A fibration with a section and of infinite genus

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Abstract

We give an example of a fibration which admits a section and of which the genus is infinite.

1 The universal fibration

Let X be a 1-connected CW-complex. Fibrations of fibre in the homotopy type of X are obtained, up to fibre homotopy equivalence, as pull back of the universal fibration $X \to B\,aut^{\bullet}\,X \to B\,aut\,X$ [1]; here $aut\,X$ denotes the topological monoid of all self-homotopy equivalences of X, $aut^{\bullet}X$ is the submonoid of $aut\,X$ consisting of pointed self-homotopy equivalences of X, and B is the Dold-Lashof functor from monoids to topological spaces [2]. Denote by $\tilde{B}\,aut\,X$ and $\tilde{B}\,aut^{\bullet}\,X$ universal coverings of $B\,aut\,X$ and $B\,aut^{\bullet}\,X$ respectively. The fibration

$$X \to \tilde{B}\,aut^{\bullet}\,X \to \tilde{B}\,aut\,X \tag{1}$$

is universal for fibrations $X \to E \xrightarrow{p} B$ for which the base space B is simply connected.

Henceforth we assume basic knowledge of rational homotopy theory for which classic references are [9, 4].

A model for the classifying space $\tilde{B}\,aut\,X$ was first given by Sullivan in [9] and later by Schlessinger-Stasheff [7] and Tanré [10]. The latter describes also a KS-extension model for the universal fibration. We use this model to compute the KS-model for the universal fibration for $X = \mathbb{C}P(2)$.

The Sullivan minimal model for $X = \mathbb{C}P(2)$ is given by $(\Lambda(x_2, x_5), d)$ with $dx_2 = 0$, $dx_5 = x_2^3$. Using derivations on the Sullivan model (see [10] for details), we deduce the following

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Proposition 1. *A model of the universal fibration $X \to \tilde{B}\,aut^{\bullet} X \to \tilde{B}\,aut\, X$ is given by the KS-extension*

$$(\Lambda(y_4, y_6), 0) \to (\Lambda(y_4, y_6) \otimes \Lambda(x_2, x_5), D) \to (\Lambda(x_2, x_5), d) \tag{2}$$

where $Dx_2 = 0,\ Dx_5 = x_2^3 + y_4 x_2 + y_6$.

Note that $\tilde{B}\,aut\, X$ has the rational homotopy type of BG where $G = SU(3)$. The KS-extension (2) looks like a model of a Borel fibration [3], but the long exact sequence of the fibration leads to

$$\pi_i(\tilde{B}\,aut^{\bullet} X) \otimes \mathbb{Q} = \mathbb{Q} \text{ for } i = 2, 4 \quad \text{and } \pi_i(\tilde{B}\,aut^{\bullet} X) \otimes \mathbb{Q} = 0 \text{ for } i \neq 2, 4.$$

Hence $H^*(\tilde{B}\,aut^{\bullet} X, \mathbb{Q}) = \Lambda(y_2, y_4)$, therefore the KS-extension (2) is not a model for a Borel fibration.

2 Lusternik-Schnirelmann category and related invariants

Let X be a topological space, the Lusternik-Schnirelmann category of X, $cat(X)$, is the least integer n such that X can be covered by $n+1$ open subsets contractible in X [6].

We define other invariants related to $cat(X)$.

Let $f : X \to Y$ be a continuous map. The category of f, denoted by $cat(f)$, is the least integer n such that X is covered by $n+1$ open subsets $U_1,\ U_2, \cdots,\ U_{n+1}$ such $f_{|U_i}$ is nullhomotopic. Note $cat(X)$ is equal to the category of the identity map. An approximation of $cat(f)$ is given by the relation

$$cat(f) \geq nil(im\, f^*), \tag{3}$$

where $f^* : H^*(Y) \to H^*(X)$ is the induced map in cohomology with any coefficient ring and $nil(R)$ denotes the nilpotency index of the ring R.

Let $p : E \to B$ be a fibration with fibre X. The sectional category $secat(p)$ is the least integer n such B can be covered by $n+1$ open subsets over each of which p admits a section. It verifies

$$secat(p) \geq nil(\ker\, p^*). \tag{4}$$

The genus of p, $genus(p)$, is the least integer n such B can be covered by $n+1$ open subsets over each of which p is a trivial fibration.

We have the following relation between the two invariants

Theorem 2. [5, 8] *Given a fibration $X \to E \xrightarrow{p} B$,*

(a) The genus of p is equal to $cat(f)$, where $f : B \to B\,aut\,X$ is the classifying map of the fibration p

(b) $secat(p) \leq genus(p)$

(c) $secat(p) = genus(p)$, if p a principal G-bundle.

In contrast with Theorem 2(c), we construct in the following section a fibration for which $secat(p) = 0$ and $genus(p) = \infty$.

3 An example

Theorem 3. *If p is the fibration for which the following KS-extension*

$$(\Lambda y_4, 0) \xrightarrow{i} (\Lambda y_4 \otimes \Lambda(x_2, x_5), D) \to (\Lambda(x_2, x_5), d) \text{ where } Dx_2 = 0,\ Dx_5 = x_2^3 + y_4 x_2, \tag{5}$$

is a model, then $secat(p) = 0$ and $genus(p) = \infty$.

Proof: Define $r : (\Lambda y_4 \otimes \Lambda(x_2, x_5), D) \to (\Lambda y_4, 0)$ by $r(y_4) = y_4, r(x_2) = r(x_5) = 0$. As r is a retraction of i, the fibration p admits a section, hence $secat(p) = 0$.

On the other hand, it is easily seen from [10] that the KS-extension is classified by the inclusion map

$$f : (\Lambda y_4, 0) \to (\Lambda(y_4, y_6), 0),$$

hence $im(f) = \Lambda y_4$. As $genus(p) = cat(f) \geq nil(im\, f) = \infty$, it follows that the genus of p is infinite.

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