The common dynamics of the tribonacci substitutions

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Abstract

The dynamical systems arising from the tribonacci substitutions: $1 \to 12$, $2 \to 13$, $3 \to 1$ and $1 \to 12$, $2 \to 31$, $3 \to 1$, have very distinctive dynamical and geometrical properties. In this article we study the dynamics that is common to these two systems, i.e. the dynamics obtained by the product of the prefix automata of these substitutions. We show the topological properties of the geometrical realization in the plane of this symbolic space. We also show that the dynamic associated to the product automaton can be realized in the plane and on the torus as a piece exchange transformation. We discuss the generalization of these results for the k-bonacci substitutions.

1 Introduction

The substitutions

$$\Pi_1 : \begin{cases} 1 & \to & 12 \\ 2 & \to & 13 \\ 3 & \to & 1 \end{cases} \quad \text{and} \quad \Pi_2 : \begin{cases} 1 & \to & 12 \\ 2 & \to & 31 \\ 3 & \to & 1 \end{cases}$$

have the same incidence matrix therefore the same recurrence relation, the tribonacci recurrence relation. The fixed points of each substitution have the same digit frequencies. In spite of that they have very different dynamical and geometrical properties, e.g. the Rauzy fractal of the first is simply connected [17] and the second is not simply connected [14]. The Rauzy fractal is defined in section 3.

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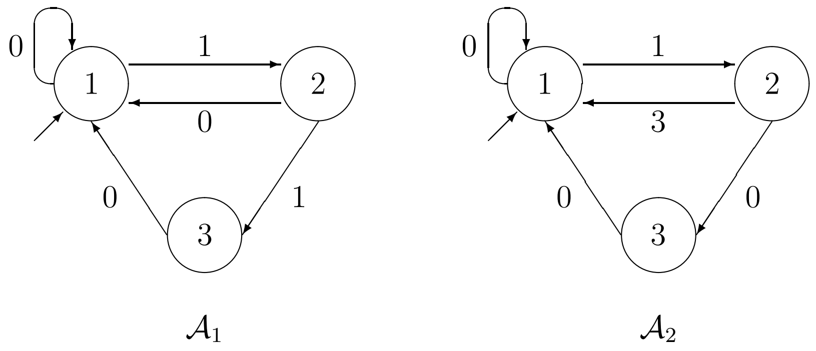


Figure 1: The prefix automata.

In this paper we shall study the common dynamics of these two substitutions. This is given by the dynamical system associated to the product of the prefix automata associated to each one of these substitutions. We will show its self-similar structure and use it to study the topological and dynamical properties of its geometric realization in the plane and on the two-dimensional torus. The topological structure of the realized symbolic space is a dendrite (Proposition 2). The realized dynamic is a piece exchange map on the dendrite (Theorem 1). In the last section we discuss a generalization of these results for k-bonacci substitutions.

Let Π be a substitution on the alphabet $\mathcal{C} = \{1, \ldots, k\}$ with a fixed point $\mathbf{u} = u_0u_1 \ldots$. We associate to it the dynamical system (Ω, σ) where σ is the shift, i.e. $\sigma(v_0v_1v_2 \ldots) = v_1v_2 \ldots$, and $\Omega = \overline{\{\sigma^n(\mathbf{u}) | n \in \mathbb{Z}^+\}}$. The closure is taken in the product topology of $\mathcal{C}^{\mathbb{Z}^+}$. This dynamical system admits an equivalent representation, but we need first to introduce the concepts of the numeration systems and the prefix automaton associated to Π.

The prefix automaton for the substitution Π, whose fixed point is obtained as $\Pi^\infty(1)$, is defined, in [19] as follows:

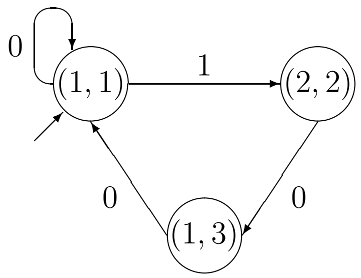
- set of states= $\mathcal{C}$
- set of labels= Pref = {the proper prefixes of the words $\Pi(1), \ldots, \Pi(k)$}. We shall denote by 0 the empty prefix.
- transitions: If p, q are states of the automaton and W a word of Pref then there exists a transition labelled by W from p to q, if and only if, the word Wq is a prefix of $\Pi(p)$.
- initial state= 1.

This automaton reads words from left to right. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the prefix automaton of Π_1 and Π_2 respectively. They are shown in figure 1.

In [19, 7] was proved the following theorem:

Let U be a non empty prefix of $\mathbf{u}$, the fixed point of the substitution Π. Then there exists a unique path in the prefix automaton, starting from 1 and labelled by $(W(n), W(n-1), \ldots, W(0))$ such that $W(n) \neq 0$ and

$$U = \Pi^n(W(n))\Pi^{n-1}(W(n-1)) \cdots W(0).$$



$\mathcal{A}_1 \times \mathcal{A}_2$

Figure 2: The product automaton of $\mathcal{A}_1$ and $\mathcal{A}_2$.

o Conversely, to any such path, there correspond a prefix of $\mathbf{u}$*, given by the above formula.*

According to this theorem we can write $u_0u_1 \dots u_{n-1}$, the word formed by the first n symbols of $\mathbf{u}$, as follows: $u_0u_1 \dots u_{n-1} = \Pi^k(W(k)) \cdots \Pi(W(1))W(0)$. Therefore, $n = |\Pi^k(W(k))| + \cdots + |\Pi(W(1))| + |W(0)|$ where $|W|$ denotes the length of the word W. Hence we have a numeration system based in the prefix automaton. See [7, 10] for more details about numeration systems based on automata.

Let

$$\mathcal{R} = \{\underline{a} = a_0a_1 \ldots \in \mathrm{Pref}^{\mathbb{Z}^+} \,|\, a_n \ldots a_0 \text{ is a path in } \mathcal{A} \text{ for all } n \in \mathbb{Z}^+\},$$

where $\mathcal{A}$ is the prefix automaton of Π. We can order lexicographically all the finite paths in $\mathcal{A}$, in a increasing way. So the addition by 1 of such a path is defined as its consecutive element. Therefore addition by 1 in $\mathcal{R}$ is well defined, we will denote this map by $\mathcal{T}$. Hence we have the dynamical system $(\mathcal{R}, \mathcal{T})$, which is called the *adic system* of Π. Using the techniques presented in [17, 18] it can be proved that this dynamical system is topological equivalent to (Ω, σ). See [12, 26, 27] for more details about adic systems. Let $(\mathcal{R}_1, \mathcal{T}_1)$ and $(\mathcal{R}_2, \mathcal{T}_2)$ be the adic systems associated with Π_1 and Π_2 respectively. As it can be seen in the automata of Π_1, the system $(\mathcal{R}_1, \mathcal{T}_1)$ is defined in the alphabet $\{0, 1\}$ and the only forbidden factor, according $\mathcal{A}_1$ is three consecutive 1's. On the other hand the system $(\mathcal{R}_2, \mathcal{T}_2)$ is defined on the alphabet $\{0, 1, 3\}$ and the forbidden factors are given by the automaton $\mathcal{A}_2$.

The automaton product $\mathcal{A}_1 \times \mathcal{A}_2$ is defined in [8] as the automaton whose set of states is given by the product of the states of $\mathcal{A}_i$, the set of labels is given by the intersection of the set of labels of $\mathcal{A}_1$ and $\mathcal{A}_2$, the initial states are the product of the initial states of each automaton and the transitions are given in the following way: If p_i and q_i are states of the automaton $\mathcal{A}_i$ and w is an element of the intersection of the labels of $\mathcal{A}_1$ and $\mathcal{A}_2$ then there exists a transition labelled by w from (p_1, p_2) to (q_1, q_2) if and only if there is a transition labelled by w from p_i to q_i for $i = 1$ and 2. The automaton $\mathcal{A}_1 \times \mathcal{A}_2$ can be seen in figure 2.

We will define the adic system corresponding to the automaton $\mathcal{A}_1 \times \mathcal{A}_2$ in the following way: Let

$$\mathcal{P} = \{\underline{a} = a_0a_1 \ldots \in \{0, 1\}^{\mathbb{Z}^+} \,|\, a_n \ldots a_0 \text{ is a path in } \mathcal{A}_1 \times \mathcal{A}_2 \text{ for all } n \in \mathbb{Z}^+\}.$$

We order lexicographically all the finite paths in $\mathcal{A}_1 \times \mathcal{A}_2$, this allow us to define a new numeration system. So we define $\mathcal{T}(a_n \dots a_0) = b_m \dots b_0$ where $b_m \dots b_0$ is the next sequence of $a_n \dots a_0$ in the increasing lexicographical order. We extend this map to the set of one sided infinite sequences on $\mathcal{A}_1 \times \mathcal{A}_2$, i.e. $\mathcal{P}$. We will denote this new map by the same symbol: $\mathcal{T}$. Hence $(\mathcal{P}, \mathcal{T})$ is the adic system associated to the product automaton $\mathcal{A}_1 \times \mathcal{A}_2$.

Remark 1. The map $\mathcal{T}$ is the induced map of $\mathcal{T}_i$ on $\mathcal{P}$, when we consider this space as a subset of $\mathcal{R}_i$, for $i = 1$ or 2.

Remark 2. By the definition of $\mathcal{P}$, this set admits the dynamic given by the shift map,

$$\begin{array}{rccc} \sigma : & \{0,1\}^{\mathbb{Z}^+} & \longrightarrow & \{0,1\}^{\mathbb{Z}^+} \\ & a_0 a_1 \dots & \mapsto & a_1 \dots \end{array}$$

From the automaton $\mathcal{A}_1 \times \mathcal{A}_2$ we obtain that the dynamical system $(\mathcal{P}, \sigma)$ is the sub-shift of the full shift where the symbols 11 and 101 are forbidden. Therefore it is also a sub-shift of the Fibonacci shift $(\mathcal{F}, \sigma)$. However the adic system $(\mathcal{P}, \mathcal{T})$ does not come from Fibonacci adic system, since this system is minimal.

2 The self-similar structure of $\mathcal{P}$.

As $\mathcal{R}_1$ is defined by all one sided infinite sequences in $\{0, 1\}$ with the forbidden factor 111. The system $(\mathcal{R}_1, \mathcal{T}_1)$ has a self-similar structure, which can be expressed by the following iterated function system, IFS, $\{F_1, F_2, F_3\}$:

$$F_1 = \tau, \quad F_2 = \mathcal{T}_1 \circ \tau^2, \quad F_3 = \mathcal{T}_1 \circ \tau \circ \mathcal{T}_1 \circ \tau^2.$$

where $\tau : \mathcal{R}_1 \to \mathcal{R}_1$, $\tau(a_0 a_1 \dots) = 0 a_0 a_1 \dots$. See [4, page 80] for the definition of IFS. Let Σ_3 be the set of all one-sided sequences in $\{1, 2, 3\}$ and $\underline{x} = x_0 x_1 \dots$ one of its elements. The sets $\{F_{x_0} F_{x_1} \cdots F_{x_n}(\mathcal{R}_1)\}_n$ form a decreasing sequence of sets which has a point as limit. Using this fact we have the following map:

$$\begin{array}{rccl} p : & \Sigma_3 & \longrightarrow & \mathcal{R}_1 \\ & \underline{x} = x_0 x_1 \dots & \mapsto & \lim_{n\to\infty} F_{x_0} \cdots F_{x_n}(\underline{b}), \end{array}$$

where $\underline{b}$ is a point in $\mathcal{R}_1$. Due to the property mentioned above the value of $p(\underline{x})$ is independent of the chosen $\underline{b}$. This map has the property

$$F_{x_0}(p(\sigma(\underline{x}))) = p(\underline{x})$$

where σ is the shift in Σ_3 and without difficulty it can be checked that it is a bijection between these two symbolic spaces.

From the automaton $\mathcal{A}_1 \times \mathcal{A}_2$ we get that the forbidden factors on $\mathcal{P}$, as a subset of $\mathcal{R}_1$ are 11 and 101, therefore the maps F_3 and F_2^2 are not allowed in its self-similar structure. Hence we can express the self-similar structure of $\mathcal{P}$ using the sub-shift of finite type associated to the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is (Σ_A, σ_A), where

$$\Sigma_A = \{\underline{x} \in \{1,2,3\}^{\mathbb{Z}^+} \mid A_{x_i x_{i+1}} = 1\}.$$

Let

$$\begin{array}{cccc} p_A & : & \Sigma_A & \longrightarrow & \mathcal{P} \\ & & \underline{x} = x_0 x_1 \ldots & \mapsto & \lim_{n\to\infty} F_{x_0} \cdots F_{x_n}(\underline{b}) \end{array}$$

where $\underline{b}$ is a point in $\mathcal{P}$. This map is bijective and has the property

$$F_{x_0}(p_A(\sigma_A(\underline{x}))) = p_A(\underline{x}).$$

Using this we can express the self-similar structure of $\mathcal{P}$ by the following IFS: $G = \{G_1, G_2\}$ where $G_1 = F_1$ and $G_2 = F_2 F_1$. We can redefine p_A in term of this new IFS:

$$\begin{array}{cccc} p_2 & : & \Sigma_2 = \{1,2\}^{\mathbb{Z}^+} & \longrightarrow & \mathcal{P} \\ & & \underline{x} = x_0 x_1 \ldots & \mapsto & \lim_{n\to\infty} G_{x_0} \cdots G_{x_n}(\underline{b}). \end{array}$$

3 Geometric realization of $(\mathcal{P}, \mathcal{T})$.

The incidence matrix of a substitution Π is the matrix $M = (m_{ij})$ such that the entry m_{ij} is the number of occurrences of the symbol i in the word $\Pi(j)$.

Let

$$\begin{array}{rcl} \delta_i & : & \mathcal{R}_i \to \mathbb{C} \\ \delta_i(\underline{a}) & = & \sum_{j\geq 0} \Delta(a_j)\beta^j \end{array}$$

where β is one of the complex eigenvalues of the incidence matrix of the substitution Π_i, with $i = 1$ or 2 ([17]) and

$$\Delta(a_j) = \begin{cases} 0 & \text{if } a_j = 0 \\ 1 & \text{if } a_j = 1 \\ \beta^{-1} & \text{if } a_j = 3 \end{cases}$$

Let $R_i = \delta_i(\mathcal{R}_i)$, for $i = 1, 2$. The sets R_1 and R_2 are known as the Rauzy fractals associated to the substitutions Π_1 and Π_2. Since the complex eigenvalues of M are conjugate, the sets R_1 and R_2 are independent of the choice of the complex eigenvalue.

Let P be the image of $\mathcal{P}$ under δ_1. If we take the image under δ_2 instead of δ_1 we get the same set P, due to $\mathcal{P}$ is obtained from the product automaton. Therefore we denote by δ the restriction of δ_1 or δ_2 to $\mathcal{P}$.

Proposition 1. *The set P is a proper subset of $R_1 \cap R_2$.*

Proof: Let $z \in P$, we can write $z = \sum_{i\geq 0} \Delta(a_i)\beta^i$ where $\underline{a} = a_0 a_1 \ldots$ is recognized by $\mathcal{A}_1 \times \mathcal{A}_2$, i.e. $\underline{a} \in \mathcal{P}$. By the definition of the product automaton, $\underline{a}$ is recognized by $\mathcal{A}_1$ and $\mathcal{A}_2$, therefore $\sum_{i\geq 0} \Delta(a_i)\beta^i \in R_1 \cap R_2$.

However $P \neq R_1 \cap R_2$, since the Lebesgue measure of P is zero (proposition 3) and we will see the Lebesgue measure of $R_1 \cap R_2$ is positive. The sets R_1 and R_2 are fixed points of an IFS with the open set condition and they have Hausdorff dimension equal to 2 ([24]), so they have locally positive measure. Since the origin is an interior point of R_1 and belongs to P, we get that for any U neighbourhood small enough of the origin, $m(U \cap R_1 \cap R_2) = m(U \cap R_2) > 0$, where m is the Lebesgue measure. Hence $m(R_1 \cap R_2) > 0$. ■

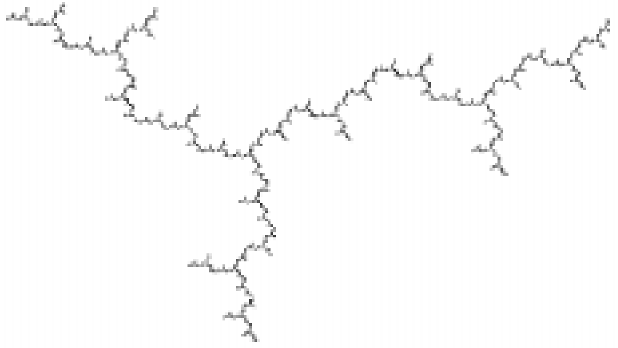


Figure 3: The set P

In order to understand the topology of $\mathcal{P}$ we need first to address the question: When are two different sequences of Σ_2 mapped to the same point in P by $\phi = \delta \circ p_2 : \Sigma_2 \to P$?

In Lemma 3.1 of [22] it was proved that the preimages under δ_1 of the points where the regions of R_1 corresponding to the symbols 1, 2 and 3 meet, are:

$$000\overline{100} = 000100100100\cdots, \quad 1000\overline{100}, \quad 11000\overline{100}.$$

Only the first two belong to $\mathcal{P}$. Since $\overline{100}$ is the fixed point of G_2, we have:

$$p_2(111\overline{2}) = 000\overline{100}, \quad p_2(21\overline{2}) = 1000\overline{100}.$$

So $\phi(111\overline{2}) = \phi(21\overline{2})$. Moreover this is the only point where the regions of $R_1 \cap P$ corresponding to the symbol 1 and to the symbol 2 meet. So it turns out from the self-similarity, that the only symbolic sequences that are mapped to the same point are of the form $w111\overline{2}$ and $w21\overline{2}$, for any finite word w. From this fact and using the techniques of [3], we get from proposition 10 of [3]:

Proposition 2. *The set P is a dendrite, i.e. a connected and locally connected set which does not contain a subset homeomorphic to a circle. More over the order of the branching points is* 3.

Proposition 3. *The Hausdorff dimension of P is $2\log\rho/\log\lambda$, where ρ is the real root of $x^3 - x^2 - 1 = 0$ and λ the real root of $x^3 - x^2 - x - 1 = 0$.*

Proof: The matrix that describes the transitions between the blocks: 00, 01 and 10 in $\mathcal{P}$ as subset of $\mathcal{R}_1$ is:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Note that the characteristic polynomial of this matrix is $x^3 - x^2 - 1 = 0$. By Corollary 1 of [21], we get that the Hausdorff dimension of $P = \delta(\mathcal{P})$ is $2\log\rho/\log\lambda$, where λ is the Perron-Frobenius eigenvalue of the matrix associated the substitution Π_1. ∎

Remark 3. The product automaton of $\mathcal{A}_1$ and $\mathcal{A}_2$ is the prefix automaton of the substitution $1 \to 12$, $2 \to 3$, $3 \to 1$. However the Rauzy fractal associated to this substitution is a set with different topological properties of the set P, e.g. has nonempty interior.

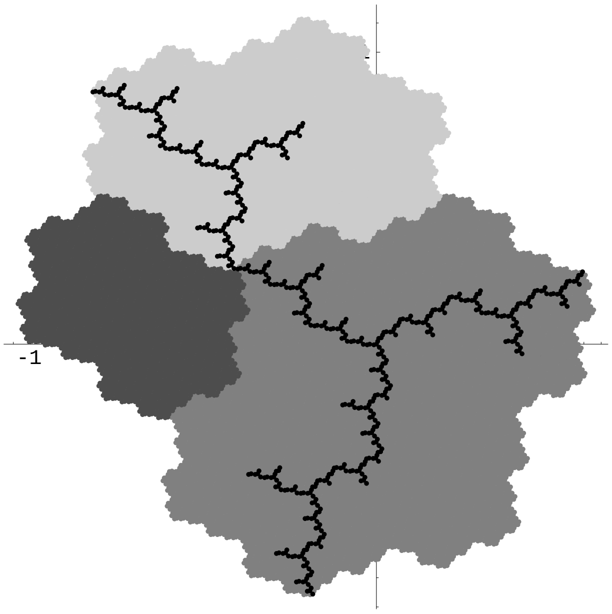


Figure 4: P as a subset of R_1

We say that a symbolic system $(\mathcal{R}, \mathcal{T})$ is *geometrically realized* as (X, T) if this is a dynamical system defined on a geometrical structure such that there exists a continuous and surjective map $\delta : \mathcal{R} \to X$ satisfying $\delta \circ \mathcal{T} = T \circ \delta$.

The adic systems $(\mathcal{R}_i, \mathcal{T}_i)$, with $i = 1$ or 2, can be realized geometrically in $\mathbb{R}^2$, as (R_i, T_i) where T_i is a piece exchange transformation in the plane [17, 14] and $(\mathcal{R}_1, \mathcal{T}_1)$ can be realized on the two-dimensional torus as an irrational translation [17]. In the construction of the (R_i, T_i)'s we identify $\mathbb{R}^2$ with $\mathbb{C}$. The following results tell us that the adic system of $\mathcal{P}$ can be realized geometrically in the plane and on the two-dimensional torus.

Let X be a connected subset of $\mathbb{R}^n$. We say that the dynamical system (X, T) is a *piece exchange transformation* if there exists $\{X_i\}_i$ a partition of X in connected sets, such that $\{T(X_i)\}_i$ is also a partition of X.

Theorem 1. *The system $(\mathcal{P}, \mathcal{T})$ can be realized in the plane as (P, T) where T is a piece exchange transformation, i.e. $\delta \circ \mathcal{T} = T \circ \delta$.*

Proof: Since P is a subset of R_1, the dynamics on this set is T_1. We define T as the induced map of T_1 on P, i.e.

$$T(z) = T_1^{n(z)}(z), \text{ where } n(z) = \min\{n \mid T_1^n(z) \in P\}.$$

Observe that this is compatible with the symbolic dynamics, since $\mathcal{T}$ is the induced map of $\mathcal{T}_1$ on $\mathcal{P}$. If we denote by $P(b_0 \dots b_j) = \delta(\{\underline{a} \in \mathcal{P} \mid a_0 = b_0, \dots, a_j = b_j\})$,

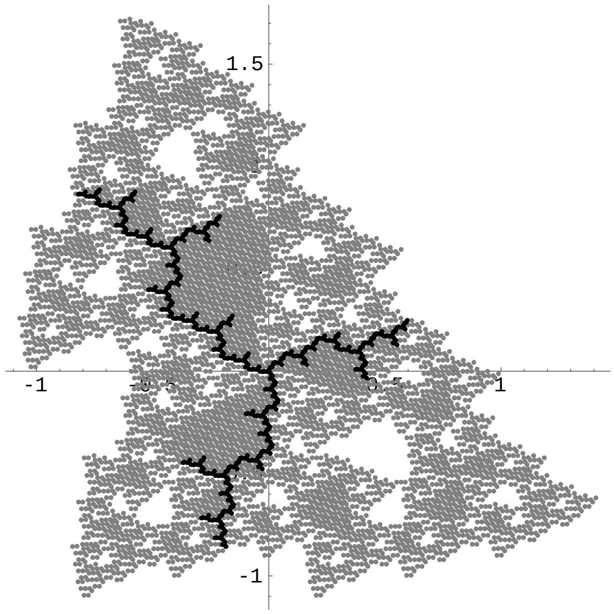


Figure 5: P as a subset of R_2

note that T is well defined on these cylinders. We have:

$$
\begin{array}{lcrcl}
P(000) & \stackrel{T}{\rightarrow} & P(100) & \text{here} & T = T_1 \\
P(001000) & \rightarrow & P(000100) & \text{''} & T = T_1^3 \\
P(001001000) & \rightarrow & P(0000000100) & \text{''} & T = T_1^{16} \\
 & \vdots & & & \\
P(01000) & \rightarrow & P(00100) & \text{''} & T = T_1^2 \\
P(01001000) & \rightarrow & P(00000100) & \text{''} & T = T_1^7 \\
 & \vdots & & & \\
P(1000) & \rightarrow & P(0100) & \text{''} & T = T_1 \\
P(1001000) & \rightarrow & P(0000100) & \text{''} & T = T_1^6 \\
 & \vdots & & &
\end{array}
$$

The map T_1 is continuous (it is an irrational translation) on the cylinders $P(0)$ and $P(10)$, so by self-similarity T is continuous on each of cylinders of the previous table. On the other hand P is connected hence each of these cylinders corresponds to a connected subset of P.

As it can be seen in the previous table the return time function, i.e. the $n(z)$ in the definition of the map T, is not bounded at the points: $\delta(\overline{001})$, $\delta(01\overline{001})$, $\delta(1\overline{001})$. The map T is well defined at these points and the image is 0, since $\mathcal{T}(\overline{001}) = \mathcal{T}(1\overline{001}) = \mathcal{T}(01\overline{001}) = \overline{0}$.

Therefore T is a piece exchange transformation in at most an infinite (countable) number of pieces.

Note, if we use in this proof (R_2, T_2) instead of (R_1, T_1) we get similar results. ∎

Corollary 1. *The system $(\mathcal{P}, \mathcal{T})$ can be realized on the two-dimensional torus as a piece exchange transformation.*

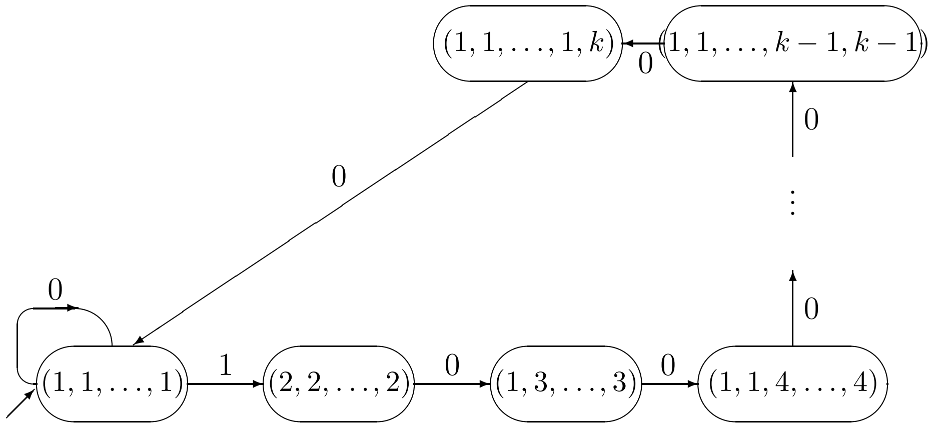


Figure 6: The product automaton of the k-bonacci substitutions.

Proof: In [16] is given a characterization of the points in R_1 that have more than one symbolic representation, from which it can be concluded that the pre-image of the boundary of R_1 under δ_1 does not contain the word 000 (the dynamical and geometrical properties of the set where the word 000 is forbidden are studied in [20, 21]). Therefore the only points of P that belong to the boundary of R_1 are $\delta(\overline{001})$, $\delta(01\overline{001})$ and $\delta(1\overline{001})$. In Lemma 3.1 of [22] we prove that these three points are mapped under $\hat{\delta} = {}^{\delta}/_{\mathbb{Z}^2}$ to the same point to $\mathbb{T}^2$. In the proof of Theorem 1 we show that the images under T of the previous points of the boundary are the same. Therefore the map T induces a map on the torus. ∎

4 The k-bonacci case

Consider the following family of substitutions, which generate different dynamical systems.

$$\Xi_{1,k}: \begin{cases} 1 & \rightarrow 12 \\ 2 & \rightarrow 13 \\ 3 & \rightarrow 14 \\ & \vdots \\ k-1 & \rightarrow 1k \\ k & \rightarrow 1 \end{cases} \quad \Xi_{2,k}: \begin{cases} 1 & \rightarrow 12 \\ 2 & \rightarrow 31 \\ 3 & \rightarrow 14 \\ & \vdots \\ k-1 & \rightarrow 1k \\ k & \rightarrow 1 \end{cases} \quad \cdots \quad \Xi_{k-1,k}: \begin{cases} 1 & \rightarrow 12 \\ 2 & \rightarrow 31 \\ 3 & \rightarrow 41 \\ & \vdots \\ k-1 & \rightarrow k1 \\ k & \rightarrow 1 \end{cases}$$

The common dynamic to the dynamical systems associated to these substitutions is given by the adic systems associated to the product automaton of the automata of $\Xi_{1,k}$, $\Xi_{2,k}$, $\ldots$, $\Xi_{k-1,k}$. Which we will denote by $\mathcal{A}(k)$. This automaton is in figure 6. A k-bonacci substitution is a substitution on a alphabet of k letters, such that its incidence matrix is the same as the matrix of the substitution $\Xi_{1,k}$.

Proposition 4. *$\mathcal{A}(k)$ is the product automaton of all k-bonacci substitutions.*

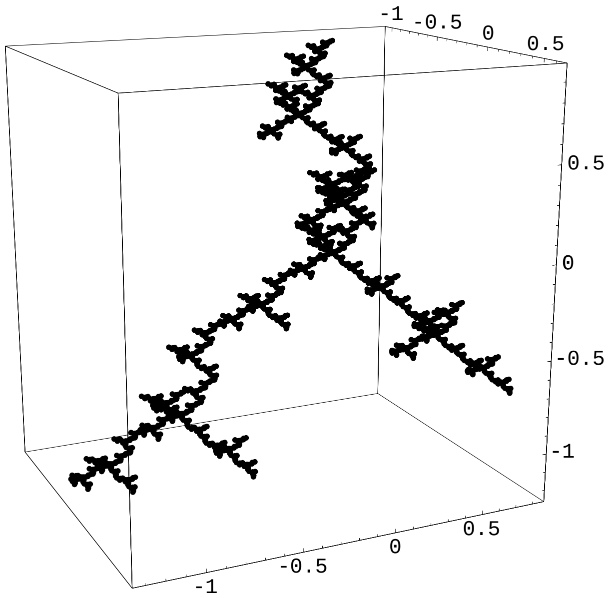


Figure 7: The set $P[4]$

Proof: Given Ξ any k-bonacci substitution such that $\Xi(1) = 12$, and let $\mathcal{B}$ be its prefix automaton. We shall show that $\mathcal{A}(k) \times \mathcal{B} = \mathcal{A}(k)$. Let us denote the states of $\mathcal{A}(k)$ as follows: $(1,1,\ldots,1) = 1, (2,\ldots,2) = 2, (1,3,\ldots,3) = 3, \ldots, (1,1,\ldots,1,k-1,k-1) = k-1, (1,\ldots,1,k) = k$. By the hypothesis on Ξ we get that the transitions on $\mathcal{B}$ are 0 between the state 1 and itself and 1 between the states 1 and 2. Therefore in $\mathcal{A}(k) \times \mathcal{B}$ there will be the transitions labelled by 0 between $(1,1)$ and itself and labelled by 1 between $(1,1)$ and $(2,2)$. Since Ξ is a k-bonacci substitution, from each state different from k there are two transitions one of them labelled by 0, and from the k-th state there is only one transition which is to the state 1, and it is labelled by 0. Therefore there is a path in $\mathcal{B}$ from the state 2 to 1 labelled by 0's. Hence $\mathcal{A}(k) \times \mathcal{B} = \mathcal{A}(k)$.

If Ξ is k-bonacci substitution such that $\Xi(1) = 21$, we define $\overline{\Xi}$ as the substitution obtained by flipping the images of the elements of the alphabet under Ξ. The dynamical systems generated by Ξ and $\overline{\Xi}$ are equivalent, and $\overline{\Xi}$ is of the type studied previously. ∎

Let $(\mathcal{P}[k], \mathcal{T}[k])$ be this adic system, and $P[k]$ the geometrical realization of $\mathcal{P}[k]$ on $\mathbb{R}^{k-1}$ according to the Rauzy map:

$$\begin{aligned} \delta \quad &: \quad \mathcal{P}[k] \to \mathbb{R}^{k-1} \\ \delta(\underline{a}) \quad &= \quad \sum_{j\geq 0} a_j B^j z \end{aligned}$$

where B is the restriction of the incidence matrix of any k-bonacci substitution (all these substitutions have the same incidence matrix) to its contractive eigenspace and $z = (1,0,\ldots,0)$. The adic system associated to the substitution $\Xi_{1,k}$ is realized in $(\mathbb{T}^{k-1}, T)$ where T is a translation by the normalized eigenvector of the matrix associated to this substitution [17, 25].

In a similar way as in the tribonacci case, we can conclude that the set $\mathcal{P}[k]$ is the fixed point of the IFS $\{\tau, \mathcal{T}[k]\tau^k\}$. Therefore $P[k]$ is the fixed point of the IFS $\{H_1, H_2\}$ where $H_1(x) = T(x)$, $H_2(x) = TB^k(x)$ and T is the previous translation, but acting on $\mathbb{R}^{k-1}$.

Hence we get that the Hausdorff dimension of this set is in the interval $[s_0, s_1]$ where s_0 is the solution of $|\gamma_0|^s + |\gamma_0|^{ks} = 1$, s_1 is the solution of $|\gamma_1|^s + |\gamma_1|^{ks} = 1$, γ_0 is the smallest eigenvalue in norm of the matrix B and γ_1 is the largest.

In a similar way we can conclude that $P[k]$ is a proper subset of the intersection of the Rauzy fractals associated to each $\Xi_{i,k}$. Theorem 1 can be generalized in the same way.

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