The André/Bruck and Bose representation in PG(2h,q): unitals and Baer subplanes

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Abstract

Many authors have used the André/Bruck and Bose representation of $\mathrm{PG}(2, q^2)$ in $\mathrm{PG}(4, q)$ to study objects in the Desarguesian plane with great success. This paper looks at the André/Bruck and Bose representation of the Desarguesian plane $\mathrm{PG}(2, q^h)$ in $\mathrm{PG}(2h, q)$ in order to determine whether this higher dimensional representation provides additional information about objects in the plane. In particular, we look at the representation of unitals and Baer subplanes in this setting.

1 Introduction: the André/Bruck-Bose representation

André[1] and Bruck and Bose[4, 5] independently developed a method for representing translation planes of order q^h with kernel containing $\mathrm{GF}(q)$ in the projective space $\mathrm{PG}(2h, q)$. André gave a construction based on group theory; Bruck and Bose gave an equivalent geometric construction. This geometric construction of Bruck and Bose is the form we use in this paper. We refer to this representation as the **André/Bruck-Bose representation**.

In this section we present the results of Bruck and Bose. In particular, we obtain a representation of the Desarguesian plane $\mathrm{PG}(2, q^h)$ in the projective space $\mathrm{PG}(2h, q)$. We also obtain a convenient and natural coordinate system for $\mathrm{PG}(2, q^h)$ in this André/Bruck-Bose representation.

Throughout this paper we shall use the following notation. An **r−space** of $\mathrm{PG}(n, q)$ is a subspace of dimension r. We shall use the term a subspace of

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$\mathrm{PG}(n,q)\backslash\Sigma_\infty$ to mean a subspace of $\mathrm{PG}(n,q)$ which is not a subspace of the hyperplane Σ_∞. An $\mathbf{r}-$**spread** of $\mathrm{PG}(n,q)$ is a set of $r-$spaces which partitions the points of $\mathrm{PG}(n,q)$. When the dimension of the subspace elements of a given $r-$spread is clear by the context, we often refer to the $r-$spread as a **spread**.

1.1 The André/Bruck-Bose construction

In this section we describe the André/Bruck-Bose representation; we follow[4, Section 4].

Let $\mathcal{S}$ be a $(h-1)-$spread of $\Sigma_\infty = \mathrm{PG}(2h-1,q)$ and embed Σ_∞ as a hyperplane in $\mathrm{PG}(2h,q)$.

Define an incidence structure *aff*(Ψ) as follows.

The *points* of *aff*(Ψ) are the points of $\mathrm{PG}(2h,q)\backslash\Sigma_\infty$.

The *lines* of *aff*(Ψ) are the $h-$spaces of $\mathrm{PG}(2h,q)$ which intersect Σ_∞ in a unique element of $\mathcal{S}$. (Note that this implies that each such $h-$space is not contained in Σ_∞.)

The *incidence* relation of *aff*(Ψ) is that induced by the incidence relation of $\mathrm{PG}(2h,q)$.

By [4, Theorem 4.1, Corollary], *aff*(Ψ) is an affine plane of order q^h.

The affine plane *aff*(Ψ) may be embedded in a projective plane Ψ by adjoining the spread $\mathcal{S}$ to *aff*(Ψ) as a **line at infinity** which we denote by ℓ_∞. Each element of $\mathcal{S}$ corresponds to a class of parallel lines of *aff*(Ψ), thus each element of $\mathcal{S}$ is adjoined to Ψ as a **point at infinity**.

By [4, Theorem 7.1, Corollary], *aff*(Ψ) is a translation plane with translation line the line at infinity. Moreover, every finite translation plane is isomorphic to at least one plane *aff*(Ψ). Also we note, by [5, Theorem 12.1, Corollary], for $h \geq 2$ and $q > 2$, the finite projective plane Ψ is Desarguesian if and only if the $(h-1)-$spread $\mathcal{S}$ of Σ_∞ is a regular spread.

1.2 The André/Bruck-Bose representation of $\mathbf{PG(2,q^h)}$ in $\mathbf{PG(2h,q)}$

Our aim is to obtain a convenient coordinate representation of $\mathrm{PG}(2,q^h)$ in the André/Bruck-Bose setting with construction Ψ as given in the previous section. We require a regular $(h-1)-$spread $\mathcal{S}$ of $\Sigma_\infty = \mathrm{PG}(2h-1,q)$. The following determination of a regular spread $\mathcal{S}$ is a special case of the work of Bruck and Bose given in [4, Section 5].

Represent $\Sigma_\infty = \mathrm{PG}(2h-1,q)$ as the $(2h)-$dimensional vector space $\mathrm{GF}(q^{2h})$ over $\mathrm{GF}(q)$; the points of $\mathrm{PG}(2h-1,q)$ corresponding to the 1-dimensional vector subspaces of $\mathrm{GF}(q^{2h})$. We do this in the following way: let

$$\{1,\alpha,\ldots,\alpha^{h-1},\beta,\beta\alpha,\ldots,\beta\alpha^{h-1}\}$$

denote a basis for $\mathrm{GF}(q^{2h})$ as a vector space over $\mathrm{GF}(q)$. Here $\alpha \in \mathrm{GF}(q^h)\backslash\mathrm{GF}(q)$ has minimal polynomial $p_\alpha(x) = x^h - c_{h-1}x^{h-1} - \ldots - c_1x - c_0$, so that $\{1,\alpha,\ldots,\alpha^{h-1}\}$ forms a basis for $\mathrm{GF}(q^h) = \mathrm{GF}(q)(\alpha)$ over $\mathrm{GF}(q)$ and $\beta \in \mathrm{GF}(q^{2h})\backslash\mathrm{GF}(q^h)$ is algebraic over $\mathrm{GF}(q^h)$ so that $\{1,\beta\}$ forms a basis for $\mathrm{GF}(q^{2h})$ over $\mathrm{GF}(q^h)$.

Let $J(\infty)$, $J(0)$, $J(1)$ be three distinct $(h-1)-$subspaces of $\mathrm{PG}(2h-1,q)$, chosen so that as vector subspaces of $\mathrm{GF}(q^{2h})$,

$J(\infty)$ has basis $\{1, \alpha, \ldots, \alpha^{h-1}\}$,
$J(0)$ has basis $\{\beta, \beta\alpha, \ldots, \beta\alpha^{h-1}\}$, and
$J(1)$ has basis $\{1+\beta, \alpha+\beta\alpha, \ldots, \alpha^{h-1}+\beta\alpha^{h-1}\}$.

Denote by $'$ the following linear transformation of $J(\infty)$ onto $J(0)$,

$$' \; : \; a \longmapsto a' = \beta a.$$

Consequently, the following linear transformation maps $J(\infty)$ onto $J(1)$,

$$a \longmapsto a + a'.$$

Note that the vector space $\mathrm{GF}(q^{2h})$ is the direct sum of $J(\infty)$ and $J(0)$.

The three vector subspaces $J(\infty)$, $J(0)$, $J(1)$ intersect pairwise in the zero vector and hence, when considered as $(h-1)-$dimensional subspaces of $\mathrm{PG}(2h-1,q)$, the three subspaces are pairwise disjoint.

Since $J(\infty)$ is the $h-$dimensional vector space $\mathrm{GF}(q^h)$ over $\mathrm{GF}(q)$ with basis $\{1, \alpha, \ldots, \alpha^{h-1}\}$, each element $a \in J(\infty)$ can be uniquely expressed in the form,

$$a \;=\; a_0 + a_1\alpha + \ldots + a_{h-1}\alpha^{h-1}$$

where the a_i are in $\mathrm{GF}(q)$.

Note that α^h is an element of $\mathrm{GF}(q^h)$ and

$$\alpha^h \;=\; c_0 + c_1\alpha + \ldots + c_{h-1}\alpha^{h-1} \tag{1}$$

since $\alpha \in \mathrm{GF}(q^h)$ has minimal polynomial $p_\alpha(x) = x^h - c_{h-1}x^{h-1} - \ldots - c_0$, where the c_i are in $\mathrm{GF}(q)$.

Similarly, for each power α^{h+i}, $i = 1, \ldots, h-2$, the element α^{h+i} is also an element of $\mathrm{GF}(q^h)$ and therefore can be uniquely expressed as a linear combination of the basis elements $\{1, \alpha, \ldots, \alpha^{h-1}\}$. Hence, let

$$\alpha^{h+i} \;=\; g_{i,0} + g_{i,1}\alpha + \ldots + g_{i,h-1}\alpha^{h-1} \tag{2}$$

where the $g_{i,j}$ are in $\mathrm{GF}(q)$.

Consider the product ba of two elements $b, a \in J(\infty)$. We have,

$$\begin{aligned} b &= b_0 + b_1\alpha + \ldots + b_{h-1}\alpha^{h-1} \\ a &= a_0 + a_1\alpha + \ldots + a_{h-1}\alpha^{h-1} \end{aligned}$$

where b_i and a_i are elements of $\mathrm{GF}(q)$. Therefore ba is given by,

$$(b_0 + b_1\alpha + \ldots + b_{h-1}\alpha^{h-1})(a_0 + a_1\alpha + \ldots + a_{h-1}\alpha^{h-1}) \tag{3}$$

and by substituting the expressions (1) and (2) into the product (3), we can simplify (3) and determine ba as a (unique) linear combination of $\{1, \alpha, \ldots, \alpha^{h-1}\}$. Denote this linear combination by,

$$\begin{aligned} ba &\equiv (b_0 + b_1\alpha + \ldots + b_{h-1}\alpha^{h-1})(a_0 + a_1\alpha + \ldots + a_{h-1}\alpha^{h-1}) \\ &= (d_0 + d_1\alpha + \ldots + d_{h-1}\alpha^{h-1}) \\ &\equiv d \end{aligned}$$

where the d_i are in GF(q) and $d \in J(\infty) = \mathrm{GF}(q^h)$.

For convenience, we represent each element $a \in J(\infty)$ as a $h-$dimensional vector $(a_0, a_1, \ldots, a_{h-1})$, where $a = a_0 + a_1\alpha + \ldots + a_{h-1}\alpha^{h-1}$ with the $a_i \in \mathrm{GF}(q)$ as usual. Then for each element $b \in J(\infty)$, $b = b_0 + b_1\alpha + \ldots + b_{h-1}\alpha^{h-1} \equiv (b_0, b_1, \ldots, b_{h-1})$, the product (3) is equivalent to a linear transformation of $J(\infty)$ defined by a $h \times h$ matrix, which we shall denote by B_b, with entries in GF(q), as follows,

$$\begin{array}{rcl} J(\infty) & \longrightarrow & J(\infty) \\ a \equiv (a_0, a_1, \ldots, a_{h-1}) & \longmapsto & (a_0, a_1, \ldots, a_{h-1})B_b = (d_0, d_1, \ldots, d_{h-1}) \equiv d. \end{array}$$

For each of these $h \times h$ matrices B_b over GF(q) defined above, define

$$J(b) \;=\; \{aB_b + a' \mid a \in J(\infty)\} \tag{4}$$

so that $J(b)$ is a $h-$dimensional vector subspace of GF(q^{2h}) and so represents a $(h-1)-$space in $\Sigma_\infty = \mathrm{PG}(2h-1, q)$.

Let $\mathcal{C}$ denote the collection of the q^h matrices B_b over GF(q), so that,

$$\mathcal{C} = \{B_b \mid b \in \mathrm{GF}(q^h)\}.$$

Let $\mathcal{S}$ be the collection

$$\{J(\infty)\} \cup \{J(b) \mid b \in \mathrm{GF}(q^h)\}$$

of q^h+1 $(h-1)-$spaces in PG$(2h-1, q)$. Note that for $b = 0$ and $b = 1$ the definition of spaces $J(0)$ and $J(1)$ is consistent with our earlier definition of these spaces. We also note by (3) and the following remarks, that $J(0)$ is defined by the zero matrix $B_0 = 0$ in $\mathcal{C}$ and $J(1)$ is defined by the identity matrix $B_1 = I$ in $\mathcal{C}$.

We now show that $\mathcal{S}$ is a regular $(h-1)-$spread of $\Sigma_\infty = \mathrm{PG}(2h-1, q)$.

First we note that since $J(\infty)$ has basis $\{1, \alpha, \ldots, \alpha^{h-1}\}$ as a vector subspace of GF(q^{2h}) and given the definition of $J(b)$, the subspaces $J(\infty)$ and $J(b)$ have only the zero vector in common and hence as $(h-1)-$spaces in PG$(2h-1, q)$ they are disjoint.

Consider a matrix B_b in $\mathcal{C}$. For any element $a \in J(\infty)$ the product aB_b corresponds to the element ba in $J(\infty) = \mathrm{GF}(q^h)$. Hence $aB_b = 0$, for $a \in J(\infty)$ and $a \neq 0$, if and only if $b = 0$. It follows that for every non-zero matrix B_b in $\mathcal{C}$, B_b is non-singular. Moreover we note that for distinct matrices B_{b_1}, B_{b_2} in $\mathcal{C}$,

$$B_{b_1} - B_{b_2} = B_{b_1 - b_2}$$

is an element of $\mathcal{C}$ since $b_1 - b_2 \in \mathrm{GF}(q^h)$. Similarly, $\mathcal{C}$ is closed under matrix multiplication. In fact $(\mathcal{C}, +, \cdot)$ is isomorphic to the field GF(q^h) under the isomorphism $B_b \mapsto b$ from $\mathcal{C}$ to GF(q^h).

For distinct matrices B_{b_1}, B_{b_2}, since $B_{b_1} - B_{b_2}$ is an element of $\mathcal{C}$, by the above discussion $B_{b_1} - B_{b_2}$ is non-singular. Next suppose that the two vector subspaces $J(b_1)$ and $J(b_2)$ of GF(q^{2h}), corresponding to the distinct matrices $B_{b_1}, B_{b_2} \in \mathcal{C}$ respectively, have a non-zero vector x in common. By the definition in (4), for some elements $a_1, a_2 \in J(\infty) = \mathrm{GF}(q^h)$,

$$x \;=\; a_1 B_{b_1} + a_1' \;=\; a_2 B_{b_2} + a_2'.$$

By equating coefficients of the basis elements of $\mathrm{GF}(q^{2h})$, we obtain $a_1^{'} = a_2^{'}$ and therefore $a_1 = a_2$. Hence we have the equality $a_1 B_{b_1} = a_1 B_{b_2}$ which implies $a_1(B_{b_1} - B_{b_2}) = 0$. Since $B_{b_1} - B_{b_2}$ is non-singular we have $a_1 = 0$ and so $x = 0$, a contradiction.

Hence $\mathcal{S}$ is a collection of $q^h + 1$ pairwise disjoint $(h-1)-$spaces in $\Sigma_\infty = \mathrm{PG}(2h-1,q)$, that is, $\mathcal{S}$ is a $(h-1)-$spread of Σ_∞. Finally, by [5, Theorem 11.3] and since $(\mathcal{C}, +, \cdot)$ is a field, the spread $\mathcal{S}$ is a regular spread of Σ_∞.

Since $\mathcal{S}$ is a regular $(h-1)-$spread of $\Sigma_\infty = \mathrm{PG}(2h-1,q)$, the André/Bruck-Bose construction Ψ, of the previous section, with spread $\mathcal{S}$, is a Desarguesian projective plane of order q^h, for $h \geq 2$ and $q > 2$.

1.3 Coordinates for the projective plane $\Psi = \mathbf{PG}(2, \mathbf{q^h})$

Let Ψ be a finite projective plane of order q^h with the construction of Section 1.1 and the notation introduced there. Let $\mathcal{S}$ be the regular $(h-1)-$spread of $\Sigma_\infty = \mathrm{PG}(2h-1,q)$ determined in the previous section and with the notation introduced there.

In this section we use the results of [4, Section 6] to obtain a coordinate system for this Desarguesian projective plane Ψ determined by $\mathcal{S}$.

First we recall a familiar coordinatisation of $\mathrm{PG}(2,q^h)$. The points of $\mathrm{PG}(2,q^h)$ have homogeneous coordinates (x,y,z), where $x,y,z \in \mathrm{GF}(q^h)$ and x,y,z are not all equal to zero. Let ℓ_∞, the line at infinity, be the line with equation $z = 0$, or in line coordinates, ℓ_∞ is the line $[0,0,1]$. Let $\mathrm{AG}(2,q^h) = \mathrm{PG}(2,q^h) \backslash \ell_\infty$ be the affine plane obtained from $\mathrm{PG}(2,q^h)$ by removing ℓ_∞ and all of its points. The points of $\mathrm{AG}(2,q^h)$ have coordinates of the form $(x,y,1)$.

The lines of $\mathrm{AG}(2,q^h)$ may be divided into two types:

(i) Lines with equation $y = d$ or, equivalently, with line coordinates $[0,1,-d]$, where $d \in \mathrm{GF}(q^h)$.
These lines constitute a parallel class of lines in $\mathrm{AG}(2,q^h)$ with point at infinity $(1,0,0)$ in $\mathrm{PG}(2,q^h)$.

(ii) Lines with equation $x = by + f$ or, equivalently, with line coordinates $[1,-b,-f]$, where $b,f \in \mathrm{GF}(q^h)$.
For each $b \in \mathrm{GF}(q^h)$ these lines constitute a parallel class of lines in $\mathrm{AG}(2,q^h)$ with point at infinity $(b,1,0)$ in $\mathrm{PG}(2,q^h)$.

We work in the André/Bruck-Bose setting to obtain a natural coordinatisation of the incidence structure Ψ, natural in the sense that the coordinatisation will correspond to the above coordinatisation of the plane $\mathrm{PG}(2,q^h)$ in a convenient way.

We have $\Sigma_\infty = \mathrm{PG}(2h-1,q)$ embedded as a hyperplane in the projective space $\mathrm{PG}(2h,q)$. We represent $\mathrm{PG}(2h-1,q)$ as a $2h-$dimensional vector space $\mathrm{GF}(q^{2h})$ over the field $\mathrm{GF}(q)$ with basis

$$\{1, \alpha, \alpha^2, \ldots, \alpha^{h-1}, \beta, \beta\alpha, \ldots, \beta\alpha^{h-1}\}.$$

Embed $\mathrm{GF}(q^{2h})$ as a hyperplane in the $(2h+1)-$dimensional vector space $\mathrm{GF}(q^{2h+1})$, and we only need to add a single element e^* say of $\mathrm{GF}(q^{2h+1})$ which is not in $\mathrm{GF}(q^{2h})$ in order to obtain a basis

$$\{1, \alpha, \alpha^2, \ldots, \alpha^{h-1}, \beta, \beta\alpha, \ldots, \beta\alpha^{h-1}, e^*\}$$

for $\mathrm{GF}(q^{2h+1})$.

The regular $(h-1)$-spread $\mathcal{S}$ of $\mathrm{PG}(2h-1,q)$ is the collection of q^h+1 $h-$dimensional vector subspaces of $\mathrm{GF}(q^{2h})$ defined in the previous section, with the notation introduced there,

$$\mathcal{S} \;=\; \{J(\infty)\} \cup \{J(b) \mid b \in \mathrm{GF}(q^h)\}.$$

Consider the construction in Section 1.1 of the finite Desarguesian projective plane Ψ. Each affine point of Ψ is a 1-dimensional vector subspace of $\mathrm{GF}(q^{2h+1})$ not contained in the hyperplane $\mathrm{GF}(q^{2h})$ and so has a unique basis element of the form

$$x+y'+e^* \quad \text{or, equivalently,} \quad (x_0,x_1,\ldots,x_{h-1},y_0,y_1,\ldots,y_{h-1},1)$$

where $y' \in J(0)$ so that $x,y \in J(\infty) = \mathrm{GF}(q^h)$ and have unique representation in the form $x=\sum_{i=0}^{h-1} x_i\alpha^i$, $y=\sum_{i=0}^{h-1} y_i\alpha^i$, where the x_i, y_i are in $\mathrm{GF}(q)$. (Note that we have used the fact that $\mathrm{GF}(q^{2h})$ is the direct sum of $J(\infty)$ and $J(0)$.) Thus we define the coordinates of the affine point of Ψ with this basis element to be $(x,y,1)$ for every ordered pair of elements $x,y \in J(\infty) = \mathrm{GF}(q^h)$. We have defined,

$$\begin{aligned}(x,y,1) &\equiv x+y'+e^* \\ &\equiv (x_0,x_1,\ldots,x_{h-1},y_0,y_1,\ldots,y_{h-1},1).\end{aligned}$$

A line of Ψ, distinct from the line at infinity, is a $(h+1)-$dimensional vector subspace of $\mathrm{GF}(q^{2h+1})$ over $\mathrm{GF}(q)$ which intersects $\mathrm{GF}(q^{2h})$ in a unique element J of $\mathcal{S}$ and so has the form,

$$\begin{aligned}\langle J,(x,y,1)\rangle &\equiv \langle J, x+y'+e^*\rangle \\ &\equiv \langle J,(x_0,x_1,\ldots,x_{h-1},y_0,y_1,\ldots,y_{h-1},1)\rangle\end{aligned}$$

provided $(x,y,1)$ is one of its points.

We divide these lines into two types:

(i) Lines with equation $y=d$. If d is in $J(\infty)=\mathrm{GF}(q^h)$, the point $(x,y,1)$ of Ψ lies on the line

$$\langle J(\infty),(0,d,1)\rangle$$

if and only if $y=d$.
These lines constitute a parallel class of lines in *aff*(Ψ) with point at infinity $J(\infty)$ in Ψ.

(ii) Lines with equation $x=by+f$. Note that f is in $J(\infty)=\mathrm{GF}(q^h)$ and $J(b)$ is in $\mathcal{S}$, hence the point $(x,y,1)$ lies on the line

$$\langle J(b),(f,0,1)\rangle \;=\; \langle \{aB_b+a' \mid a \in J(\infty)\}, f+0'+e^*\rangle$$

if and only if $(x-f)+y'$ is in $J(b)$, that is, if and only if

$$(x_0-f_0,x_1-f_1,\ldots,x_{h-1}-f_{h-1}) = (y_0,y_1,\ldots,y_{h-1})B_b$$

where $f=f_0+f_1\alpha+\ldots+f_{h-1}\alpha^{h-1}$.
For each $b \in \mathrm{GF}(q^h)$ these lines constitute a parallel class of lines in *aff*(Ψ) with point at infinity $J(b)$ in Ψ.

Now we can consider the line at infinity ℓ_∞ of Ψ as being the line with equation $z = 0$, or in line coordinates the line $[0, 0, 1]$. Each element of the regular spread $\mathcal{S} = \{J(\infty)\} \cup \{J(b) \mid b \in \mathrm{GF}(q^h)\}$ is a point on the line at infinity and it is convenient to associate $J(b)$ with the coordinates $(b, 1, 0)$ for all $b \in \mathrm{GF}(q) \cup \{\infty\}$, so that in particular $J(\infty)$ is associated with $(1, 0, 0)$.

2 The André/Bruck-Bose representation in $\mathbf{PG(2h, q)}$ of Baer subplanes and Baer sublines of $\mathbf{PG(2, q^h)}$

In this section we consider the André/Bruck-Bose representation in projective space of dimension greater than 4. In other words we consider the André/Bruck-Bose representation defined by spreads other than 1−spreads of $\mathrm{PG}(3, q)$.

Our motivation is to determine the representation in André/Bruck-Bose of the Baer subplanes and Baer sublines in $\mathrm{PG}(2, q^h)$, when h is even. In our treatment we consider the case $h = 4$, that is the André/Bruck-Bose representation of $\mathrm{PG}(2, q^4)$ in $\mathrm{PG}(8, q)$, in greatest detail.

2.1 Preliminaries

A **Baer subplane** of $\mathrm{PG}(2, q^{2s})$ is a subplane of order q^s. Each line ℓ in $\mathrm{PG}(2, q^{2s})$ intersects a given Baer subplane B in exactly 1 or $q^s + 1$ points; if $|\ell \cap B| = q^s + 1$, then the intersection $\ell \cap B$ is called a **Baer subline** (of ℓ) in B. A Baer subplane B in $\mathrm{PG}(2, q^{2s})$ is therefore a blocking $(q^{2s} + q^s + 1)-$set in $\mathrm{PG}(2, q^{2s})$; conversely, by [6, Theorem 13.2.2], a blocking $(q^{2s} + q^s + 1)-$set in $\mathrm{PG}(2, q^{2s})$ is a Baer subplane of $\mathrm{PG}(2, q^{2s})$.

For a fixed chosen line ℓ_∞ of $\mathrm{PG}(2, q^{2s})$, which we call the **line at infinity**, the Baer subplanes of $\mathrm{PG}(2, q^{2s})$ which contain $q^s + 1$ points of ℓ_∞ will be called **affine Baer subplanes** of $\mathrm{PG}(2, q^{2s})$. The remaining Baer subplanes, which each contain a unique point of ℓ_∞, will be called **non-affine Baer subplanes** of $\mathrm{PG}(2, q^{2s})$.

In this section we will be discussing spreads of $\mathrm{PG}(2h - 1, q)$ in detail. We shall require the following results on spreads.

Theorem 2.1.

1. [6, Theorem 4.1.1] *A spread of $r-$spaces of $\mathrm{PG}(n, q)$ exists if and only if $(r+1)$ divides $(n + 1)$.*

2. [7, Theorem 25.6.7] *The group $\mathrm{PGL}(2n + 2, q)$ acts transitively on the set of all regular $n-$spreads of $\mathrm{PG}(2n + 1, q)$.*

3. [3, Theorem 5.3] *Let $\mathrm{PG}(3, q)$ be embedded as a subgeometry of $\mathrm{PG}(3, q^2)$. Let $^-$ denote the Fröbenius automorphism of $\mathrm{PG}(3, q^2)$ which fixes every point in $\mathrm{PG}(3, q)$. If x is a subspace of $\mathrm{PG}(3, q^2)$, then $\overline{x}$ is called the subspace of $\mathrm{PG}(3, q^2)$* **conjugate** *to x with respect to the quadratic extension $\mathrm{GF}(q^2)$ of $\mathrm{GF}(q)$. Let g be any line of $\mathrm{PG}(3, q^2)$ which contains no point of $\mathrm{PG}(3, q)$. For each such line g, let $\mathcal{S}_g$ denote the set of all lines of $\mathrm{PG}(3, q)$ which meet g.*

Then, $\mathcal{S}_g = \mathcal{S}_{\overline{g}}$ and $\mathcal{S}_g$ is a regular spread of $\mathrm{PG}(3, q)$. *Moreover, each regular spread of* $\mathrm{PG}(3, q)$ *can be represented in this manner for a unique pair of lines* $g, \overline{g}$.

In the following, we also use some properties of reguli and Segre varieties; concepts which we will not define here, but for which we provide references where appropriate. For definitions and results regarding Segre varieties see for example [7, Sections 25.5, 25.6].

2.2 The main construction

In [7, page 206] a method for constructing spreads is given; a particular case of which is the following. Note that since $2h$ divides $4h$, a $(2h-1)-$spread $\mathcal{S}_{2h-1,q^2}$ exists in $\mathrm{PG}(4h-1, q^2)$ and since $\mathcal{S}_{2h-1,q^2}$ has more elements than there are points in $\mathrm{PG}(4h-1, q)$, there exists an element of $\mathcal{S}_{2h-1,q^2}$ which is disjoint from the Baer subspace $\mathrm{PG}(4h-1, q)$ of $\mathrm{PG}(4h-1, q^2)$. It is therefore possible to embed $\mathrm{PG}(2h-1, q^2)$ in the extension $\mathrm{PG}(4h-1, q^2)$ of $\mathrm{PG}(4h-1, q)$ in such a way that $\mathrm{PG}(2h-1, q^2)$ does not contain a point of $\mathrm{PG}(4h-1, q)$.

Construction 2.2. : a construction of a $(2h-1)-$spread of $\mathrm{PG}(4h-1, q)$ from a $(h-1)-$spread of $\mathrm{PG}(2h-1, q^2)$.
Consider a projective space $\mathrm{PG}(2h-1, q^2)$, $h \geq 1$. Since h divides $2h$, there exists a $(h-1)-$spread $\mathcal{S}'$ of $\mathrm{PG}(2h-1, q^2)$ which contains $q^{2h}+1$ spread elements Π^j_{h-1,q^2}, $j = 1, \ldots, q^{2h}+1$, of dimension $h-1$ over $\mathrm{GF}(q^2)$. Embed $\mathrm{PG}(2h-1, q^2)$ in the extension $\mathrm{PG}(4h-1, q^2)$ of $\mathrm{PG}(4h-1, q)$ so that $\mathrm{PG}(2h-1, q^2)$ does not contain a point of $\mathrm{PG}(4h-1, q)$. The $(h-1)-$space Π^j_{h-1,q^2} and its conjugate $\overline{\Pi}^j_{h-1,q^2}$ generate a $(2h-1)-$space Π^j_{2h-1,q^2} of $\mathrm{PG}(4h-1, q^2)$ and $\Pi^j_{2h-1,q^2} \cap \mathrm{PG}(4h-1, q)$ is a $(2h-1)-$space $\Pi^j_{2h-1,q}$ of $\mathrm{PG}(4h-1, q)$. The $q^{2h}+1$ spaces $\Pi^j_{2h-1,q}$ form a partition of $\mathrm{PG}(4h-1, q)$ and we denote this $(2h-1)-$spread of $\mathrm{PG}(4h-1, q)$ by $\mathcal{S}$.

We prove that $\mathcal{S}$ is regular if $\mathcal{S}'$ is regular.

Theorem 2.3. *In Construction 2.2, if the $(h-1)-$spread $\mathcal{S}'$ of* $\mathrm{PG}(2h-1, q^2)$ *is regular, then the $(2h-1)-$spread $\mathcal{S}$ of* $\mathrm{PG}(4h-1, q)$ *is regular.*

Proof Let Π^1_{h-1,q^2}, Π^2_{h-1,q^2}, Π^3_{h-1,q^2} be three distinct elements of $\mathcal{S}'$. Denote by $R' = \mathcal{R}(\Pi^1_{h-1,q^2}, \Pi^2_{h-1,q^2}, \Pi^3_{h-1,q^2})$ the unique $(h-1)-$regulus of $\mathrm{PG}(2h-1, q^2)$ containing these three spread elements. Let $\Pi^1_{2h-1,q}$, $\Pi^2_{2h-1,q}$, $\Pi^3_{2h-1,q}$ be the three distinct elements of $\mathcal{S}$ corresponding to Π^1_{h-1,q^2}, Π^2_{h-1,q^2}, Π^3_{h-1,q^2} respectively in the given construction. Let $R = \mathcal{R}(\Pi^1_{2h-1,q}, \Pi^2_{2h-1,q}, \Pi^3_{2h-1,q})$ denote the unique $(2h-1)-$regulus of $\mathrm{PG}(4h-1, q)$ containing $\Pi^1_{2h-1,q}$, $\Pi^2_{2h-1,q}$ and $\Pi^3_{2h-1,q}$. So R is a system of maximal $(2h-1)-$spaces of a Segre variety $\zeta_{1,2h-1}$ in $\mathrm{PG}(4h-1, q)$. Over $\mathrm{GF}(q^2)$, R becomes a $(2h-1)-$regulus R_{q^2} of $\mathrm{PG}(4h-1, q^2)$. Due to the above construction of the spread $\mathcal{S}$ we have for $j = 1, 2, 3$, Π^j_{h-1,q^2} is contained in Π^j_{2h-1,q^2}, where Π^j_{2h-1,q^2} is the unique element of the regulus R_{q^2} which contains $\Pi^j_{2h-1,q}$. Thus the line transversals of R' in $\mathrm{PG}(2h-1, q^2)$ are line transversals of R_{q^2} and therefore

R' is a Segre subvariety $\zeta_{1,h-1}$ of R_{q^2} and by [7, Theorem 25.5.12], the regulus R' is precisely the intersection $R_{q^2} \cap \mathrm{PG}(2h-1, q^2)$.

It now follows that for any $(2h-1)-$space $\Pi^j_{2h-1,q}$ in R, where $\Pi^j_{2h-1,q}$ is distinct from $\Pi^1_{2h-1,q}, \Pi^2_{2h-1,q}$ and $\Pi^3_{2h-1,q}$, the unique element Π^j_{2h-1,q^2} of R_{q^2} which contains $\Pi^j_{2h-1,q}$ has the property that $\Pi^j_{2h-1,q^2} \cap \mathrm{PG}(2h-1, q^2) = \Pi^j_{h-1,q^2}$, for some element Π^j_{h-1,q^2} of R'. By the Construction 2.2 of $\mathcal{S}$ from $\mathcal{S}'$, if $\Pi^j_{2h-1,q}$ ($\in R$) is an element of $\mathcal{S}$, then Π^j_{h-1,q^2} ($\in R'$) is an element of $\mathcal{S}'$. The converse of the preceding statement is true if Π^j_{h-1,q^2} ($\in R'$) is one of the $q+1$ elements of R' associated to the elements of R via the construction of the spread $\mathcal{S}$. (Note that R' has q^2+1 elements and R has $q+1$ elements).

If $\mathcal{S}'$ is a regular spread, then the regulus R' defined by $\Pi^1_{h-1,q^2}, \Pi^2_{h-1,q^2}, \Pi^3_{h-1,q^2}$ is contained in $\mathcal{S}'$ and therefore, by the preceding argument, the regulus R of $\mathrm{PG}(4h-1, q)$ defined by $\Pi^1_{2h-1,q}, \Pi^2_{2h-1,q}, \Pi^3_{2h-1,q}$ is contained in $\mathcal{S}$. The result now follows. ∎

Consider a translation plane π of order $q^{2h} = (q^2)^h$ defined by the André/Bruck-Bose construction of Section 1.1 with a $(h-1)-$spread $\mathcal{S}'$ of $\Sigma'_\infty = \mathrm{PG}(2h-1, q^2)$. We now have a convenient correspondence between this André/Bruck-Bose representation of π and a second André/Bruck-Bose representation of π defined by a $(2h-1)-$spread $\mathcal{S}$ of $\Sigma_\infty = \mathrm{PG}(4h-1, q)$, where $\mathcal{S}'$ and $\mathcal{S}$ are associated by Construction 2.2.

For Desarguesian planes of certain orders which have an André/Bruck-Bose representation, the above Construction 2.2 and Theorem 2.3 provide us with a convenient method to obtain an André/Bruck-Bose representation of the plane in a space of higher dimension and lower order.

To illustrate this, we consider the Desarguesian plane $\mathrm{PG}(2, q^4)$. The plane $\mathrm{PG}(2, q^4)$ has an André/Bruck-Bose representation in $\mathrm{PG}(4, q^2)$ defined by a regular line spread $\mathcal{S}'$ of $\mathrm{PG}(3, q^2)$ and an André/Bruck-Bose representation in $\mathrm{PG}(8, q)$ defined by a regular $3-$spread $\mathcal{S}$ of $\mathrm{PG}(7, q)$. In the following sections we determine properties concerning the André/Bruck-Bose representation in $\mathrm{PG}(8, q)$ of the Baer substructures of $\mathrm{PG}(2, q^4)$ and provide some generalisations to higher dimensions.

2.3 Affine Baer subplanes and induced spreads in the André/Bruck-Bose representation of $\mathbf{PG(2, q^4)}$ in $\mathbf{PG(8, q)}$

Theorem 2.4. *A regular* $3-$*spread* $\mathcal{S}$ *in* $\mathrm{PG}(7, q)$ *has a well-defined and unique set of induced regular* $1-$*spreads, one in each element of* $\mathcal{S}$.

Proof The regular $3-$spreads of $\mathrm{PG}(7, q)$ are projectively equivalent. Therefore, we can assume that $\mathcal{S}$ is the regular $3-$spread of $\mathrm{PG}(7, q)$ obtained from a regular $1-$spread $\mathcal{S}'$ of $\mathrm{PG}(3, q^2)$ by the Construction 2.2 with $h = 2$. We repeat the construction for this special case to establish notation.

Embed $\mathrm{PG}(7, q)$ in $\mathrm{PG}(7, q^2)$ and let Σ_{3,q^2} be a $3-$space over $\mathrm{GF}(q^2)$ in $\mathrm{PG}(7, q^2)$ which has no point in common with $\mathrm{PG}(7, q)$. Let $\mathcal{S}'$ be a regular $1-$spread of Σ_{3,q^2}. Consider the conjugate space $\overline{\Sigma}_{3,q^2}$ of Σ_{3,q^2}. For each element Π^j_{1,q^2} in $\mathcal{S}'$,

$j = 1, \ldots, q^4+1$, the $3-$space Π^j_{3,q^2} spanned by Π^j_{1,q^2} and its conjugate $\overline{\Pi}^j_{1,q^2}$ intersects $PG(7, q)$ in a $3-$space which we denote by $\Pi^j_{3,q}$. By Construction 2.2, these $q^4 + 1$ $3-$spaces $\Pi^j_{3,q}$ form a $3-$spread $\mathcal{S}$ of $PG(7, q)$ which by Theorem 2.3 is regular.

Each element $\Pi^j_{3,q}$ of $\mathcal{S}$ is the intersection $\langle \Pi^j_{1,q^2}, \overline{\Pi}^j_{1,q^2} \rangle \cap PG(7, q)$ for a unique line Π^j_{1,q^2} of $\mathcal{S}'$. For j fixed, the join of each point P of Π^j_{1,q^2} to its conjugate $\overline{P}$ yields a line of $\Pi^j_{3,q}$ and the collection of these $q^2 + 1$ lines constitutes a regular $1-$spread $\mathcal{S}^j_1$ of $\Pi^j_{3,q}$.

Hence each element $\Pi^j_{3,q}$ of the regular $3-$spread $\mathcal{S}$ of $PG(7, q)$ has a well defined induced regular $1-$spread which we denote by $\mathcal{S}^j_1$.

It remains to prove that the set of induced spreads $\{\mathcal{S}^j_1\}$ obtained above for the regular $3-$spread $\mathcal{S}$ is unique.

Consider the regular line spread $\mathcal{S}'$ of $\Sigma_{3,q^2} \cong PG(3, q^2)$ and the regular $3-$spread $\mathcal{S}$ of $PG(7, q)$ associated to $\mathcal{S}'$ by the Construction 2.2. By the André/Bruck-Bose construction of Section 1.1, these spreads correspond to an André/Bruck-Bose representation of $PG(2, q^4)$ in $PG(4, q^2)$ and an André/Bruck-Bose representation of $PG(2, q^4)$ in $PG(8, q)$ respectively. Denote these André/Bruck-Bose incidence structures by Ψ_{4,q^2} and $\Psi_{8,q}$ respectively.

From the first part of this proof, there exists a well defined 1-1 correspondence between the points of Σ_{3,q^2} and the (line) elements of the induced $1-$spreads $\{\mathcal{S}^j_1\}$ of elements of $\mathcal{S}$ in $PG(7, q)$.

For $\mathcal{S}$ a regular $3-$spread of $PG(7, q)$, the (line) elements of the $q^4 + 1$ induced regular $1-$spreads $\{\mathcal{S}^j_1\}$ shall be called **induced spread lines**. That is, for each $3-$space $\Pi^j_{3,q} \in \mathcal{S}$, a line ℓ of $\Pi^j_{3,q}$ is an induced spread line if and only if $\ell \in \mathcal{S}^j_1$. The lines of $PG(7, q)$ not contained in any element of $\mathcal{S}$ shall be called **transversal lines**.

Consider the André/Bruck-Bose representation Ψ_{4,q^2} of $PG(2, q^4)$ in $PG(4, q^2)$; let $\Sigma'_\infty = \Sigma_{3,q^2}$ denote the hyperplane at infinity of the construction, so that Σ'_∞ contains a regular spread $\mathcal{S}'$, and the elements of $\mathcal{S}'$ correspond to the points on the line at infinity ℓ_∞ of $PG(2, q^4)$ in this representation. By [4, Section 9], an affine Baer subplane B of $PG(2, q^4)$ is represented in Ψ_{4,q^2} by a plane not contained in $\Sigma'_\infty = \Sigma_{3,q^2}$ and which meets Σ'_∞ in a line ℓ which is not an element of $\mathcal{S}'$. Consider such a line ℓ in Σ_{3,q^2}. Using the setting of the first part of this proof, the line ℓ and its conjugate $\overline{\ell}$ generate a $3-$space $\langle \ell, \overline{\ell} \rangle$ of $PG(7, q^2)$ and the intersection $\langle \ell, \overline{\ell} \rangle \cap PG(7, q)$ is a $3-$space Σ_ℓ of $PG(7, q)$. Since ℓ is incident with exactly $q^2 + 1$ $1-$spread elements in Σ_{3,q^2}, the $3-$space Σ_ℓ intersects exactly $q^2 + 1$ of the $3-$spaces in the spread $\mathcal{S}$ of $PG(7, q)$, meeting each in an induced spread line. So in particular Σ_ℓ is disjoint from the remaining spread elements in $\mathcal{S}$.

Consider the André/Bruck-Bose representation, $\Psi_{8,q}$, of $PG(2, q^4)$ in $PG(8, q)$ defined by the regular $3-$spread $\mathcal{S}$ of $PG(7, q)$. Consider a $4-$dimensional subspace B^* of $\Psi_{8,q}$ which intersects $PG(7, q)$ in the $3-$space Σ_ℓ. Any $4-$space l^* in $\Psi_{8,q}$, not contained in $PG(7, q)$, which intersects $PG(7, q)$ in a unique element of $\mathcal{S}$, either intersects B^* in a unique affine point, or the spread element contained in l^* is one of the $q^2 + 1$ incident with B^*. Since B^* and the $q^2 + 1$ $3-$spread elements incident with B^* constitute a $(q^4 + q^2 + 1)-$blocking set in $PG(2, q^4)$, it follows that B^* represents an affine Baer subplane of $PG(2, q^4)$.

By considering all lines ℓ in Σ_{3,q^2} which are not elements of the $1-$spread $\mathcal{S}'$ and repeating the above procedure, we obtain the André/Bruck-Bose representation in $\mathrm{PG}(8,q)$ of all $q^4(q^8+q^6+q^4+q^2)$ affine Baer subplanes of $\mathrm{PG}(2,q^4)$.

Intrinsic to this representation is the existence of exactly $q^8+q^6+q^4+q^2$ 3−spaces of $\mathrm{PG}(7,q)$ which each intersect precisely q^2+1 elements of the regular 3−spread $\mathcal{S}$ of $\mathrm{PG}(7,q)$ and such that the intersection in each case is a unique induced spread line, namely an element of the induced regular 1−spread of that 3−space spread element. That is, since $\mathrm{PG}(2,q^4)$ contains no further affine Baer subplanes, it follows that the set of induced spreads $\{\mathcal{S}_1^j\}$ is unique. ■

Theorem 2.5. *Let $\mathcal{S}$ be a regular 3−spread of $\mathrm{PG}(7,q)$. For each 3−space Σ of $\mathrm{PG}(7,q)$ one of the following holds:*

(1) Σ is an element of $\mathcal{S}$ and therefore $\Sigma=\Pi_{3,q}^j$ has an induced regular 1−spread $\mathcal{S}_1^j$. By definition $\Sigma=\Pi_{3,q}^j$ contains exactly q^2+1 induced spread lines.

There are q^4+1 3−spaces Σ of this type in $\mathrm{PG}(7,q)$.

(2) Σ intersects exactly q^2+1 elements of $\mathcal{S}$, in which case it meets each in an induced spread line. This set of q^2+1 induced spread lines constitutes a regular 1−spread of Σ which we shall call a **partition 1−spread**.

Any two induced spread lines, contained in distinct elements of $\mathcal{S}$, span such a 3−space. There are $q^8+q^6+q^4+q^2$ 3−spaces Σ of this type in $\mathrm{PG}(7,q)$.

(3) Σ intersects x elements of $\mathcal{S}$ where $x>q^2+1$. In this case either:

(i) $x=q^3+1$ and Σ intersects one element of $\mathcal{S}$ in a plane (which necessarily contains an induced spread line) and Σ intersects a further q^3 elements of $\mathcal{S}$, meeting each in a point,

or,

(ii) Σ intersects y elements of $\mathcal{S}$ in a line (with $0\le y<q^2+1$) and Σ intersects a further $x-y=(q^3+q^2+q+1)-y(q+1)>0$ elements of $\mathcal{S}$ meeting each in a point.

In this case Σ contains at most one induced spread line.

Moreover, if $\Pi_{3,q}^1,\Pi_{3,q}^2,\Pi_{3,q}^3$ are three distinct elements of $\mathcal{S}$ which each intersects Σ in a line, then Σ has a non-trivial intersection with each element of $\mathcal{S}$ in the 3−regulus $\mathcal{R}(\Pi_{3,q}^1,\Pi_{3,q}^2,\Pi_{3,q}^3)$ defined by $\Pi_{3,q}^1,\Pi_{3,q}^2,\Pi_{3,q}^3$; indeed Σ intersects each such element of $\mathcal{S}$ in a line.

Proof By Theorem 2.4, the q^4+1 elements of $\mathcal{S}$ constitute the 3−spaces of $\mathrm{PG}(7,q)$ of type (1).

By the remarks preceding Theorem 2.5, there exist $q^8+q^6+q^4+q^2$ 3−spaces of $\mathrm{PG}(7,q)$ which each intersect q^2+1 distinct elements of $\mathcal{S}$ and which contain a partition 1−spread. We shall call these 3−spaces **partition 3−spaces** of $\mathrm{PG}(7,q)$. We must show that these are the only 3−spaces of $\mathrm{PG}(7,q)$ which intersect exactly q^2+1 distinct elements of $\mathcal{S}$.

$\Psi_{8,q}$ is the André/Bruck-Bose representation of $\mathrm{PG}(2,q^4)$ in $\mathrm{PG}(8,q)$. The line at infinity ℓ_∞ is the line with "points" the elements of $\mathcal{S}$ in $\mathrm{PG}(7,q)$. As usual, the

Baer subplanes of $\mathrm{PG}(2,q^4)$ which are secant to ℓ_∞ are called *affine* Baer subplanes. There exist precisely $q^4(q^8+q^6+q^4+q^2)$ affine Baer subplanes of $\mathrm{PG}(2,q^4)$.

Consider a 4−space B^* in $\mathrm{PG}(8,q)$ not contained in $\mathrm{PG}(7,q)$ and which intersects $\mathrm{PG}(7,q)$ in a 3−space Σ where Σ intersects exactly q^2+1 elements of $\mathcal{S}$. Necessarily, Σ intersects each of these q^2+1 elements of $\mathcal{S}$ in a line. By the incidence in $\Psi_{8,q}$, B^* intersects ℓ_∞ in exactly q^2+1 points. Each 4−space ℓ of $\mathrm{PG}(8,q)$ which represents a line of $\mathrm{PG}(2,q^4)$ distinct from ℓ_∞ is not contained in $\mathrm{PG}(7,q)$ and meets $\mathrm{PG}(7,q)$ in an element of $\mathcal{S}$. Such a 4−space ℓ either intersects B^* in a point of $\mathrm{PG}(8,q)\backslash\mathrm{PG}(7,q)$ or the element of $\mathcal{S}$ incident with ℓ is one of the q^2+1 3−spread elements incident with B^*. It follows that B^* represents a (q^4+q^2+1)−blocking set B in $\mathrm{PG}(2,q^4)$ and hence B is an affine Baer subplane of $\mathrm{PG}(2,q^4)$.

Therefore any 4−space of $\mathrm{PG}(8,q)$, not contained in $\mathrm{PG}(7,q)$ and which meets $\mathrm{PG}(7,q)$ in a partition 3−space represents an affine Baer subplane of $\mathrm{PG}(2,q^4)$. There are $q^4(q^8+q^6+q^4+q^2)$ such 4−spaces of $\mathrm{PG}(8,q)$. Since this is also the number of affine Baer subplanes of $\mathrm{PG}(2,q^4)$, there exist no further 3−spaces of $\mathrm{PG}(7,q)$ (besides the partition 3−spaces) which intersect exactly q^2+1 elements of $\mathcal{S}$.

Let Σ be a 3−space of $\mathrm{PG}(7,q)$ spanned by induced spread lines ℓ_1 and ℓ_2 where ℓ_1 and ℓ_2 lie in distinct elements of $\mathcal{S}$. In the quadratic extension, the lines ℓ_1 and ℓ_2 intersect Σ_{3,q^2} in distinct points L_1 and L_2 respectively. The 3−space (over $\mathrm{GF}(q^2)$) spanned by the line L_1L_2 and its conjugate $\overline{L_1L_2}$ is the quadratic extension Σ_{q^2} of Σ. By joining each point P on L_1L_2 to its conjugate $\overline{P}$ on $\overline{L_1L_2}$ we obtain a set of q^2+1 lines of Σ which constitute a regular 1−spread of Σ. The elements of this 1−spread are all induced spread lines and hence by definition, this regular 1−spread is a partition 1−spread. Thus Σ is a partition 3−space. We have that two induced spread lines from distinct elements of $\mathcal{S}$ span a partition 3−space.

The 3−spaces of $\mathrm{PG}(7,q)$ of type (1) and (2) have now been classified. The type (3) 3−spaces include all possible exceptions. It remains to prove the final remark regarding a 3−space of type (3)(ii).

Consider a 3−space Σ of $\mathrm{PG}(7,q)$ which intersects strictly greater than q^2+1 elements of $\mathcal{S}$ but which meets no element of $\mathcal{S}$ in a plane. Suppose Σ intersects the 3−spread elements $\Pi^1_{3,q},\Pi^2_{3,q},\Pi^3_{3,q}$ each in a line ℓ_1,ℓ_2,ℓ_3 respectively. The lines ℓ_1,ℓ_2,ℓ_3 define a unique 1−regulus $R_1=\mathcal{R}(\ell_1,\ell_2,\ell_3)$ in Σ. The 3−spread elements $\Pi^1_{3,q},\Pi^2_{3,q},\Pi^3_{3,q}$ define a unique 3−regulus $R_3=\mathcal{R}(\Pi^1_{3,q},\Pi^2_{3,q},\Pi^3_{3,q})$ which is contained in $\mathcal{S}$ since $\mathcal{S}$ is regular. The line transversals of R_1 are contained in Σ and are necessarily transversals of the regulus R_3. Hence each spread element in R_3 intersects Σ in a line, namely a maximal space of the Segre variety R_1. ∎

Corollary 2.6. *Let $\Psi_{8,q}$ denote the André/Bruck-Bose representation of $\mathrm{PG}(2,q^4)$ in $\mathrm{PG}(8,q)$ and let Σ_∞ denote the hyperplane at infinity of $\mathrm{PG}(8,q)$.*

1. *B is an affine Baer subplane of $\mathrm{PG}(2,q^4)$ if and only if in $\Psi_{8,q}$ B is a 4−space not contained in Σ_∞ which intersects Σ_∞ in a partition 3−space, that is a 3−space which meets exactly q^2+1 elements of the 3−spread $\mathcal{S}$ of Σ_∞.*

2. *b_ℓ is a Baer subline of $\mathrm{PG}(2,q^4)$ that contains a point of ℓ_∞ if and only if in $\Psi_{8,q}$ b_ℓ is a plane not contained in Σ_∞ which intersects Σ_∞ in an induced spread line.*

Proof The affine Baer subplane structure in $\Psi_{8,q}$ was determined in the proof of Theorem 2.5. A line ℓ, distinct from ℓ_∞, of an affine Baer subplane B of $\mathrm{PG}(2,q^4)$ intersects B in a Baer subline b_ℓ that contains a point of ℓ_∞.

In $\Psi_{8,q}$, ℓ is a 4−space which intersects Σ_∞ in an element $\Pi^j_{3,q}$ of the 3−spread $\mathcal{S}$ and B is a 4−space which intersects Σ_∞ in a partition 3−space Σ. The 3−spaces $\Pi^j_{3,q}$ and Σ intersect in an induced spread line, hence in $\Psi_{8,q}$, the intersection $\ell \cap B$ is a plane, not contained in Σ_∞ and which contains an induced spread line. This plane is then the André/Bruck-Bose representation of the Baer subline b_ℓ. By counting the number of such planes, the result follows. ■

2.4 Some generalisations to higher dimensions

We investigate some generalisations to higher dimensions of the results determined for $\mathrm{PG}(2,q^4)$.

Theorem 2.7. *Consider the Desarguesian plane* $\mathrm{PG}(2,q^{2^n})$ *($n \geq 1$) and the n André/Bruck-Bose representations* $\Psi_{2^{i+1}} = \Psi_{2^{i+1},q^{2^{n-i}}}$ $(1 \leq i \leq n)$ *which are determined by a*

$$\begin{array}{rcl}
\textit{regular } 1-\textit{spread} & \textit{in} & \mathrm{PG}(3,q^{2^{n-1}}), \\
\textit{regular } 3-\textit{spread} & \textit{in} & \mathrm{PG}(7,q^{2^{n-2}}), \\
\vdots & & \\
\textit{regular } (2^i-1)-\textit{spread} & \textit{in} & \mathrm{PG}(2^{i+1}-1,q^{2^{n-i}}), \\
\vdots & & \\
\textit{regular } (2^n-1)-\textit{spread} & \textit{in} & \mathrm{PG}(2^{n+1}-1,q) \textit{ respectively.}
\end{array}$$

Then the regular $(2^i-1)-$*spread in the hyperplane* $\mathrm{PG}(2^{i+1}-1,q^{2^{n-i}})$ *at infinity of* $\Psi_{2^{i+1}}$ *has a set of induced regular* $(2^{i-1}-1)-$*spreads, one in each element of the* $(2^i-1)-$*spread. Furthermore, for each such induced regular* $(2^{i-1}-1)-$*spread, there exists a set of induced regular* $(2^{i-2}-1)-$*spreads, and so on, until finally there exist induced regular* $1-$*spreads.*

Proof Let $\mathcal{S}_1$ be regular 1−spread of $\mathrm{PG}(3,q^{2^{n-1}})$ and embed $\mathrm{PG}(3,q^{2^{n-1}})$ as a subspace in $\mathrm{PG}(7,q^{2^{n-1}})$ in such a way that it is skew to $\mathrm{PG}(7,q^{2^{n-2}})$. By Theorems 2.3, 2.4 and the Construction 2.2, $\mathcal{S}_1$ determines a regular 3−spread $\mathcal{S}_3$ of $\mathrm{PG}(7,q^{2^{n-2}})$ which has a set of induced regular 1−spreads, one in each element of $\mathcal{S}_3$. Embed $\mathrm{PG}(7,q^{2^{n-2}})$ as a subspace in $\mathrm{PG}(15,q^{2^{n-2}})$ in such a way that $\mathrm{PG}(7,q^{2^{n-2}})$ is skew to the Baer subspace $\mathrm{PG}(15,q^{2^{n-3}})$ of $\mathrm{PG}(15,q^{2^{n-2}})$ and recursively repeat the above procedure using Construction 2.2. At the final stage we obtain a regular $(2^n-1)-$spread in $\mathrm{PG}(2^{n+1}-1,q)$ which contains the nested induced regular spreads of each stage. If we stop the procedure before the final stage we have a regular $(2^i-1)-$spread in $\mathrm{PG}(2^{i+1}-1,q^{2^{n-i}})$ with the nested induced regular spreads obtained up until that stage. Since regular $(2^i-1)-$spreads in $\mathrm{PG}(2^{i+1}-1,q^{2^{n-i}})$ are projectively equivalent, the regular spread we have constructed, which contains nested induced spreads, is representative. ■

Corollary 2.8. *For each $1 \le i \le n$, embed $\mathrm{PG}(2^{i+1}-1, q^{2^{n-i}}) = \Sigma_\infty^{2^{i+1}-1}$ as a hyperplane in $\mathrm{PG}(2^{i+1}, q^{2^{n-i}})$ and let $\Psi_{2^{i+1}}$ denote the André/Bruck-Bose representation of $\mathrm{PG}(2, q^{2^n})$ in $\mathrm{PG}(2^{i+1}, q^{2^{n-i}})$ determined by the regular $(2^i-1)-$spread $\mathcal{S}_{2^i-1}$ of $\Sigma_\infty^{2^{i+1}-1}$, as in Theorem 2.7.*

Then B is an affine Baer subplane of $\mathrm{PG}(2, q^{2^n})$ if and only if in $\Psi_{2^{i+1}}$, B is a $(2^i)-$space B^* of $\mathrm{PG}(2^{i+1}, q^{2^{n-i}})$ not contained in $\Sigma_\infty^{2^{i+1}-1}$ and which intersects $\Sigma_\infty^{2^{i+1}-1}$ in exactly $q^{2^{n-1}}+1$ elements of $\mathcal{S}_{2^i-1}$.

Furthermore, each element $\Lambda \in \mathcal{S}_{2^i-1}$ is either disjoint to B^* or intersects B^* in a unique element (a $(2^{i-1}-1)-$space of order $q^{2^{n-i}}$) of the induced regular $(2^{i-1}-1)-$spread $\mathcal{S}^j_{2^{i-1}-1}$ in Λ.

Proof Similar to the proof of Theorem 2.4; use Construction 2.2 repeatedly and then a counting argument using the known number of affine Baer subplanes of $\mathrm{PG}(2, q^{2^n})$. ∎

Note that the André/Bruck-Bose representation B^* of an affine Baer subplane B of $\mathrm{PG}(2, q^{2^n})$ is determined in Corollary 2.8, regardless of which of the n possible André/Bruck-Bose representations of $\mathrm{PG}(2, q^{2^n})$ is being considered. Moreover, implicit to Theorem 2.7 and its Corollary 2.8 is the André/Bruck-Bose representation of subplanes (which are not necessarily Baer subplanes) of order $q^{2^{n-j}}$ of $\mathrm{PG}(2, q^{2^n})$ which contain the line at infinity as a line. Due to the existence of the induced spreads determined in Theorem 2.7, in an André/Bruck-Bose representation $\Psi_{2^{i+1}}$ of a Desarguesian plane $\mathrm{PG}(2, q^{2^n})$ we have the André/Bruck-Bose representations of the subplanes, which contain the line at infinity as a line, nested in $\Psi_{2^{i+1}}$ as linear subspaces.

Finally, let ℓ be a subline of order $q^{2^{n-j}}$ $(1 \le j \le n)$ of a line L of $\mathrm{PG}(2, q^{2^n})$ such that ℓ contains a unique point on the line at infinity. It follows from the above discussion that the representation of ℓ in any André/Bruck-Bose representation $\Psi_{2^{i+1}}$ of $\mathrm{PG}(2, q^{2^n})$ is determined.

Corollary 2.9. *Let ℓ be a subline of order $q^{2^{n-j}}$ $(1 \le j \le n)$ of a line L of $\mathrm{PG}(2, q^{2^n})$ such that ℓ contains a unique point on the line at infinity ℓ_∞ of $\mathrm{PG}(2, q^{2^n})$. Let $\Psi_{2^{i+1}}$ $(1 \le i \le n)$ denote the André/Bruck-Bose representation of $\mathrm{PG}(2, q^{2^n})$ defined by a regular $(2^i-1)-$spread of $\mathrm{PG}(2^{i+1}-1, q^{2^{n-i}})$.*

Then the subline ℓ is represented by a $(2^{i-j})-$subspace ℓ^* of the $(2^i)-$space L^*, which represents L, in $\Psi_{2^{i+1}}$. Moreover, ℓ^* intersects the hyperplane at infinity $\mathrm{PG}(2^{i+1}-1, q^{2^{n-i}})$ in exactly a unique induced spread element of dimension $2^{i-j}-1$ and order $q^{2^{n-i}}$. ∎

In this way we obtain the André/Bruck-Bose representations of any (not just a Baer) subplane of $\mathrm{PG}(2, q^{2^n})$ which contains the line at infinity as a line and any (not just a Baer) subline of a line of $\mathrm{PG}(2, q^{2^n})$ such that the subline contains a unique point on the line at infinity.

2.5 Non-Affine Baer subplanes in the André/Bruck-Bose representation of $\mathbf{PG(2, q^4)}$ in $\mathbf{PG(8, q)}$

The André/Bruck-Bose representation of $\mathrm{PG}(2, q^4)$ in $\mathrm{PG}(8, q)$ is determined by a regular $3-$spread $\mathcal{S}_3$ in a fixed hyperplane $\Sigma_{7,q}$ of $\mathrm{PG}(8, q)$. We denote this representation by $\Psi_{8,q}$ and we denote by ℓ_∞ the line at infinity of $\mathrm{PG}(2,q^4)$ which corresponds to the spread $\mathcal{S}_3$ in $\Sigma_{7,q}$.

In Section 2.3 we investigated the representation of the affine Baer subplanes of $\mathrm{PG}(2,q^4)$ in $\Psi_{8,q}$. In Corollary 2.6 we characterised the affine Baer subplanes of $\mathrm{PG}(2,q^4)$ in terms of this representation. Moreover we determined how Baer sublines b_ℓ of lines of $\mathrm{PG}(2,q^4)$, such that b_ℓ is incident with ℓ_∞, are represented in $\Psi_{8,q}$. We now consider the non-affine Baer subplanes of $\mathrm{PG}(2,q^4)$ and the Baer sublines which are disjoint from ℓ_∞.

Let ℓ be a line in $\mathrm{PG}(2,q^4)$ distinct from ℓ_∞ and let P be the unique point of intersection of ℓ and ℓ_∞. Let b_ℓ be a Baer subline of ℓ such that b_ℓ is disjoint from ℓ_∞, so that P is not incident with b_ℓ. Also let Ψ_{4,q^2} be the André/Bruck-Bose representation of $\mathrm{PG}(2,q^4)$ in $\mathrm{PG}(4,q^2)$ defined by a regular 1−spread $\mathcal{S}_1$ of a hyperplane Σ_{3,q^2} of $\mathrm{PG}(4,q^2)$. In Ψ_{4,q^2}, ℓ is represented by a plane ℓ^* of $\mathrm{PG}(4,q^2)\backslash\Sigma_{3,q^2}$ and the intersection $\ell^*\cap\Sigma_{3,q^2}$ is a line P^* which is an element of the spread $\mathcal{S}_1$. It is known that a Baer subline in $\mathrm{PG}(2,q^4)$ which is disjoint from the line at infinity is represented in Ψ_{4,q^2} by a certain non-degenerate conic, hence the Baer subline b_ℓ is represented by a non-degenerate conic b_ℓ^* in the plane ℓ^* such that b_ℓ^* is disjoint from P^*; we call such a conic in Ψ_{4,q^2} a **Baer conic**. Note that the plane $\ell^*\backslash\{P^*\}$, namely ℓ^* with the line P^* and all its points removed, is isomorphic to the affine plane $\mathrm{AG}(2,q^2)$. We have the following correspondence due to the André/Bruck-Bose representation: the points of $\ell^*\backslash\{P^*\}$ correspond to the points of ℓ distinct from P; the lines of $\ell^*\backslash\{P^*\}$ correspond to the Baer sublines of ℓ which contain P; incidence is containment.

In $\Psi_{8,q}$, the line ℓ is represented by a 4−space ℓ^{**} of $\mathrm{PG}(8,q)\backslash\Sigma_{7,q}$ and the intersection $\ell^{**}\cap\Sigma_{7,q}$ is a 3−space element P^{**} of the regular 3−spread $\mathcal{S}_3$ of $\Sigma_{7,q}$. By Theorem 2.4, there exists a fixed induced regular 1−spread $\mathcal{S}_\ell^1$ in P^{**}. By Corollary 2.6 the planes of $\ell^{**}\backslash P^{**}$ which intersect P^{**} in a unique line of $\mathcal{S}_\ell^1$ represent the Baer sublines of ℓ which contain the point P. Hence this regular 1−spread $\mathcal{S}_\ell^1$ in P^{**} defines an André/Bruck-Bose representation of $\mathrm{PG}(2,q^2)$ in the 4−space ℓ^{**}; this is the André/Bruck-Bose representation of the plane ℓ^* in a 4−dimensional projective space.

Let b_ℓ^{**} denote the representation in $\Psi_{8,q}$ of the Baer subline b_ℓ of ℓ. In Ψ_{4,q^2}, since b_ℓ^* is a non-degenerate conic in ℓ^*, disjoint from the line P^* in ℓ^*, it follows from the preceding paragraph that $b_\ell^{**}\subseteq\ell^{**}$ is precisely an André/Bruck-Bose representation in 4−dimensions of a non-degenerate conic in the plane $\ell^*=\mathrm{PG}(2,q^2)$ and disjoint from the line at infinity, P^*.

By [2], [9] (see also [8]), a Baer subplane B of $\mathrm{PG}(2,q^4)$ which intersects ℓ_∞ in a unique point R is represented in Ψ_{4,q^2} by a ruled cubic surface B^* with line directrix R^* where R^* is an element of $\mathcal{S}_1$. Moreover, the intersection $B^*\cap\Sigma_{3,q^2}$ in Ψ_{4,q^2} is exactly the line R^* and the points of B^* lie on q^2+1 distinct lines of $\Psi_{4,q^2}\backslash\Sigma_{3,q^2}$, one through each point of R^*. These lines represent the Baer sublines in B which are incident with R. The remaining Baer sublines in B are represented in Ψ_{4,q^2} by the q^2 (Baer) conics on the ruled cubic surface B^*.

In $\Psi_{8,q}$, the Baer subplane B is represented by a structure B^{**} in $\mathrm{PG}(8,q)$. The point R in $\mathrm{PG}(2,q^4)$ is represented by an element R^{**} of the spread $\mathcal{S}_3$ of $\Sigma_{7,q}$. By Theorem 2.4 and Corollary 2.6 and by considering the situation in Ψ_{4,q^2} above, the Baer sublines in B incident with R are represented in $\Psi_{8,q}$ by q^2+1 distinct planes in $\mathrm{PG}(8,q)\backslash\Sigma_{7,q}$ each of which intersect $\Sigma_{7,q}$ in a distinct line of the induced $1-$spread $\mathcal{S}_1^j$ in R^{**}. Moreover, as B^* contains q^2 Baer conics in Ψ_{4,q^2}, the structure B^{**} contains q^2 representations of Baer conics in $\Psi_{8,q}$, where each has the structure discussed above for a given $4-$space of $\Psi_{8,q}$, which corresponds to a line of $\mathrm{PG}(2,q^4)$.

It is difficult to determine in more helpful geometric detail the André/Bruck-Bose representation of the non-affine Baer subplanes of $\mathrm{PG}(2,q^4)$ without appealing to further deep theory.

3 Unitals

In this section we study the André/Bruck-Bose representation in $\mathrm{PG}(4h,q)$ of unitals in $\mathrm{PG}(2,q^{2h})$. We treat the case $h=2$ in greatest detail, and discuss the results for general h. We also present an idea for the possible construction of non-Buekenhout-Metz unitals in Desarguesian planes of order q^4, however, so far we have been unable to determine whether a set so constructed is a unital.

A *unital* U in a projective plane $\mathcal{P}$ of order q is a set of $q\sqrt{q}+1$ points, such that every line of $\mathcal{P}$ meets U in 1 or $\sqrt{q}+1$ points. Lines of $\mathcal{P}$ are called *tangent* or *secant* lines if they meet U in 1 or $\sqrt{q}+1$ points respectively. A unital is called *hyperbolic* if the line at infinity ℓ_∞ is a secant, and *parabolic* if ℓ_∞ is a tangent. The *classical* unital is the set of points in $\mathrm{PG}(2,q)$ projectively equivalent to the Hermitian curve with equation

$$xx^{\sqrt{q}}+yy^{\sqrt{q}}+zz^{\sqrt{q}}=0.$$

3.1 Coordinates for the André/Bruck-Bose representation of $\mathbf{PG(2,q^4)}$ in $\mathbf{PG(8,q)}$

In order to study unitals of $\mathrm{PG}(2,q^4)$ in $\mathrm{PG}(8,q)$ we use the coordinate representation described in Section 1.1. To simplify our calculations, we use an appropriate but less general form for the basis of $\mathrm{GF}(q^4)$ as a vector space over $\mathrm{GF}(q)$. In this section we describe the coordinate representation for this case in detail in order to establish notation.

Let Σ_∞ be a hyperplane of $\mathrm{PG}(8,q)$ and let $\mathcal{S}$ be a regular 3-spread of Σ_∞. Consider the incidence structure with: *points* the points of $\mathrm{PG}(8,q)\setminus\Sigma_\infty$, together with the elements of $\mathcal{S}$; *lines* the 4-spaces of $\mathrm{PG}(8,q)$ not in Σ_∞ which contain an element of $\mathcal{S}$, together with the *line at infinity* ℓ_∞ containing all the spread elements of $\mathcal{S}$; and *incidence* is the natural incidence. This structure is isomorphic to $\mathrm{PG}(2,q^4)$. (Note that $\mathrm{PG}(2,q^4)$ also corresponds to a regular $1-$spread of $\mathrm{PG}(3,q^2)$ by the usual André/Bruck-Bose representation in $\mathrm{PG}(4,q^2)$.) Recall that by Theorem 2.4, each element of $\mathcal{S}$ contains an induced regular $1-$spread. Planes of $\mathrm{PG}(8,q)\setminus\Sigma_\infty$ about an induced spread line correspond to Baer sublines of $\mathrm{PG}(2,q^4)$ which contain a point on ℓ_∞ and conversely.

We coordinatise $\mathrm{PG}(2,q^4)$ as follows. Let the points of $\mathrm{PG}(2,q^4)$ have homogeneous coordinates (x,y,z) where $x,y,z\in\mathrm{GF}(q)$. Let $z=0$ be the line at infinity ℓ_∞, then the affine points have coordinates $(x,y,1)$ where $x,y\in\mathrm{GF}(q^4)$. Let $\{1,\beta\}$ be a basis for $\mathrm{GF}(q^4)$ as a vector space over $\mathrm{GF}(q^2)$ where β has minimum polynomial $x^2+x+\gamma=0$, $\gamma\in\mathrm{GF}(q^2)\setminus\mathrm{GF}(q)$. Let $\{1,\alpha\}$ be a basis for $\mathrm{GF}(q^2)$ as a vector space over $\mathrm{GF}(q)$ where α has minimum polynomial $x^2+x+c=0$, $c \in \mathrm{GF}(q)$. Hence $\{1,\alpha,\beta,\alpha\beta\}$ is a basis for $\mathrm{GF}(q^4)$ as a vector space over $\mathrm{GF}(q)$. We can write γ as $\gamma = a+\alpha b$ for unique $a,b \in \mathrm{GF}(q)$, where $b \neq 0$. Note that for q even, both γ and c are category 1; and for q odd, $1-4\gamma$ is a non-square in $\mathrm{GF}(q^2)$ and $1-4c$ is a non-square in $\mathrm{GF}(q)$.

Let the homogeneous coordinates of a point in $\mathrm{PG}(8,q)$ be $(x_0,x_1,x_2,x_3,y_0,y_1,y_2,y_3,z)$, where $x_i,y_i,z \in \mathrm{GF}(q)$. Let Σ_∞ be the hyperplane with equation $z=0$.

If $x,y \in \mathrm{GF}(q^4)$ then we can write $x = x_0+x_1\alpha+x_2\beta+x_3\alpha\beta$ and $y = y_0+y_1\alpha+y_2\beta+y_3\alpha\beta$ for unique $x_0,x_1,x_2,x_3,y_0,y_1,y_2,y_3 \in \mathrm{GF}(q)$. We map the affine point $(x,y,1)$ of $\mathrm{PG}(2,q^4)$ to the point $(x_0,x_1,x_2,x_3,y_0,y_1,y_2,y_3,1)$ in $\mathrm{PG}(8,q)\setminus\Sigma_\infty$.

We use this mapping to show that the lines of $\mathrm{PG}(2,q^4)$ distinct from ℓ_∞ correspond to 4-spaces of $\mathrm{PG}(8,q)\setminus\Sigma_\infty$ about a spread element. Consider the line $x=my+fz$ in $\mathrm{PG}(2,q^4)$ which meets ℓ_∞ in the point $(m,1,0)$. This line has affine equation $x=my+f$. Writing $x=x_0+x_1\alpha+x_2\beta+x_3\alpha\beta$, $y=y_0+y_1\alpha+y_2\beta+y_3\alpha\beta$, $m=m_0+m_1\alpha+m_2\beta+m_3\alpha\beta$ and $f=f_0+f_1\alpha+f_2\beta+f_3\alpha\beta$ where $x_i,y_i,m_i,f_i \in \mathrm{GF}(q)$, the line corresponds to the set of points in $\mathrm{PG}(8,q)\setminus\Sigma_\infty$ satisfying

$$\begin{aligned}x_0+x_1\alpha+x_2\beta+x_3\alpha\beta = (m_0+m_1\alpha+m_2\beta+m_3\alpha\beta)(y_0+y_1\alpha+y_2\beta+y_3\alpha\beta)+\\ (f_0+f_1\alpha+f_2\beta+f_3\alpha\beta).\end{aligned}$$

This can be simplified using the minimum polynomials of α and β. Equating coefficients of basis elements and homogenising gives four linearly independent hyperplanes in $\mathrm{PG}(8,q)$. The line corresponds to the 4-space intersection of the following four linearly independent hyperplanes:

$$m_0y_0 - cm_1y_1 + (bcm_3 - am_2)y_2 + (bcm_2 + c(a-b)m_3)y_3 - x_0 + f_0z = 0$$

$$m_1y_0+(m_0-m_1)y_1+(b(m_3-m_2)-am_3)y_2+((a-b)(m_3-m_2)+bcm_3)y_3-x_1+f_1z=0$$

$$m_2y_0 - cm_3y_1 + (m_0-m_2)y_2 + c(m_3-m_1)y_3 - x_2 + f_2z = 0$$

$$m_3y_0 + (m_2-m_3)y_1 + (m_1-m_3)y_2 + (m_0-m_1-m_2+m_3)y_3 - x_3 + f_3z = 0.$$

This 4-space meets Σ_∞ in the 3-space $J(m)$, the element of $\mathcal{S}$ which corresponds to the point at infinity of all affine lines with equation $x=my+f$, $f \in \mathrm{GF}(q^4)$.

Finally, the lines $y=dz$ with point at infinity (1,0,0) correspond to the points of $\mathrm{PG}(8,q)$ in the 4-space $y_0=d_0z, y_1=d_1z, y_2=d_2z, y_3=d_3z$, where $d=d_0+d_1\alpha+d_2\beta+d_3\alpha\beta$ for unique $d_i \in \mathrm{GF}(q)$. These 4-spaces all contain the spread element $J(\infty)$ defined by $y_0=0, y_1=0, y_2=0, y_3=0, z=0$.

3.2 The hyperbolic classical unital in $\mathrm{PG}(2,q^4)$

Let U be a classical unital secant to ℓ_∞ in $\mathrm{PG}(2,q^4)$. We can describe its point set in $\mathrm{PG}(8,q)$ by using the two André/Bruck-Bose correspondences of $\mathrm{PG}(2,q^4)$ in $\mathrm{PG}(4,q^2)$ and $\mathrm{PG}(8,q)$ respectively. In $\mathrm{PG}(8,q)$, the classical unital secant to ℓ_∞ in $\mathrm{PG}(2,q^4)$ corresponds to q^2+1 elements of the spread $\mathcal{S}$ and q^2-1 planes of $\mathrm{PG}(8,q)\setminus\Sigma_\infty$ about each induced spread line in these spread elements; a total of

$(q^2-1)(q^2+1)(q^2+1)$ planes. No two of these planes lie in a 4-space that corresponds to either a Baer subplane or a line of $\mathrm{PG}(2,q^4)$.

We investigate this structure further using coordinates and show:

Theorem 3.1. *Let U be the hyperbolic classical unital in* $\mathrm{PG}(2,q^4)$. *In the André/ Bruck and Bose representation of* $\mathrm{PG}(2,q^4)$ *in* $\mathrm{PG}(8,q)$, *U corresponds to the set of points in the intersection of two quadrics in* $\mathrm{PG}(8,q)$.

Proof. We use the coordinatisation described in Section 3.1. The classical unital secant to ℓ_∞ in $\mathrm{PG}(2,q^4)$ is isomorphic to the set of points satisfying $x\overline{x}+y\overline{y}+z\overline{z}=0$ where $\overline{x}=x^{q^2}$. This is equivalent to the set of points $(x,y,1)$ satisfying the affine equation $x\overline{x}+y\overline{y}+1=0$ together with the q^2+1 points of the Baer subline $x\overline{x}+y\overline{y}=0$ of ℓ_∞. Writing $x=x_0+x_1\alpha+x_2\beta+x_3\alpha\beta$ and $y=y_0+y_1\alpha+y_2\beta+y_3\alpha\beta$ and substituting into $x\overline{x}+y\overline{y}+1=0$ we obtain the equation:

$$F_0(x)+F_0(y)+(F_1(x)+F_1(y))\alpha+1=0$$

where

$$F_0(x)=x_0^2-cx_1^2+ax_2^2-c(a-b)x_3^2-x_0x_2+cx_1x_3-2bcx_2x_3$$

$$F_1(x)=-x_1^2+bx_2^2-(a-b+bc)x_3^2+2x_0x_1-x_0x_3-x_1x_2+x_1x_3+2(a-b)x_2x_3.$$

This is the equation of a quadric in the quadratic extension $\mathrm{PG}(8,q^2)$ of $\mathrm{PG}(8,q)$.

By equating coefficients of powers of α and homogenising we obtain two quadrics in $\mathrm{PG}(8,q)$, namely

$$\mathcal{Q}: F_0(x)+F_0(y)+z^2=0$$

$$\mathcal{Q}': F_1(x)+F_1(y)=0$$

The points of the unital correspond to points in the intersection of these two quadrics and conversely. ∎

The two quadrics $\mathcal{Q}$, $\mathcal{Q}'$ both contain the q^2+1 elements of $\mathcal{S}$ which correspond to the points of ℓ_∞ that satisfy the equation $x\overline{x}+y\overline{y}=0$. They contain no further common spread element as the unital contains no further point of ℓ_∞ in $\mathrm{PG}(2,q^4)$.

By determining the points at which all the partial derivatives of a quadric vanish, we can find the space of singular points of the quadric (or in the case q even and the quadric non-singular, we obtain the nucleus of the quadric). We look at the quadrics $\mathcal{Q}$, $\mathcal{Q}'$ in the cases q odd and q even separately.

If q is even, $\mathcal{Q}$ is non-singular with nucleus $(0,0,0,0,0,0,0,0,1)$ and $\mathcal{Q}'$ is singular with singular space the point $(0,0,0,0,0,0,0,0,1)$. However, if we let $\mathcal{Q}''=\mathcal{Q}+\mathcal{Q}'$, then $\mathcal{Q}''$ is non-singular with nucleus $(0,0,0,0,0,0,0,0,1)$ and the unital corresponds to the points on the intersection of two quadrics $\mathcal{Q}$ and $\mathcal{Q}''$ which are both non-singular with a common nucleus.

If q is odd, $\mathcal{Q}$ is singular or non-singular depending on the choice of basis for $\mathrm{GF}(q^4)$ over $\mathrm{GF}(q)$; similarly for $\mathcal{Q}''=\mathcal{Q}+\mathcal{Q}'$. The quadric $\mathcal{Q}'$ is always singular, with the dimension of the space of singular points depending on the choice of basis.

3.3 The parabolic classical unital in $\mathrm{PG}(2, q^4)$

Let U be a Buekenhout-Metz unital in $\mathrm{PG}(2,q^4)$ tangent to ℓ_∞, the parabolic classical unital is an example of such a unital. By considering the André/Bruck-Bose representations of $\mathrm{PG}(2,q^4)$ in $\mathrm{PG}(4,q^2)$ and in $\mathrm{PG}(8,q)$, we observe that U corresponds to the set of points of $\mathrm{PG}(8,q)$ lying on q^4+1 planes about an induced spread line ℓ together with the spread element J containing ℓ. These planes are not contained in Σ_∞; no three lie in a 4-space which corresponds to a Baer subplane of $\mathrm{PG}(2,q^4)$; and no two lie in a 4-space which corresponds to a line of $\mathrm{PG}(2,q^4)$. These points form a cone $\mathcal{U}$ which is the join of the *vertex* ℓ, and a *base* a set of q^4 points of $\mathrm{PG}(8,q) \setminus \Sigma_\infty$ (no three in a plane about an induced spread line) in a 6-space that meets Σ_∞ in a 5-space and meets the spread element J in an induced spread line m distinct from ℓ; we also include m in the base of the cone.

We now investigate the parabolic classical unital using coordinates. We use the quadric notation of [7, Chapter 22]. In $\mathrm{PG}(n,q)$, n even, we denote by $\mathcal{P}_n$ a non-singular parabolic quadric. In $\mathrm{PG}(n,q)$, n odd, we denote by $\mathcal{E}_n$ (respectively $\mathcal{H}_n$) a non-singular elliptic (respectively hyperbolic) quadric. If a quadric in $\mathrm{PG}(n,q)$ is singular, then it is a *cone* denoted $\Pi_{n-s-1}\mathcal{Q}_s$, that is, the join of a $(n-s-1)$-space *vertex* Π_{n-s-1} to a *base* a non-singular quadric $\mathcal{Q}_s$ in an s-space Π_s, where Π_s and Π_{n-s-1} are disjoint. Note that the vertex Π_{n-s-1} is the space of singular points of the quadric. We use the phrase an *s-dimensional quadric* to mean a quadric $\mathcal{Q}_s$ that lies in an s-space but is not contained in any $(s-1)$-space.

Theorem 3.2. *Let U be the parabolic classical unital in $\mathrm{PG}(2,q^4)$.*
In the André/Bruck-Bose representation in $\mathrm{PG}(8,q)$, U corresponds to a set of points $\mathcal{U}$ which is the intersection of two quadrics $\mathcal{Q}_0$, $\mathcal{Q}_1$ of $\mathrm{PG}(8,q)$, that is, $\mathcal{U} = \mathcal{Q}_0 \cap \mathcal{Q}_1$. The characters of $\mathcal{Q}_0$ and $\mathcal{Q}_1$ depend on the basis chosen for $GF(q^4)$ as a vector space over $GF(q)$ as follows:

1. *If q is even, then*
   - *(i) if b is category 1, then $\mathcal{Q}_0$ and $\mathcal{Q}_1$ both have form $\Pi_2\mathcal{E}_5$;*
   - *(ii) if b is category 0, then $\mathcal{Q}_0$ and $\mathcal{Q}_1$ both have form $\Pi_2\mathcal{H}_5$.*
2. *If q is odd, let $k = 16b^2c + (4a-1)(4a-4b-1)$, then*
   - *(i) if $k \neq 0$ and k is a square in $GF(q)$, then $\mathcal{Q}_0$ and $\mathcal{Q}_1$ both have form $\Pi_2\mathcal{H}_5$;*
   - *(ii) if $k \neq 0$ and k is a non-square in $GF(q)$, then $\mathcal{Q}_0$ and $\mathcal{Q}_1$ both have form $\Pi_2\mathcal{E}_5$;*
   - *(iii) if $k = 0$, then $\mathcal{Q}_0$ and $\mathcal{Q}_1$ both have form $\Pi_3\mathcal{P}_4$.*

In every case, the singular spaces of $\mathcal{Q}_0$ and $\mathcal{Q}_1$ contain the induced spread line ℓ which is the vertex of $\mathcal{U}$.

Proof. We use the coordinatisation described in Section 3.1. The parabolic classical unital in $\mathrm{PG}(2,q^4)$ is isomorphic to the set of points satisfying $x\overline{x}+y\overline{z}+\overline{y}z = 0$ where $\overline{x} = x^{q^2}$. This corresponds to the set of affine points $(x,y,1)$ satisfying $x\overline{x}+y+\overline{y} = 0$

together with the point at infinity $(0,1,0)$. Writing $x = x_0 + x_1\alpha + x_2\beta + x_3\alpha\beta$ and $y = y_0 + y_1\alpha + y_2\beta + y_3\alpha\beta$ we obtain a quadric in $\mathrm{PG}(8,q^2)$ with equation:

$$F_0(x) + 2y_0 - y_2 + \alpha(F_1(x) + 2y_1 - y_3) = 0$$

where $F_0(x), F_1(x)$ are given in Section 3.2. Equating coefficients of basis elements and homogenising gives two quadrics $\mathcal{Q}_0, \mathcal{Q}_1$ in $\mathrm{PG}(8,q)$ whose intersection corresponds to the points of the unital. The quadrics $\mathcal{Q}_0$, $\mathcal{Q}_1$ are as follows:

$$\mathcal{Q}_0 : x_0^2 - cx_1^2 + ax_2^2 - c(a-b)x_3^2 - x_0x_2 + cx_1x_3 - 2bcx_2x_3 + 2y_0z - y_2z = 0$$

$$\mathcal{Q}_1 : -x_1^2 + bx_2^2 - (a-b+bc)x_3^2 + 2x_0x_1 - x_0x_3 - x_1x_2 + x_1x_3 + 2(a-b)x_2x_3 + 2y_1z - y_3z = 0.$$

These two quadrics both contain the 3-space spread element $J(0)$ which is the intersection of the five linearly independent hyperplanes $x_0 = 0, x_1 = 0, x_2 = 0$, $x_3 = 0, z = 0$. This spread element corresponds to the point $(0,1,0)$ of ℓ_∞ in $\mathrm{PG}(2,q^4)$. The quadrics contain no further common spread element.

By determining the points at which all the partial derivatives of $\mathcal{Q}_i$ vanish, we can find the space of singular points of the quadric $\mathcal{Q}_i$, $i = 0,1$.

1. Consider the case q even.

If q is even then the quadrics are given by the equations:

$$\mathcal{Q}_0 : x_0^2 + cx_1^2 + ax_2^2 + (ac+bc)x_3^2 + x_0x_2 + cx_1x_3 + y_2z = 0$$

$$\mathcal{Q}_1 : x_1^2 + bx_2^2 + (a+b+bc)x_3^2 + x_0x_3 + x_1x_2 + x_1x_3 + y_3z = 0.$$

The space of singular points of $\mathcal{Q}_0$ is a plane π_0 in Σ_∞, where π_0 is the intersection of six linearly independent hyperplanes:

$$\pi_0 : x_0 = 0, x_1 = 0, x_2 = 0, x_3 = 0, y_2 = 0, z = 0.$$

The space of singular points of $\mathcal{Q}_1$ is a plane π_1 in Σ_∞, where π_1 is the intersection of six linearly independent hyperplanes:

$$\pi_1 : x_0 = 0, x_1 = 0, x_2 = 0, x_3 = 0, y_3 = 0, z = 0.$$

Using the technique described in [7, Section 22.2] we can determine whether the base of the quadric $\mathcal{Q}_i$ is elliptic or hyperbolic. We need to look at the intersection of each quadric $\mathcal{Q}_i$ with a 5-space skew to the space of singular points of the quadric.

For $\mathcal{Q}_0$, the 5-space defined by $y_0 = 0, y_1 = 0, y_3 = 0$ is disjoint from π_0, so we consider the points of the quadric with coordinates $(x_0,x_1,x_2,x_3,0,0,y_2,0,z)$, or for convenience, $\mathbf{x} = (x_0,x_1,x_2,x_3,y_2,z)$. We write the quadric as $\frac{1}{2}\mathbf{x}(A+B)\mathbf{x}^t$ as described in [7], with

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2c & 0 & c & 0 & 0 \\ 1 & 0 & 2a & 0 & 0 & 0 \\ 0 & c & 0 & 2c(a+b) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Now $|B| = c^2$ and $|A| = c^2(-4a+1)(4a+4b-1)$. Note that since $|A| \equiv -c^2 \neq 0$, the quadric is non-singular. Moreover,

$$\frac{|B| - (-1)^{\frac{5+1}{2}}|A|}{4|B|} = -4a^2 - 4ab + 2a + b \equiv b \qquad \text{since } q \text{ is even.}$$

Hence the base quadric of $\mathcal{Q}_0$ is a non-singular 5-dimensional quadric which is elliptic if b is category 1 and is hyperbolic if b is category 0.

Similarly, for $\mathcal{Q}_1$ we intersect $\mathcal{Q}_1$ with the 5-space given by $y_0 = 0, y_1 = 0, y_2 = 0$ and consider the quadric thus obtained with coordinates $\mathbf{x} = (x_0, x_1, x_2, x_3, y_3, z)$. The matrices A and B are as follows:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2b & 0 & 0 & 0 \\ 1 & 1 & 0 & 2(a+b+bc) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

and $|A| = 4b - 1$, $|B| = 1$. This quadric is non-singular as $|A| \equiv -1 \neq 0$. Further,

$$\frac{|B| - (-1)^{\frac{5+1}{2}}|A|}{4|B|} = b.$$

Hence the base quadric of $\mathcal{Q}_1$ is a non-singular 5-dimensional quadric which is elliptic if b is category 1 and hyperbolic if b is category 0.

Thus both $\mathcal{Q}_0$ and $\mathcal{Q}_1$ have base a 5-dimensional elliptic quadric if b is in category 1, else they both have base a 5-dimensional hyperbolic quadric.

Hence, if q is even, the unital corresponds to the intersection of two quadrics $\mathcal{Q}_0$ with plane vertex π_0 and $\mathcal{Q}_1$ with plane vertex π_1, both quadrics with a base 5-dimensional quadric of the same character t. The two vertex planes π_0, π_1 are contained in $J(0)$ and meet in a line which is an induced spread line of $J(0)$ (since the unital consists of q^4 planes of $\mathrm{PG}(8,q) \setminus \Sigma_\infty$ about an induced spread line). The base of the unital lies in a 6-space disjoint from this induced spread line. Such a 6-space meets $\mathcal{Q}_0$ (respectively $\mathcal{Q}_1$) in a cone with vertex a point of π_0 (respectively π_1) and base a non-singular 5-dimensional quadric of character t. Thus the base of the unital is the intersection of two 6-dimensional quadric cones with distinct point vertices in Σ_∞ and base quadrics of character t.

2. We now consider the case q odd. Here the dimension of the space of singular points of each quadric depends on the choice of basis.

The partial derivatives of $\mathcal{Q}_0$ all vanish when the following equations are satisfied.

$$\begin{aligned}
2x_0 \qquad\qquad - \quad x_2 \qquad\qquad\qquad\qquad\qquad &= 0 \\
- 2cx_1 \qquad\qquad + \qquad cx_3 \qquad\qquad &= 0 \\
-x_0 \qquad\qquad + 2ax_2 - 2bcx_3 \qquad\qquad &= 0 \\
cx_1 - 2bcx_2 + 2c(b-a)x_3 \qquad\qquad &= 0 \\
z &= 0 \\
2y_0 - y_2 &= 0
\end{aligned}$$

These equations are linearly independent if and only if

$$k = 16b^2c + (4a-1)(4a-4b-1) \neq 0.$$

In this case, $\mathcal{Q}_0$ has vertex a plane and base a non-singular 5-dimensional quadric (as derived below). In order to determine the character of the base quadric we need to find a 5-space that is disjoint from the vertex of $\mathcal{Q}_0$. The 5-space $y_1 = 0, y_3 = 0, y_0 - y_2 = 0$ is suitable. $\mathcal{Q}_0$ meets this 5-space in the quadric

$$x_0^2 - cx_1^2 + ax_2^2 - c(a-b)x_3^2 - x_0x_2 + cx_1x_3 - 2bcx_2x_3 + y_0z = 0$$

of points with coordinates $\mathbf{x} = (x_0, x_1, x_2, x_3, y_0, z)$. Writing this as $\frac{1}{2}\mathbf{x}(A+B)\mathbf{x}^t$ yields:

$$A = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -2c & 0 & c & 0 & 0 \\ -1 & 0 & 2a & -2bc & 0 & 0 \\ 0 & c & -2bc & 2c(b-a) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $|A| = -c^2(16b^2c + (4a-1)(4a-4b-1)) = -c^2k$. Hence the base quadric is hyperbolic if and only if $-|A| = c^2k$ is a non-zero square of $\mathrm{GF}(q)$, that is, if and only if k is a non-zero square of $\mathrm{GF}(q)$; else (with $k \neq 0$) the base quadric is elliptic.

The partial derivatives of $\mathcal{Q}_1$ all vanish when the following equations are satisfied.

$$\begin{array}{rcrcrcrrr} & & 2x_1 & & & - & x_3 & & = 0 \\ 2x_0 & - & 2x_1 & - & x_2 & + & x_3 & & = 0 \\ & - & x_1 & + & 2bx_2 & + & 2(a-b)x_3 & & = 0 \\ -x_0 & + & x_1 & + & 2(a-b)x_2 & - & 2(a-b+bc)x_3 & & = 0 \\ & & & & & & & z & = 0 \\ & & & & & & & 2y_1 - y_3 & = 0 \end{array}$$

These equations are linearly independent if and only if

$$k = 16b^2c + (4a-1)(4a-4b-1) \neq 0.$$

In this case, $\mathcal{Q}_1$ has vertex a plane and base a non-singular 5-dimensional quadric (as derived below). To determine the character of this base quadric we need to find a 5-space that is disjoint from the vertex of $\mathcal{Q}_1$. The 5-space $y_0 = 0, y_2 = 0, y_1 - y_3 = 0$ is suitable. $\mathcal{Q}_1$ meets this 5-space in the quadric

$$-x_1^2 + bx_2^2 + (b-a-bc)x_3^2 + 2x_0x_1 - x_0x_3 - x_1x_2 + x_1x_3 + 2(a-b)x_2x_3 + y_1z = 0$$

of points with coordinates $\mathbf{x} = (x_0, x_1, x_2, x_3, y_1, z)$. Writing this as $\frac{1}{2}\mathbf{x}(A+B)\mathbf{x}^t$ yields:

$$A = \begin{bmatrix} 0 & 2 & 0 & -1 & 0 & 0 \\ 2 & -2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2b & 2(a-b) & 0 & 0 \\ -1 & 1 & 2(a-b) & 2(b-a-bc) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $|A| = -(4a-1)(4a-4b-1) - 16b^2c = -k$. Hence the base quadric of $\mathcal{Q}_1$ is hyperbolic if and only if $-|A| = k$ is a non-zero square of $\mathrm{GF}(q)$, that is, if and only if $\mathcal{Q}_0$ has a hyperbolic base quadric; else (with $k \neq 0$) the base quadric of $\mathcal{Q}_1$ is elliptic.

In summary, if $k = 16b^2c + (4a-1)(4a-4b-1) \neq 0$ the space of singular points of $\mathcal{Q}_0$ is a plane:

$$\pi_0 : x_0 = 0, x_1 = 0, x_2 = 0, x_3 = 0, z = 0, 2y_0 - y_2 = 0$$

and the space of singular points of $\mathcal{Q}_1$ is a plane:

$$\pi_1 : x_0 = 0, x_1 = 0, x_2 = 0, x_3 = 0, z = 0, 2y_1 - y_3 = 0.$$

Both these planes lie in the spread element $J(0)$: $x_0 = 0, x_1 = 0, x_2 = 0, x_3 = 0, z = 0$ which corresponds to the point $(0,1,0)$ of $\mathrm{PG}(2,q^4)$. The planes meet in the induced spread line $\ell : x_0 = 0, x_1 = 0, x_2 = 0, x_3 = 0, z = 0, 2y_0 - y_2 = 0, 2y_1 - y_3 = 0$. The quadrics $\mathcal{Q}_0$ and $\mathcal{Q}_1$ each have as base non-singular 5-dimensional quadrics of the same character. The base quadrics are hyperbolic if k is a square in $\mathrm{GF}(q)$, else they are elliptic quadrics.

We now consider the case $k = 16b^2c + (4a-1)(4a-4b-1) = 0$. Note that in this case, $4a - 1 \neq 0$ and $4a - 4b - 1 \neq 0$.

The space of singular points of $\mathcal{Q}_0$ is a 3-space Γ_0 given by the following five linearly independent hyperplanes:

$$\Gamma_0 : 2x_0 - x_2 = 0, 2x_1 - x_3 = 0, 4bx_2 + (4a-4b-1)x_3 = 0, z = 0, 2y_0 - y_2 = 0.$$

It can be shown that the base quadric of $\mathcal{Q}_0$ is a non-singular 4-dimensional quadric, that is, $\mathcal{Q}_0$ has form $\Pi_3\mathcal{P}_4$.

The space of singular points of $\mathcal{Q}_1$ is a 3-space Γ_1 given by the following five linearly independent hyperplanes:

$$\Gamma_1 : 2x_0 - 2x_1 - x_2 + x_3 = 0, 2x_1 - x_3 = 0, 4bx_2 + (4a-4b-1)x_3 = 0, z = 0, 2y_1 - y_3 = 0.$$

The base quadric of $\mathcal{Q}_1$ is a non-singular 4-dimensional quadric, that is, $\mathcal{Q}_1$ has form $\Pi_3\mathcal{P}_4$.

The two 3-space vertices Γ_0, Γ_1 meet the spread element $J(0)$ in the planes π_0 and π_1 respectively (where π_0 and π_1 are defined as above). In addition, $\Gamma_0 \cap \Gamma_1$ is given by the intersection of six linearly independent hyperplanes, hence it is a plane that meets $J(0)$ in the induced spread line ℓ.

Note that in this case $\mathcal{U}$ contains 3-spaces of $\mathrm{PG}(8,q)$. We noted earlier that the unital consists of q^4 planes of $\mathrm{PG}(8,q) \setminus \Sigma_\infty$ about an induced spread line, no three in a 4-space that corresponds to a Baer subplane of $\mathrm{PG}(2,q^4)$. As shown above, the vertices Γ_0 and Γ_1 each contain the induced spread line ℓ. Moreover, since Γ_0 and Γ_1 each intersect $J(0)$ in a plane, neither vertex Γ_0 nor Γ_1 contains a further induced spread line (see Theorem 2.5). So any 4 space containing one of these vertices does not correspond to either a Baer subplane or a line of $\mathrm{PG}(2,q^4)$. ■

3.4 Searching for new unitals

Consider the two André/Bruck-Bose representations of $\mathrm{PG}(2,q^4)$ in $\mathrm{PG}(4,q^2)$ and $\mathrm{PG}(8,q)$ respectively. In the $\mathrm{PG}(4,q^2)$ representation, every plane and line of $\mathrm{PG}(4,q^2)\setminus\Sigma_\infty$ has a natural correspondence to either a line, a Baer subplane or a Baer subline of $\mathrm{PG}(2,q^4)$. However, in $\mathrm{PG}(8,q)$ there are planes and 4-spaces that have no "nice" correspondence with objects in $\mathrm{PG}(2,q^4)$. In particular, a plane of $\mathrm{PG}(8,q)$ that meets Σ_∞ in a line which is not an induced spread line does not correspond to a Baer subline of $\mathrm{PG}(2,q^4)$.

Recall that in the André/Bruck-Bose representation of $\mathrm{PG}(2,q^4)$ in $\mathrm{PG}(8,q)$, a Buekenhout-Metz unital in $\mathrm{PG}(2,q^4)$ corresponds to a set of q^4 planes of $\mathrm{PG}(8,q)\setminus\Sigma_\infty$ about an induced spread line. Consider a transformation of the unital that moves its q^4 planes to a set of q^4 planes of $\mathrm{PG}(8,q)\setminus\Sigma_\infty$ about a line ℓ' which is contained in an element of the spread $\mathcal{S}$, but such that ℓ' is not an induced spread line of $\mathcal{S}$. Suppose that the resulting set of points corresponds to a unital of $\mathrm{PG}(2,q^4)$. Then this unital contains a point on ℓ_∞ such that none of the q^2 secants through this point meets the unital in a Baer subline. This unital is not Buekenhout-Metz since, by construction, through each point of a Buekenhout-Metz unital there exists at least one secant line which meets the unital in a Baer subline.

We have run some computer searches in small projective planes testing several different types of transformations which gave a set of q^4 planes in $\mathrm{PG}(8,q)\setminus\Sigma_\infty$ through a line that lies in an element of $\mathcal{S}$ but is not an induced spread line. In each case we considered, the resulting set was not a unital of $\mathrm{PG}(2,q^4)$, however the set did have an interesting structure. Recall that a unital in $\mathrm{PG}(2,q^4)$ is a set of type $(1,q^2+1)$. The sets we investigated were sets of type $(1,q+1,2q+1,\ldots,q^2+1,(q+1)q+1)$ and for each such set, every one of these intersection numbers occurred.

It seems plausible that a search for non-Buekenhout-Metz unitals in the Desarguesian plane should be carried out in the André/Bruck-Bose representation of $\mathrm{PG}(2,q^4)$ in $\mathrm{PG}(8,q)$ based on the above observations.

3.5 Unitals in $\mathrm{PG}(2,q^{2h})$

The results of Sections 3.2 and 3.3 can be generalized to the classical unital of $\mathrm{PG}(2,q^{2h})$ in the André/Bruck-Bose representation in $\mathrm{PG}(4h,q)$ as follows.

Theorem 3.3. *The classical unital in* $\mathrm{PG}(2,q^{2h})$ *corresponds to the intersection of* h *quadrics in the André/Bruck-Bose representation of* $\mathrm{PG}(2,q^{2h})$ *in* $\mathrm{PG}(4h,q)$. ∎

Theorem 3.4. *In the André/Bruck-Bose representation of* $\mathrm{PG}(2,q^{2^n})$ *in* $\mathrm{PG}(2^{i+1},q^{2^{n-i}})$, $1\le i\le n$, *a Buekenhout-Metz unital in* $\mathrm{PG}(2,q^{2^n})$ *corresponds to a set of* q^{2^n} 2^{i-1}*-dimensional subspaces of* $\mathrm{PG}(2^{i+1},q^{2^{n-i}})$ *(not in* Σ_∞*) about an induced spread element of dimension* $2^{i-1}-1$. ∎

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