ for $a + b \neq 0$ and $\mu' \neq 0$ for $b \neq a + 2$.

Proof. The points (a), (b) and (c) are obvious. The values of the coefficients λ and λ' are

$$\begin{array}{l} \lambda = \lim I_2/(I_1 - \frac{2(a+2)}{2-b} I_3) \\ = \frac{2-b}{2}\left[\frac{1}{\sqrt{a(b-2)}} - \frac{1}{\sqrt{(a+2)b}}\right] / \left[\frac{ab-3b+4}{\sqrt{a(b-2)}} - \frac{b(a-1)}{\sqrt{(a+2)b}}\right], \\ \lambda' = \frac{2-b}{2}\left[\frac{1}{\sqrt{a(b-2)}} + \frac{1}{\sqrt{(a+2)b}}\right] / \left[\frac{ab-3b+4}{\sqrt{a(b-2)}} + \frac{b(a-1)}{\sqrt{(a+2)b}}\right] \end{array}$$

From this we easily get that

$$\lambda - \lambda' = const \cdot [(ab - 3b + 4) - b(a-1)] = const \cdot 2(2-b) \neq 0$$

To prove (e) we firstly show that $\mu \neq 0$. (Then $\mu' \neq 0$ by Lemma 4). This property is equivalent to the linear independence of the functions J_0, J_1, J_2 or that

$$\det \begin{bmatrix} J_0 & J_1 & J_2 \\ \dot{J}_0 & \dot{J}_1 & \dot{J}_2 \\ \ddot{J}_0 & \ddot{J}_1 & \ddot{J}_2 \end{bmatrix} \neq 0 \tag{14}$$

The leading term in (14) has growth order $h^{1-1/a}h^0h^{1/a-1}$ for $-2 < a < -1$ and $h^{1-1/a}h^0h^{-1/(a+2)-1}$ for $-1 < a < 0$. The coefficient can be computed from (14), when we replace J_0 with $K_0h^{1-1/a}$ and $J_{1.2}$ with the first two terms of their asymptotics. It is equal $DAL\gamma^\tau$, $(-2 < a < -1)$, and $DBM\alpha^\sigma$, $(-1 < a < 0)$, where $D \neq 0$,

$A = \frac{-2a}{a+2}\gamma K_1 + \beta K_2 = \frac{-\pi}{\sqrt{(a+2)b}}\left[\sqrt{\frac{b(b-2)}{a(a+2)}} + \frac{b-1}{a+1}\right]$ and

$B = -\beta K_1 + 2\frac{a+2}{a}\alpha K_2 = \frac{-\pi}{\sqrt{a(b-2)}}\left[\frac{b-1}{a+1} + \sqrt{\frac{b(b-2)}{a(a+2)}}\right]$.

Here if $b < 1, a < -1$, then $(b-1)/(a+1) > 0$ and $A < 0$ and if $b > 1, a < -1$, then

$$\frac{b(b-2)}{a(a+2)} - \left(\frac{b-1}{a+1}\right)^2 = \frac{(a+b)(b-a-2)}{a(a+1)^2(a+2)} \tag{15}$$

vanishes at $a + b = 0$, (Q_3^{LV}). Similarily, $B < 0$ for $b > 1$ and for $b < 1$ we have the formula (15). Therefore $\mu \neq 0$ for $a + b \neq 0$. The property $\mu' \neq 0$ is proved in the same way, (it also follows from $\mu \neq 0$ and from Lemma 4). ■

The exact sign of μ can be easily calculated but we shall not do it here. Besides, we can use the fact that, for the case $S_3 : a = -1/2, b = 0$, the curve Ω has no inflection points (see [3]).

Remark 7. Unfortunately, in the case of large cycles we do not have automatic statement that the zeroes of the Abelian integral determine the position of limit cycles of the perturbed system. The reason is that the integrals behave singularly as $h \to \infty$ and the linear approximations can be not sufficient.

For example, one can extend the statements of Lemmas 8 and 9 to the case $a = -2$; the formulas acquire some logarithmic factors $\ln h$. The curve $\Omega(-2, b)$ remains convex near the corresponding endpoint and the Abelian integral has at most 2 zeroes.

On the other hand, it has been proven in [7] that the cyclicity of the corresponding contour H_{11}^1 is 3.

6 The global bifurcation diagram

The results of Sections 5 and 6 describe the positions of the curves Ω and Ω' near their endpoints for parameters in the square $-2 < a < 0 < b < 2$. We believe that the following conjecture holds.

Conjecture 1. *The curve Ω has at most one inflection point away from its ends, (for any $(a, b) \in \mathbb{R}^2$).*

Theorem 1. *If the Conjecture is true and $c = -2 < a < 0 < b < 2$, then any perturbation of the system (1) has at most 4 limit cycles of amplitude $O(1)$, (i.e. away from the equator at infinity), with possible configurations $(0,0), (1,0), (1,1), (2,0), (2,1), (3,1)$. The bifurcation diagram of the curves Ω and Ω' is the same as at Figure 3.*

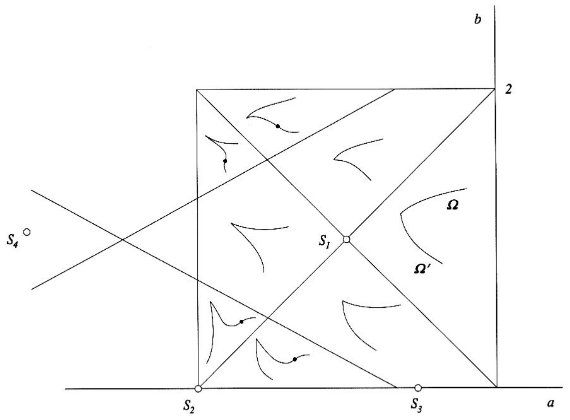

Figure 3

Remark 8. For $a < -2$ and for $a > 0$, the curves Ω and Ω' diverge. Their ends have no simple expressions. They are ratios of definite integrals $\int u^{a+i} x^j du$ along quadratic curve $x^2 + \alpha u^2 - \beta u + \gamma = 0$.

One may also ask what happens with the inflection point p of Ω (below the line $3a + 5b + 2 = 0$ at Fig. 3). We know that for the point $S_4 : a = -4, b = 1$ the curve Ω is convex (see [3]). So, the only possibility is that it bifurcates to infinity, i.e. the corresponding triple limit cycle tends to a separatrix contour in the Poincare sphere with cyclicity 3. The evidence of this phenomenon is proved in [10].

Remark 9. One can try to combine the result of this paper (extended to the case $a = -2, 0 < b < 2$) with the result of [7] about cyclicity 3 of the infinite contour H_{11}^1. When $a = -2$, the curves Ω and Ω' become tangent at their common enpoint.

It is shown in [7] that, if there are 3 large limit cycles around one focus, then there can be at most 1 large cycle around the other focus.

The situation with 2 or 3 large cycles around one focus occurs for perturbations such that the line L (see (7)) is almost equal to the common line tangent to the curves Ω nad Ω' at their point of intersection.

The position of the curves Ω, Ω' shows that we cannot obtain additional limit cycle arising from points of the intersection $L \cap (\Omega \cup \Omega')$.

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