On the continuous dependence on parameters of solutions of the second order periodic problem

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1 Introduction

This paper is devoted to the continuous dependence on functional parameters of solutions of the second order periodic problem. Sufficient conditions for the existence of solutions of this problem and their continuous dependence on parameters are presented.

The question of the existence and uniqueness of solutions for the periodic problem was widely discussed by Mawhin and Willem in many monographs and papers ([3], [4], [5], [6]). Some interesting results about the existence of periodic solutions of ordinary differential equations we can be found in papers ([2],[9]). The problem of the continuous dependence on parameters for scalar equations was investigated in papers ([7],[8]). In the case of the functional parameter from L^∞, sufficient conditions for the existence of solutions of the second order differential equations with Dirichlet-type boundary conditions and their continuous dependence on parameters, are given in paper ([10]).

In this paper we consider a periodic problem of the second order with functional

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parameter of the form

$$\begin{aligned} \frac{d}{dt} f_{\dot{u}}(t,u,\dot{u},\omega) &= f_u(t,u,\dot{u},\omega) \\ u(0)-u(T) &= 0 \\ v(0)-v(T) &= 0, \end{aligned} \tag{1.1}$$

where $v = f_{\dot{u}}(t,u,\dot{u},\omega)$ for $t \in [0,T]$ *a.e.*, the parameter ω belongs to L^∞, and $u \in H_T^1$. Under some suitable assumptions, we prove that the set $\tilde{V}_k$ of solutions of (1.1) is not empty, for any $\omega_k \in W$ and $\tilde{V}_{\omega_k}$ tends to $\tilde{V}_{\omega_0}$ in the sense of Painlevé-Kuratowski as ω_k tends to ω_0 in the strong topology of L^∞. In many situations, it is more natural to consider the normal form of (1.1)

$$\begin{aligned} \ddot{u} &= \nabla F(t,u,\omega) \\ u(0)-u(T) &= \dot{u}(0)-\dot{u}(T) = 0 \end{aligned} \tag{1.2}$$

We give sufficient conditions under which (1.2) continuously depends on the parameter ω. We occupy ourselves with cases when ∇F is bounded and F is convex.

2 Formulation of the second order problem

By H_T^1 we shall denote the space of absolutely continuous functions $u : [0,T] \to \mathbb{R}^n$ such that $\dot{u} \in L^2([0,T],\mathbb{R}^n)$ and $u(0) = u(T)$, where $T > 0$. In the space H_T^1 the norm is given by the formula $||u|| = \left(\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}}$, but it is easy to calculate that :

Lemma 1. *In the space H_T^1 the following norms are equivalent:*

1. $||u||_1 = \left(\int\limits_0^T |u(t)|^2 dt + \int\limits_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}}$,
2. $||u||_2 = \left(\int\limits_0^T |u(t)|^2 dt\right)^{\frac{1}{2}} + \left(\int\limits_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}}$,
3. $||u||_3 = |u_0| + \left(\int\limits_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}}$, $u_0 = u(0)$,
4. $||u||_4 = |\overline{u}| + \left(\int\limits_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}}$, $\overline{u} = \frac{1}{T}\int\limits_0^T u(s)ds$.

Lemma 2. *If the sequence u_k converges weakly to u_0 in H_T^1 , then $\dot{u}_k$ converges weakly to $\dot{u}_0$ in $L^2([0,T],\mathbb{R}^n)$ and u_k converges uniformly to u_0 on $[0,T]$.*

Let M be a convex and bounded subset of $\mathbb{R}^r$. By W we shall denote the set $W = \{\omega \in L^\infty([0,T],\mathbb{R}^r) : \omega(t) \in M\}$. The set W will be referred to as a set of parameters.

Let $f = f(t,x,p,w)$ be any scalar function defined on the set $[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times M$ which satisfies the following assumptions:

1-a the functions f, f_x, f_p, f_w are measurable with respect to $t \in [0,T]$ for any $(x,p,w) \in \mathbb{R}^n \times \mathbb{R}^n \times M$ and continuous with respect to $(x,p,w) \in \mathbb{R}^n \times \mathbb{R}^n \times M$ for $t \in [0,T] a.e.$,

1-b $f(t,x,\cdot,w)$ is convex for $t \in [0,T]\, a.e.$ and any $(x,w) \in \mathbb{R}^n \times M$,

1-c there exist some functions $a(\cdot) \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b(\cdot) \in L^1([0,T],\mathbb{R})$ and $c(\cdot) \in L^2([0,T],\mathbb{R})$, such that
$|f(t,x,p,w)| \le a(|x|)\left(b(t)+|p|^2\right)$,
$|f_x(t,x,p,w)| \le a(|x|)\left(b(t)+|p|^2\right)$,
$|f_p(t,x,p,w)| \le a(|x|)\left(c(t)+|p|\right)$,
$|f_w(t,x,p,w)| \le a(|x|)\left(b(t)+|p|^2\right)$
for all $(x,p,w) \in \mathbb{R}^n \times \mathbb{R}^n \times M$, $t \in [0,T]\, a.e.$.

Now, let us consider a boundary value problem, with a parameter $\omega \in W$, of the form

$$\begin{aligned} \frac{d}{dt} f_{\dot u}\left(t, u(t), \dot u(t), \omega(t)\right) &= f_u\left(t, u(t), \dot u(t), \omega(t)\right) \\ u(0) - u(T) &= 0 \\ v(0) - v(T) &= 0, \qquad t \in [0,T]\, a.e., \end{aligned} \tag{2.1}$$

where $u \in H_T^1$ and $v(t) = f_{\dot u}\left(t, u(t), \dot u(t), \omega(t)\right)$ for $t \in [0,T]\, a.e.$.

For this problem, the corresponding functional φ is given by

$$\varphi_\omega(u) = \int_0^T f\left(t, u(t), \dot u(t), \omega(t)\right) dt \tag{2.2}$$

It is easy to see that, under the above assumptions, (2.1) is a system of Euler equations for functional (2.2).

Definition 1. *We say that the functional $\varphi_\omega(\cdot)$ defined by (2.2) is uniformly coercive with respect to ω when there exists $x_0 \in X$ and number $K_1 > 0$, such that $\varphi_\omega(x_0) < K_1$ for $\omega \in W$ and*

$$\forall\, K > 0 \quad \exists R \quad \forall\, |x| > R \quad \forall\, \omega \in W \quad \varphi_\omega(x) > K$$

3 The principal lemma

Let $\varphi_k(\cdot) = \varphi_{\omega_k}(\cdot)$, $k = 0,1,2,...$, be a sequence of functionals defined by (2.2) with $\omega = \omega_k$, i.e.

$$\varphi_k(u) = \int_0^T f\left(t, u(t), \dot u(t), \omega_k(t)\right) dt$$

where $\{\omega_k\}$ is a sequence of admissible parameters. Denote by Z_k the set of all minimizers of the functional φ_k, i.e.

$$Z_k = \left\{ x \in H_T^1 : \varphi_k(x) = \min \varphi_k(h) \quad h \in H_T^1 \right\} \tag{3.1}$$

Definition 2. *We say that the sequence of sets Z_k defined by (3.1) tends to Z_0 in the weak topology of H_T^1 if any sequence $\{x_k\}$, $x_k \in Z_k$, $k = 1, 2, \ldots$ possesses cluster points (in the sense of the weak topology of H_T^1) in the set Z_0 only.*

The set of all cluster points of the sequence $\{x_k\}$ is often referred to as the upper limit (in the sense of Painlevé - Kuratowski) of the sets Z_k and denoted by $\limsup Z_k$.

In the case when the sets Z_k are singletons i.e. $Z_k = \{x_k\}$, $k = 0, 1, 2, \ldots,$ the convergence of the sets is identical with the convergence of points in the weak topology of H_T^1 (see lemma (2)).

Now, we prove

Lemma 3. *If the sequence $\{\omega_k\} \subset W$, $k = 1, 2, \ldots,$ tends to $\omega_0 \in W$ in the strong topology of L^∞, then the sequence $\varphi_k(\cdot)$ tends to $\varphi_0(\cdot)$ uniformly on the ball $B(0, R) \subset H_T^1$ for any fixed $R > 0$.*

Proof. Let $\varepsilon_1 > 0$ be an arbitrary number. Since $\omega_k \to \omega_0$ in L^∞, we have that there exists some K such that, for $k > K$ and $t \in [0, T]$, we have

$$|\omega_k(t) - \omega_0(t)| < \varepsilon_1$$

From the mean value theorem and assumption (1-c) we obtain

$$\begin{aligned} |\varphi_k(u) - \varphi_0(u)| &\leq \int_0^T |f_w(t, u(t), \dot{u}(t), \tilde{\omega}_k(t))| \, |\omega_k(t) - \omega_0(t)| \, dt \\ &\leq \int_0^T a(|u(t)|) \left(b(t) + |\dot{u}(t)|^2\right) |\omega_k(t) - \omega_0(t)| \, dt \end{aligned}$$

where $\tilde{\omega}_k(t) = \omega_0(t) + \Theta(t)(\omega_k(t) - \omega_0(t))$ and $0 \leq \Theta(t) \leq 1$. Since $||u|| \leq R$, there exists a constant $C > 0$ such that $a(|u(t)|) \leq C$ and $\int_0^T |\dot{u}(t)|^2 dt \leq ||u||^2 \leq R^2$ for any $u \in B(0, R)$.

Let us take $\varepsilon > 0$ sufficiently small. For $k > K$ and $u \in B(0, R)$ we have

$$|\varphi_k(u) - \varphi_0(u)| \leq C\varepsilon_1 \int_0^T b(t) \, dt + C\varepsilon_1 \int_0^T |\dot{u}(t)|^2 \, dt \leq C_1\varepsilon_1 + C\varepsilon_1 R^2 < \varepsilon$$

for some constant C_1. This ends the proof. ∎

We shall prove the main lemma

Lemma 4. *If*

1. *the function f satisfies assumptions (1-a)-(1-c) ,*
2. *$\varphi_k(\cdot)$ are weakly lower semicontinuous and uniformly coercive with respect to ω_k for $k = 0, 1, 2, \ldots,$*

then

a) *for any admissable parameter ω_k, the set Z_k of minimizers of the functional $\varphi_k(\cdot)$ is not empty ,*

b) *there exists a ball $B(0,R) \subset H_T^1$ such that $Z_k \subset B(0,R)$ for $k = 0,1,2,\ldots$.*

Proof. Since, by assumption 2, there exists at least one minimizer u_k of $\varphi_k(\cdot)$, therefore Z_k , $k = 0,1,2,\ldots$ is a non-empty set. Hence $\varphi_k(u_k) \le \varphi_k(0)$ for $\omega_k \in W$. Let us put $P = \sup_{\omega_k \in W} \varphi_k(0) < \infty$. So there exists an $R > 0$, such that for all $\omega_k \in W$

$$u_k \in Z_k \subset A_k = \left\{u \in H_T^1 : \varphi_k(u) \le P\right\} \subset B(0,R) \tag{3.2}$$

Indeed, suppose that the second inclusion in (3.2) does not hold. Then, for all $R > 0$ say $R = n$, $n = 1,2,\ldots$, there exists a parameter $\omega_n \in W$ such that $A_n \not\subseteq B(0,R)$ .. Thus there exists a sequence $\{u_n\}$ of elements from A_n such that $||u_n|| > R = n$, $n = 1,2,\ldots$. Since $\varphi_n(\cdot)$ is uniformly coercive, therefore for $||u_n|| \to \infty$, $n \to \infty$, we have $\varphi_n(u_n) \to \infty$. Hence $u_n \notin A_n$ for $n > P$ and we have got a contradiction. It means that $A_n \subset B(0,R)$ for some $R > 0$ and $k = 0,1,2,\ldots$. ∎

4 Theorem on the continuous dependence on parameters for the second order equation

Let $\{\omega_k\} \subset W$ be an arbitrary sequence and let us denote by $\tilde{V}_k \subset H_T^1$ a set of solutions of the periodic problem

$$\begin{gathered} \frac{d}{dt} f_{\dot{u}}\left(t, u(t), \dot{u}(t), \omega_k(t)\right) = f_u\left(t, u(t), \dot{u}(t), \omega_k(t)\right) \\ u(0) - u(T) = 0 \\ v(0) - v(T) = 0 \\ t \in [0,T] a.e., \quad k = 0,1,2,\ldots \end{gathered} \tag{4.1}$$

and by

$$V_k = \left\{u \in H_T^1 : \varphi_{\omega_k}(u) = \min \varphi_{\omega_k}(x) \quad x \in H_T^1\right\}, \quad k = 0,1,2,\ldots$$

the set of minimizers of the functional $\varphi_{\omega_k}(\cdot)$ for $k = 0,1,2,\ldots$. Now, we prove

Theorem 1. *If*

1. *f satisfies assumptions (1-a)-(1-c) ,*
2. *for any admissible parameter ω_k, the set V_k of minimizers of the functional $\varphi_{\omega_k}(\cdot)$ is not empty ,*
3. *there exists a ball $B(0,R) \subset H_T^1$ such that $V_k \subset B(0,R)$ for $k = 0,1,2,\ldots$,*

4. *$\varphi_{\omega_k}(\cdot)$ is convex for any $\omega_k \in W$, $k = 0, 1, 2, \ldots$,*

5. *the sequence $\{\omega_k\} \subset W$ tends to $\omega_0 \in W$ in the strong topology of L^∞,*

then $\limsup \tilde{V}_k$ *is a non-empty set and* $\limsup \tilde{V}_k \subset \tilde{V}_0$.

Proof. Let $\{u_k\} \subset H_T^1$ be a sequence such that $u_k \in V_k$ for $k = 1, 2, \ldots$. Since $V_k \subset B(0, R)$, $k = 1, 2, \ldots$, with some $R > 0$, we may assume that u_k tends to u_0 in H_T^1. Denote

$$m_k = \varphi_k(u_k) = \inf\left\{\varphi_k(x) : x \in H_T^1\right\} = \inf\left\{\varphi_k(x) : x \in B(0, R)\right\}$$

Since, by assumption, 5 , $\varphi_k(\cdot)$ tends to $\varphi_0(\cdot)$ uniformly on ball $B(0, R)$ therefore

$$m_k \to m_0. \tag{4.2}$$

Suppose that u_0 does not belong to V_0. The set is not empty, thus there exists $x \in V_0$ such that $u_0 \neq x$. We have

$$m_k - m_0 = \varphi_k(u_k) - \varphi_0(x) = [\varphi_k(u_k) - \varphi_0(u_k)] + [\varphi_0(u_k) - \varphi_0(x)] \tag{4.3}$$

It is easy to notice that $\varphi_0(u_k) - \varphi_0(x) > 0$. So, letting $k \to \infty$ in (4.3), we get a contradiction with (4.2). Hence $\limsup V_k \subset V_0$.

Moreover, the functionals $\varphi_{\omega_k}(\cdot)$ are convex and Gâteaux-differentiable, therefore $\tilde{V}_k = V_k$, $\quad k = 0, 1, 2, \ldots$. This ends the proof. ■

Corollary 1. *If the functional $\varphi_{\omega_k}(\cdot)$ is strictly convex, then problem (4.1) possesses a unique solution, i.e. the set $\tilde{V}_k = \{u_k\}$ is a singleton for $k = 0, 1, 2, \ldots$ and u_k tends to u_0 in the weak topology of H_T^1.*

Theorem 2. *If*

1. *f satisfies assumptions (1-a)-(1-c) ,*

2. *there exist some constants $a_1, b_1 > 0$, $a_0, b_0 \geq 0$ and a function $c_0 \in L^1([0, T], \mathbb{R})$, such that*

$$f(t, x, p, w) \geq a_1 |p|^2 - a_0 |p| + b_1 |x|^2 - b_0 |x| - c_0(t)$$

for all $(x, p, w) \in \mathbb{R}^n \times \mathbb{R}^n \times M$,

3. *$\varphi_{\omega_k}(\cdot)$ is convex for any $\omega_k \in W$,*

4. *the sequence $\{\omega_k\} \subset W$ tends to $\omega_0 \in W$ in the strong topology of L^∞,*

then $\limsup \tilde{V}_k$ *is a non-empty set and* $\limsup \tilde{V}_k \subset \tilde{V}_0$.

Proof. To prove this theorem, we must show that $\varphi_{\omega_k}(\cdot)$ is uniformly coercive with respect to $\omega_k \in W$, $k = 0, 1, 2, \ldots$, and weakly lower semicontinuous. By assumption (2)

$$f(t, u(t), \dot{u}(t), \omega_k(t)) \geq a_1 |\dot{u}(t)|^2 - a_0 |\dot{u}(t)| + b_1 |u(t)|^2 - b_0 |u(t)| - c_0(t) \tag{4.4}$$

for any $\omega_k \in W$. So,

$$\varphi_{\omega_k}(u) \geq a_{\min}||u||^2 - (a_0 - b_0)\sqrt{T}||u|| - \overline{c},$$

where $||u|| = \left(\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}}$, $a_{\min} = \min\{a_1, b_1\}$, $\overline{c} = \int_0^T c_0(t)dt$ ..
Since $a_1, b_1 > 0$, the functional $\varphi_{\omega_k}(\cdot)$ is uniformly coercive with respect to ω_k.

Our next step is to prove that the functional $\varphi_{\omega_k}(\cdot)$ is weakly lower semicontinuous. By assumption (2),

$$f(t, x, p, w) \geq -\psi(t)$$

for some positive and integrable function ψ. So the fact that $\varphi_{\omega_k}(\cdot)$ is weakly lower semicontinuous is obtained from theorem 10.8.i (see [1]).

By lemma (4), the set V_k of minimizers of the functional $\varphi_{\omega_k}(\cdot)$ for $k = 0, 1, 2, \ldots$ is not empty and there exists a ball $B(0, R) \subset H^1_T$ such that $V_k \subset B(0, R)$ for $k = 0, 1, 2, \ldots$. Now, we may apply theorem (1) to obtain the assertion of this theorem. ■

5 The normal form of the second order equation

Now, let $W = \{w \in L^\infty([0, T], \mathbb{R}^r) : w(t) \in M\}$ where M is any convex and bounded subset of $\mathbb{R}^r$ and let $F : [0, T] \times \mathbb{R}^n \times M \to \mathbb{R}$ be a function which satisfies the following assumptions :

4-a $F(t, x, w)$ is measurable with respect to $t \in [0, T]$ for any $(x, w) \in \mathbb{R}^n \times M$ and continuously differentiable in x for $t \in [0, T]$,

4-b there exist functions $a(\cdot) \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b(\cdot) \in L^1([0, T], \mathbb{R})$ such that
$|F(t, x, w)| \leq a(|x|)b(t)$,
$|\nabla F(t, x, w)| \leq a(|x|)b(t)$,
for all $(x, w) \in \mathbb{R}^n \times M$, $t \in [0, T] a.e.$

Let us consider the functional

$$\Phi_\omega(u) = \int_0^T \left(\frac{1}{2}|\dot{u}(t)|^2 + F(t, u(t), \omega(t))\right) dt \tag{5.1}$$

By corollary 1.1 (see [5]), the corresponding Euler equations are of the form

$$\begin{array}{ll} \ddot{u}(t) = \nabla F(t, u(t), \omega(t)), & t \in [0, T]\ a.e. \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0 \end{array} \tag{5.2}$$

where $u \in H^1_T$ and $\omega \in W$.

Lemma 5. *The functional $\Phi_\omega(\cdot)$ given by (5.1) is weakly lower semicontinuous in H_T^1.*

Proof. Since the functional

$$H_T^1 \ni u \longmapsto \int_0^T \frac{1}{2}|\dot{u}(t)|^2 dt$$

is convex and continuous, then it is weakly lower semicontinuous, and the functional

$$H_T^1 \ni u \longmapsto \int_0^T F\left(t, u(t), \omega(t)\right) dt$$

is weakly continuous (see lemma (2)). Thus the functional $\Phi_\omega(\cdot)$, as the sum of weak lower semicontinuous functionals, is weakly lower semicontinuous in H_T^1. ∎

For our later considerations, in the space H_T^1 we shall use the norm given by

$$||u|| = |\overline{u}| + \left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}}, \overline{u} = \frac{1}{T}\int_0^T u(s)ds$$

Let us denote by $\tilde{V}_k \subset H_T^1$ the set of solutions of the periodic problem of the form

$$\begin{array}{ll} \ddot{u}(t) = \nabla F\left(t, u(t), \omega_k(t)\right), & t \in [0,T] a.e. \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 & \end{array} \tag{5.3}$$

where $\omega_k \in W$, $k = 0, 1, 2, \dots$, and by

$$V_k = \left\{ u \in H_T^1 : \Phi_{\omega_k}(u) = \min \Phi_{\omega_k}(v) \quad v \in H_T^1 \right\}, \quad k = 0, 1, 2, \dots$$

the set of minimizers of the functional $\Phi_{\omega_k}(\cdot)$ for $k = 0, 1, 2, \dots$.

Now, we make some assumptions about F under which, we can prove some sufficient conditions for the continuous dependence on the parameters $\omega_k \in W$ for problem (5.3). The first – when ∇F is bounded by an integrable function (theorem (3)) and the second – in case when F is convex (theorem (4)).

Theorem 3. *If*

1. *F satisfies assumptions (4-a)-(4-b) ,*
2. *there exists $g \in L^1\left([0,T], \mathbb{R}^+\right)$ such that $|\nabla F(t, x, \omega_k)| \le g(t) \quad \forall x \in \mathbb{R}^n$ and $\omega_k \in M$, $k = 0, 1, 2, \dots$,*
3. *$\int_0^T F\left(t, x, \omega_k(t)\right) dt \to \infty$ uniformly with respect to $\omega_k \in W$ as $|x| \to \infty$, for $k = 0, 1, 2, \dots$,*
4. *$\Phi_{\omega_k}(\cdot)$ is convex for any $\omega_k \in W$,*

5. *the sequence* $\{\omega_k\} \subset W$ *tends to* $\omega_0 \in W$ *in the strong topology of* L^∞,

then $\limsup \tilde{V}_k$ *is a non-empty set and* $\limsup \tilde{V}_k \subset \tilde{V}_0$..

Proof. We have to prove that $\Phi_{\omega_k}(\cdot)$ is uniformly coercive with respect to ω_k . Let $\omega_k \in W$. For $u \in H_T^1$, we have $u = \overline{u} + \tilde{u}$ where $\overline{u} = \int_0^T u(s)ds$. From Sobolev's inequality we obtain that

$$\Phi_{\omega_k}(u) =$$

$$
\begin{aligned}
&= \int_0^T \left(\frac{1}{2}|\dot{u}(t)|^2 + F(t, \overline{u}, \omega_k(t))\right) dt + \int_0^T \left(F(t, u(t), \omega_k(t)) - F(t, \overline{u}, \omega_k(t))\right) dt \\
&= \int_0^T \left(\frac{1}{2}|\dot{u}(t)|^2 + F(t, \overline{u}, \omega_k(t))\right) dt + \int_0^T \int_0^1 \left(\nabla F(t, \overline{u} + s\tilde{u}(t), \omega_k(t)), \tilde{u}(t)\right) ds dt \\
&\geq \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T g(t)dt ||\tilde{u}||_\infty + \int_0^T F(t, \overline{u}, \omega_k(t)) dt \\
&\geq \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - C\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}} + \int_0^T F(t, \overline{u}, \omega_k(t)) dt \\
&= \frac{1}{2}||\dot{u}||_{L^2}^2 - C||\dot{u}||_{L^2} + \int_0^T F(t, \overline{u}, \omega_k(t)) dt
\end{aligned}
$$

where C is some constant. So if $||u|| \to \infty$, then $\Phi_{\omega_k}(u) \to \infty$ uniformly with respect to ω_k .. By lemma (5) and lemma (4), the set V_k of minimizers of the functional $\Phi_{\omega_k}(\cdot)$ for $k = 0, 1, 2, \ldots$ is not empty and there exists a ball $B(0, R) \subset H_T^1$ such that $V_k \subset B(0, R)$ for $k = 0, 1, 2, \ldots$. Now, we can apply theorem (1) and get the proposition of this theorem. ■

Theorem 4. *If*

1. F *satisfies assumptions (4-a)-(4-b)* ,
2. $F(t, \cdot, \omega_k)$ *is convex for* $t \in [0, T]$ *a.e. and for all* $\omega_k \in M$, $k = 0, 1, 2, \ldots$,
3. $F(t, x, \omega_k) \geq \alpha|x| - \beta$ *for all* $x \in \mathbb{R}^n$ *and* $\omega_k \in M$, $k = 0, 1, 2, \ldots$, *where* $\alpha > 0$ *and* $\beta \geq 0$ *are some constants,*
4. *the sequence* $\{\omega_k\} \subset W$ *tends to* $\omega_0 \in W$ *in the strong topology of* L^∞,

then $\limsup \tilde{V}_k$ *is a non-empty set and* $\limsup \tilde{V}_k \subset \tilde{V}_0$..

Proof. Let $\omega_k \in W$, $k = 0, 1, 2, \ldots$. Directly from the assumptions we can conclude that the real function

$$g_k : \mathbb{R}^n \ni x \to \int_0^T F(t, x, \omega_k(t)) dt, \qquad k = 0, 1, 2, \ldots$$

has a minimum at some point $\overline{x}_{\omega_k}$ for which

$$\int_0^T \nabla F\left(t, \overline{x}_{\omega_k}, \omega_k(t)\right) dt = 0, \qquad k = 0, 1, 2, \ldots$$

and

$$\int_0^T F\left(t, x, \omega_k(t)\right) dt \to \infty \quad \text{as } |x| \to \infty \qquad k = 0, 1, 2, \ldots$$

Moreover,

$$g_k\left(x\right) = \int_0^T F\left(t, x, \omega_k(t)\right) dt \geq \int_0^T \left(\alpha|x| - \beta\right) dt = \alpha_0|x| - \beta_0$$

where $\alpha_0 > 0$.
Let us denote $A = a\left(0\right) \int_0^T b\left(t\right) dt$ and $B\left(0, \rho\right) = \{x \in \mathbb{R}^n : \alpha_0|x| - \beta_0 \leq A\}$. By assumption (4-b)

$$g_k(\overline{x}_{\omega_k}) \leq g_k(0) \leq A\,, \qquad k = 0, 1, 2, \ldots$$

Note that all minimizers $\overline{x}_{\omega_k} \in B(0, \rho)$ for $k = 0, 1, 2, \ldots$. Indeed,

$$\overline{x}_{\omega_k} \in \{x \in \mathbb{R}^n : g_k\left(x\right) \leq A\} \subset \{x \in \mathbb{R}^n : \alpha_0|x| - \beta_0 \leq A\} = B\left(0, \rho\right)$$

Moreover, by Theorem 1.7. [5], the problem (5.3) has at least one solution which minimizes Φ_{ω_k} on H_T^1 for $k = 0, 1, 2, \ldots$. Thus the set of minimizers V_k of $\Phi_{\omega_k}(\cdot)$ is not empty for $k = 0, 1, 2, \ldots$

Now, let $u_k = \overline{u}_k + \tilde{u}_k$ be a minimizer of $\Phi_{\omega_k}(\cdot)$. We have to show that there exists a ball $B(0, R)$ such that $V_k \subset B(0, R)$, $k = 0, 1, 2, \ldots$.

By the assumption (4-b)

$$\Phi_{\omega_k}(0) \leq A, \qquad k = 0, 1, 2, \ldots$$

Since u_k is a minimizer of $\Phi_{\omega_k}(\cdot)$, we obtain

$$
\begin{aligned}
A &\geq \Phi_{\omega_k}(u_k) \\
&\geq \frac{1}{2}\int_0^T |\dot{u}_k(t)|^2 dt + \int_0^T F\left(t, \overline{x}_{\omega_k}, \omega_k(t)\right) dt \\
&\qquad\qquad + \int_0^T \left(\nabla F\left(t, \overline{x}_{\omega_k}, \omega_k(t)\right), u_k(t) - \overline{x}_{\omega_k}\right) dt \\
&\geq \frac{1}{2}\int_0^T |\dot{u}_k(t)|^2 dt + \int_0^T F\left(t, \overline{x}_{\omega_k}, \omega_k(t)\right) dt \\
&\qquad\qquad - \int_0^T \left|\nabla F\left(t, \overline{x}_{\omega_k}, \omega_k(t)\right)\right| dt ||\tilde{u}_k||_\infty \\
&\geq \frac{1}{2}\int_0^T |\dot{u}_k(t)|^2 dt + \int_0^T F\left(t, \overline{x}_{\omega_k}, \omega_k(t)\right) dt - ||\tilde{u}_k||_\infty \left|\int_0^T \nabla F\left(t, \overline{x}_{\omega_k}, \omega_k(t)\right) dt\right| \\
&\geq \frac{1}{2}||\dot{u}_k||_{L^2}^2 - C_1
\end{aligned}
\tag{5.4}
$$

where $C_1 > 0$ and $k = 0, 1, 2, \ldots$.
We have proved (by(5.4)) that

$$||\dot{u}_k||_{L^2} \leq D_1, \qquad k = 0, 1, 2, \ldots$$

for some constant $D_1 > 0$.
Sobolev's inequality implies that there exists a constant $C_2 > 0$ such that $||\tilde{u}_k||_\infty \leq C_3$. From the convexity of F we obtain :

$$
\begin{aligned}
F\left(t, \frac{1}{2}\overline{u}_k, \omega_k(t)\right) = F\left(t, \frac{1}{2}\left(u_k(t) - \tilde{u}_k(t)\right), \omega_k(t)\right) &\leq \frac{1}{2}F\left(t, u_k(t), \omega_k(t)\right) \\
&+ \frac{1}{2}F\left(t, -\tilde{u}_k(t), \omega_k(t)\right)
\end{aligned}
$$

for $t \in [0, T] a.e.$ and $k = 0, 1, 2, \ldots$. Hence

$$
\begin{aligned}
A &\geq \Phi_{\omega_k}(u_k) \\
&\geq \frac{1}{2}\int_0^T |\dot{u}_k(t)|^2 dt + 2\int_0^T F\left(t, \frac{1}{2}\overline{u}_k, \omega_k(t)\right) dt - \int_0^T F\left(t, -\tilde{u}_k(t), \omega_k(t)\right) dt \\
&\geq 2\int_0^T F\left(t, \frac{1}{2}\overline{u}_k, \omega_k(t)\right) dt - C_4 \geq \alpha\left|\bar{u}_k\right| - C_5
\end{aligned}
\tag{5.5}
$$

for some $C_4, C_5 > 0$.
From (5.5) we have $|\bar{u}_k| \leq D_2$ for $k = 0, 1, 2, \ldots$. So,

$$||u_k|| \leq D_1 + D_2 = R$$

Hence V_k is bounded for $k = 0, 1, 2, \ldots$. Of course by assumption 2, $\Phi_{\omega_k}(\cdot)$ is convex for any $\omega_k \in W$. From theorem (1) we get the assertion. ∎

Basing ourselves on theorems (2) and (4), we consider two examples.

Example 1. Let $f : [0,T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times M \to \mathbb{R}$ be a function defined by the formula

$$\begin{aligned} f(t,x,p,w) &= \left(p^1\right)^2 + \left(p^2\right)^2 + p^2 \sin t + 4(1+|w|)\left[\left(x^1\right)^2 + \left(x^2\right)^2\right]^2 \\ &+ \frac{1}{2}\left[\left(p^1\right)^2 + \left(p^2\right)^2\right]\left[\left(x^1\right)^2 + \left(x^2\right)^2\right] + w \sin x^1 \sin x^2 + t x^2 w \qquad (5.6) \\ &= |p|^2 + p^2 \sin t + 4(1+|w|)|x|^2 + \frac{1}{2}|p|^2|x|^2 + w \sin x^1 \sin x^2 + t x^2 w \end{aligned}$$

where $p = (p^1, p^2)$, $x = (x^1, x^2)$, $M = [-1,1]$. Let us notice that

$$f(t,x,p,w) \geq |p|^2 - |p| + 4|x|^2 - T|x| - 1$$

Consider the functional

$$\varphi_\omega(u) = \int_0^T f(t, u(t), \dot{u}(t), \omega(t))\, dt$$

where f is given by (5.6). One can show that φ_ω is strictly convex for any $w \in W = \{\omega \in L^\infty([0,T], \mathbb{R}^r) : \omega(t) \in M\}$. Let $\{\omega_k\} \subset W$ be any sequence strongly converging to $\omega_0 \in W$. Consider a periodic problem with parameters ω_k, $k = 0,1,2,\ldots$.

$$\begin{gathered} \tfrac{d}{dt}\left(2\dot{u}^1 + |u|^2 \dot{u}^1\right) = 8u^1(1+|\omega_k|) + u^1|\dot{u}|^2 + \omega_k \cos u^1 \sin u^2 \\ \tfrac{d}{dt}\left(2\dot{u}^2 + |u|^2 \dot{u}^2 + \sin t\right) = 8u^2(1+|\omega_k|) + u^2|\dot{u}|^2 + \omega_k \sin u^1 \cos u^2 + t\omega_k \\ u(0) - u(T) = 0 \\ \dot{u}^1(0) - \dot{u}^1(T) = 0, \qquad \dot{u}^2(0) - \dot{u}^2(T) = \left(2 + |u(0)|^2\right)^{-1} \sin T \end{gathered}$$

From theorem(2) and corollary(1) it follows that, for each ω_k, $k = 0,1,2,\ldots$, this problem possesses a uniquely defined solution $u_k \in H_T^1$, and that the sequence $\{u_k\}$ tends to u_0 in the weak topology of H_T^1.

Example 2. Let $W = \{\omega \in L^\infty([0,T], \mathbb{R}) : 0 \leq \omega(t) \leq 1\}$ and let $\{\omega_k\} \subset W$, $k = 1,2,\ldots$ be any sequence strongly converging to ω_0. For $k = 0,1,2,\ldots$, consider the scalar problem

$$\begin{gathered} \ddot{u} = u e^{|u|^2 + \omega_k} + 2u[2+\omega_k]^2 + u^2 \cos u + 2u \sin u + e(t) \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{gathered} \qquad (5.7)$$

where function $e : [0,T] \to \mathbb{R}$ is bounded. In this case, F is of the form

$$F(t,x,w) = \frac{1}{2} e^{|x|^2 + w} + [2+w]^2 |x|^2 + |x|^2 \sin x + e(t)\, x$$

Let us notice that

$$F(t,x,w) \geq 3|x|^2 - K|x| \geq K|x| - \frac{K^2}{3}$$

where $K = \max|e(t)|$ for $t \in [0,T]$. If $e \equiv 0$, then $F(t,x,w) \geq 6|x| - 3$. Of course, F is strictly convex, so, using theorem(4) and corollary(1), we have that problem (5.7) has a unique solution $u_k \in H_T^1$, and that the sequence $\{u_k\}$ tends to u_0 in the weak topology of H_T^1.

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