On the number of nonisomorphic glued near hexagons

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Abstract

We recall the construction of the so-called glued geometries. Besides other things, two generalized quadrangles are needed for this construction. In order to obtain a near hexagon, both generalized quadrangles must be derivable from an admissible triple. For several pairs $(\mathcal{Q}_1, \mathcal{Q}_2)$, where each $\mathcal{Q}_i$ is a generalized quadrangle with this property, we give all the glued near hexagons arising. However, some of them can be isomorphic and we make estimates for the number of nonisomorphic ones. As a consequence, many new near hexagons will occur.

1 Introduction and overview

A *generalized quadrangle* of order (s, t) is an incidence structure of points and lines satisfying the following conditions.

(GQ1) Every two different points are incident with at most one line, or equivalently every two different lines are incident with at most one point.

(GQ2) Every point is incident with exactly $t+1$ lines ($t \geq 1$) and every line is incident with exactly $s + 1$ points ($s \geq 1$).

(GQ3) For every line L and every point p not incident with L, there exists a unique point q and a unique line M such that $p \mathrm{~I~} M \mathrm{~I~} q \mathrm{~I~} L$ (here I denotes the incidence relation).

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The point q in (GQ3) is called the *projection* of p on L. If $p \mathrm{I} L$, then the projection of p on L is p itself. We may identify each line with its set of points and take I as the symmetrized containment. Generalized quadrangles (GQ's for short) were introduced in [12]. We refer to [8] and [10] for the most important results about GQ's. A GQ of order (s,t) is sometimes denoted by GQ(s,t). The point-line dual of a GQ(s,t) is a GQ(t,s). *Grids*, respectively *dual grids*, are exactly the GQ's of order $(s,1)$, respectively $(1,t)$. A GQ is called *trivial* when it is a grid or a dual grid. An *ovoid* (respectively *spread*) of a GQ is a set of points (respectively lines) such that every line (respectively point) of the GQ is incident with exactly one element of the set.

A *near hexagon* is an incidence structure of points and lines such that the following conditions are satisfied.

(NH1) Every two different points are incident with at most one line.

(NH2) For every point p and every line L, there is a unique point q on L, nearest to p (distances are measured in the collinearity graph).

(NH3) The maximal distance between two points is three.

Near hexagons were introduced in [9]. We refer to [5] for an overview and a list of references concerning these geometries.

In the next section, we will define the classes of GQ's which we will use in this paper. In Section 3, we study automorphisms of a GQ(s,t) which fix a spread and we apply this to the classes of GQ's and certain spreads in them. In Section 4, we define a class of near hexagons, the so-called glued near hexagons. The construction given there makes use of (i) two GQ's with a spread in each of them, (ii) a certain map θ between the base lines of the two spreads. In order to obtain a near hexagon, a certain condition must be satisfied. This paper deals with two important problems. The first problem is the following: given the two $GQ's$, the two spreads and the two base lines, find all maps θ for which the condition is satisfied. This problem is solved in Section 5. The question that now rises is: can different maps yield isomorphic near hexagons? The answer to this question is affirmative. The second problem is then the following: given the two GQ's and the two spreads, how many nonisomorphic glued near hexagons can be obtained? In order to tackle this problem, we take a closer look at isomorphisms between glued near hexagons in Section 6. Using the results of Sections 3, 5 and 6, we then make estimates for the number of nonisomorphic glued near hexagons in Section 7. In this final section, we also construct a glued near hexagon for every prime of the form $2^{2^n}+1$.

2 Some classes of GQ's

In this section, we define some classes of GQ's. Besides the usual definition, we also give a construction that makes use of an admissible triple. This construction was introduced in [2] and goes as follows.

An *admissible triple* (AT for short) is a triple $(\mathcal{D}, K, \Delta)$ satisfying the following properties.

(AT1) $\mathcal{D}$ is a linear space with constant line size $s+1$ ($s \geq 1$). The point set $\mathcal{P}$ has then order $st+1$ with t some nonzero integer.

(AT2) K is a group of order $s+1$.

(AT3) Δ is a map from $\mathcal{P} \times \mathcal{P}$ to K, such that the points x, y, z are on a line if and only if $\Delta(x, y)\,\Delta(y, z) = \Delta(x, z)$ (multiplicative notation).

With each AT, there is associated a GQ(s, t), whose collinearity graph Γ has vertex set $K \times \mathcal{P}$. Two vertices (k_1, x) and (k_2, y) are adjacent whenever

(a) $x = y$ and $k_1 \neq k_2$, or

(b) $x \neq y$ and $k_2 = k_1\Delta(x, y)$.

The lines of the GQ correspond with the maximal cliques of Γ. For every point x of $\mathcal{D}$, we define $L_x = \{(k, x)|k \in K\}$. The set $S = \{L_x|x \in \mathcal{P}\}$ is then a spread of the corresponding GQ, and it is called the *associated spread* of the admissible triple. We will mention a special property of this spread after we have given the following definitions.

(i) If A is a set of lines of a GQ(s, t), then $A^\perp$ denotes the set of all lines intersecting all members of A. If L and M are two disjoint lines, then $(\{L, M\}^\perp)^\perp$ is called the *hyperbolic line* through L and M. (These hyperbolic lines are usually defined on the point set of a GQ, but since the point-line dual of a GQ is again a GQ, we can do it as well for the line set.)

(ii) A spread S of a GQ(s, t) is called *normal* if the hyperbolic line through each pair $\{L, M\} \subseteq S$ consists of $s+1$ lines all belonging to S.

(iii) A spread S of a GQ $\mathcal{Q}$ is called a *spread of symmetry*, if for every two different lines $K, L \in S$ and for every two lines M and N meeting K and L, there exists an automorphism of $\mathcal{Q}$ fixing each line of S and mapping M to N. Every spread of symmetry is a normal spread.

The associated spread of an AT is a spread of symmetry and hence is normal; conversely, if S is a spread of symmetry of $\mathcal{Q}$, then there exists an AT with $\mathcal{Q}$ as corresponding GQ and S as associated spread (see [2]). With each normal spread S of a GQ(s, t), there is associated a linear space $\mathcal{L}(S)$: the points of $\mathcal{L}(S)$ are the $st+1$ lines of S; the lines of $\mathcal{L}(S)$ are the hyperbolic lines consisting of $s+1$ lines of S; incidence is the natural one. If θ is an automorphism of the GQ fixing S, then θ induces an automorphism of $\mathcal{L}(S)$. If S is the associated spread of an AT $(\mathcal{D}, K, \Delta)$, then $\mathcal{L}(S) \simeq \mathcal{D}$ (see [2]). Hence every automorphism of the GQ fixing S induces an automorphism of $\mathcal{D}$.

2.1 The trivial GQ$'s$

There is up to isomorphism only one GQ of order $(s, 1)$, called the $(s+1)\times(s+1)$-grid. A description is as follows. The points are the elements x_{ij} with $1 \leq i, j \leq s+1$ and there are two types of lines, namely the lines L_i and M_j with $1 \leq i, j \leq s+1$. The point x_{ij} is incident with L_k, respectively M_k, if and only if $i = k$, respectively $j = k$. Dually, there exists up to isomorphism only one GQ of order $(1, t)$; a description follows by interchanging the roles of points and lines in the above description of the grids.

The AT-model for the $(s+1) \times (s+1)$-grid is as follows: $\mathcal{D}$ consists of one line of length $s+1$, K is an arbitrary group of order $s+1$ and $\Delta(x, y)$ is the identity element for all points x, y of $\mathcal{D}$. The AT-model for the dual grid of order $(1, t)$ is as follows: $\mathcal{D}$ is the complete graph K_{t+1}, K is the group of order 2 and $\Delta(x, y)$ is the identity element if and only if $x = y$.

Every grid has two spreads and a dual grid of order $(1, t)$ has $(t+1)!$ spreads. All these spreads are spreads of symmetry.

2.2 The GQ $T_2^*(O)$

With each hyperoval O in a Desarguesian projective plane $\mathrm{PG}(2, q)$ (hence q even) corresponds a GQ $T_2^*(O)$ of order $(q-1, q+1)$ as follows ([1],[6],[8]). Embed $\mathrm{PG}(2, q)$ as a hyperplane π in $\mathrm{PG}(3, q)$. The points of the GQ are the affine points of $\mathrm{PG}(3, q)$, i.e. the points not in π; the lines of the GQ are the lines of $\mathrm{PG}(3, q)$ intersecting the hyperoval in a unique point; incidence is the natural one. The standard example of a hyperoval is the *regular* hyperoval, which consists of a conic and its nucleus. For a survey of other (irregular) hyperovals, we refer to [11].

The lines of $\mathrm{PG}(3, q)$ through a fixed point of O define a spread of $T_2^*(O)$ and the affine points of a plane of $\mathrm{PG}(3, q)$, which intersects O in exactly two points, induce a grid as subquadrangle of $T_2^*(O)$. If $q \neq 2$, then every grid of $T_2^*(O)$ is obtained this way (see Theorem 3.3.4 of [8]). Let θ be any automorphism of $T_2^*(O)$ and let L be any line of $\mathrm{PG}(3, q)$ not contained in π. Suppose $q \neq 2$. If L meets O, then $\theta(L)$ is again a line of $\mathrm{PG}(3, q)$. If L does not meet O, then it is contained in exactly $\frac{q+2}{2}$ grids of $T_2^*(O)$ and the same property holds for $\theta(L)$, which proves that $\theta(L)$ is again a line of $\mathrm{PG}(3, q)$. Hence, the automorphisms of $T_2^*(O)$ are induced by automorphisms of $\mathrm{PG}(3, q)$ which fix the hyperoval.

We give now an AT-model for $T_2^*(O)$. Let L be a line of π disjoint with O. We will consider this line as the line at infinity. The points of O are then contained in the associated affine plane $\mathcal{D} = \mathrm{AG}(2, q)$. Let K be the additive group of $\mathrm{GF}(q)$. We define now the map Δ. Let a be a fixed point of O and let x, y be two points of $\mathcal{D}$. If $x = y$, then we put $\Delta(x, y) = 0$. If $x \neq y$, then $\overline{xy} = \Delta(x, y)\, \overline{ao_{xy}}$, where o_{xy} is the unique point of O different from a such that ao_{xy} is parallel with xy. It is proved in [2] that $(\mathcal{D}, K, \Delta)$ is an AT and that the corresponding GQ is isomorphic to $T_2^*(O)$. The associated spread corresponds with the set of lines of $\mathrm{PG}(3, q)$ through $a \in O$.

If $q = 2$, then $T_2^*(O)$ is a dual grid and hence all 24 spreads are spreads of symmetry. If $q \neq 2$, then $T_2^*(O)$ has $q+2$ normal spreads (see [2]), namely the spreads defined by the points of O; all these spreads are spreads of symmetry.

2.3 The GQ $Q(5, q)$

Consider in $\mathrm{PG}(5, q)$ a nonsingular elliptic quadric $Q^-(5, q)$. The canonical equation of such a quadric is $F(X_0, X_1) + X_2X_3 + X_4X_5 = 0$, where F is an irreducible homogeneous quadratic polynomial. The points of $Q(5,q)$ are the points of $Q^-(5,q)$, the lines are the lines of $\mathrm{PG}(5,q)$ which lie on the quadric and the incidence is the natural one. One can prove that this GQ has order (q,q^2). The point-line dual of $Q(5,q)$ is a GQ denoted by $H(3,q^2)$. Just as $Q(5,q)$ is the GQ of the points and the lines of $Q^-(5,q)$, the GQ $H(3,q^2)$ is the GQ of the points and the lines of a nonsingular Hermitian variety in $\mathrm{PG}(3,q^2)$. Such a variety has canonical equation $X_0^{q+1}+X_1^{q+1}+X_2^{q+1}+X_3^{q+1}=0$.

We give now an AT-model for $Q(5,q)$. Let the vector space $V(3,q^2)$ be equiped with a nonsingular Hermitian form $(\cdot,\cdot)$ (i.e. $(\sum \mu_i v_i, \sum \lambda_j w_j) = \sum\sum \mu_i \lambda_j^q (v_i,w_j)$) and let U be the corresponding unital of $\mathrm{PG}(2,q^2)$. There is a Steiner system $\mathcal{D}=S(2,q+1,q^3+1)$ related to U (the blocks are the intersections of U with nontangent lines). Let $K=\{\alpha\in \mathrm{GF}(q^2) | \alpha^{q+1}=1\}$ with multiplication inherited from $\mathrm{GF}(q^2)$. Let $z=\langle \bar{a}\rangle$ be a fixed point of U. For two points $x=\langle \bar{b}\rangle$ and $y=\langle \bar{c}\rangle$ of U, we define

$$\begin{aligned} \Delta(x,y) &= -(\bar{a},\bar{b})^{q-1}(\bar{b},\bar{c})^{q-1}(\bar{c},\bar{a})^{q-1}\in K \text{ if } x,y,z \text{ are mutually different;} \\ &= 1 \text{ otherwise.} \end{aligned}$$

This is a good definition. For, if we replace $\bar{b}$ by $\mu\bar{b}$ and $\bar{c}$ by $\lambda\bar{c}$ with $\mu,\lambda\in \mathrm{GF}(q^2)\setminus\{0\}$, then the above value for $\Delta(x,y)$ is unaltered. It is proved in [2] that $(\mathcal{D},K,\Delta)$ is an AT and that the corresponding GQ is isomorphic to $Q(5,q)$.

A normal spread of $Q(5,q)$ dualizes to a normal ovoid of $H(3,q^2)$. Such an ovoid is always the intersection of $H(3,q^2)$ with a nontangent plane (see [2]). There are hence $q^3(q-1)(q^2+1)$ normal spreads in $Q(5,q)$ and all these spreads are spreads of symmetry.

2.4 The GQ $P(W(q),x)$

For each prime power q, the GQ $W(q)$ of order (q,q) is defined as follows: the points of $W(q)$ are the points of $\mathrm{PG}(3,q)$; the lines of $W(q)$ are the totally isotropic lines of $\mathrm{PG}(3,q)$ with respect to a symplectic polarity ζ; incidence is the natural one.

If x is a point of $W(q)$, then $P(W(q),x)$ is the GQ of order $(q-1,q+1)$ obtained as follows: the points of $P(W(q),x)$ are the points of $\mathrm{PG}(3,q)$ not in x^ζ; the lines of $P(W(q),x)$ are the lines of $W(q)$ not in x^ζ together with the lines of $\mathrm{PG}(3,q)$ through x but not in x^ζ; incidence is the natural one. If q is even then $P(W(q),x)$ is isomorphic to $T_2^*(O)$ with O a regular hyperoval (see [8]). If q is odd then $P(W(q),x)$ is isomorphic to $AS(q)$, a GQ which is usually defined as follows ([1],[8]).

The points of $AS(q)$, q odd, are the points of the affine space $\mathrm{AG}(3,q)$. The lines are the following curves of $\mathrm{AG}(3,q)$:

(i) $x=\sigma, y=a, z=b$;

(ii) $x=a, y=\sigma, z=b$;

(iii) $x=c\sigma^2-b\sigma+a, y=-2c\sigma+b, z=\sigma$.

In all cases $a,b,c\in \mathrm{GF}(q)$ are fixed and σ ranges over all elements of $\mathrm{GF}(q)$. If $q=3$, then $AS(3)\simeq Q(5,2)$ (see [8]).

We give now an AT-model. $\mathcal{D}$ is the Desarguesian affine plane $\mathrm{AG}(2,q)$, K is the additive group of $\mathrm{GF}(q)$ and $\Delta((x_1,y_1),(x_2,y_2)) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$ for the points (x_1,y_1) and (x_2,y_2) of $\mathrm{AG}(2,q)$. It was proved in [2] that the GQ from this AT is indeed $P(W(q),x)$. The associated spread S corresponds in the $P(W(q),x)$-model with the lines of $\mathrm{PG}(3,q)$ through x. If q is even, then S corresponds in the $T_2^*(O)$-model with the lines through a point $a \in O$ for which $O \setminus \{a\}$ is a conic (a is unique if $q \neq 2,4$). If q is odd, then S corresponds in the $AS(q)$-model with the lines of type (i). If $q \neq 3$ and odd, then S is the unique normal spread of $AS(q)$ (see [2]).

3 Automorphisms of a GQ fixing a spread

Notations. If L and M are two disjoint lines of a GQ, then $[L,M]$ denotes the projection from L onto M. If $f : B \to C$ and $g : A \to B$ are two maps, then we denote $f \circ g$ by fg.

Let $\mathcal{Q} = (\mathcal{P},\mathcal{L},\mathrm{I})$ be a $\mathrm{GQ}(s,t)$ and let $S = \{L_1,\ldots,L_{1+st}\}$ be a spread of $\mathcal{Q}$ (Notice that we number the lines of S). We call L_1 the *base line* of the spread. The set of all automorphisms of $\mathcal{Q}$ fixing S is denoted by $\mathcal{A}_S$ and it is a subgroup of $\mathcal{A} = \mathrm{Aut}(\mathcal{Q})$. The map $\Delta_{ij} := [L_j,L_1][L_i,L_j][L_1,L_i]$ $(i,j \in \{1,\ldots,1+st\})$ defines a permutation on the points of L_1. If β is a permutation of $\{1,\ldots,1+st\}$, then we define $\Delta_{ij}^{\beta} := \Delta_{\beta(i)\beta(j)}$. The group of permutations of L_1 generated by all elements Δ_{ij} is called *the group of projectivities of* L_1 *with respect to* S and is denoted by G. If $\theta \in \mathcal{A}_S$, then the following permutations can be defined.

(a) A permutation β of $\{1,\ldots,1+st\}$, such that
$$\theta(L_i) = L_{\beta(i)}, \tag{1}$$
for all $i \in \{1,\ldots,1+st\}$.

(b) Permutations α_i of L_1 such that
$$\theta[L_1,L_i](x) = [L_1,L_{\beta(i)}]\alpha_i(x), \tag{2}$$
for all $i \in \{1,\ldots,1+st\}$ and all $x \in L_1$.

Consider now two adjacent vertices $[L_1,L_i](x)$ and $[L_1,L_j](x')$ with $x,x' \in L_1$ and $i \neq j$. Hence $x' = \Delta_{ij}(x)$ and from $[L_1,L_{\beta(i)}]\alpha_i(x) \sim [L_1,L_{\beta(j)}]\alpha_j(x') = [L_1,L_{\beta(j)}]\alpha_j\Delta_{ij}(x)$, it follows that $\alpha_i = \Delta_{ji}^{\beta}\alpha_j\Delta_{ij}$. Define
$$\alpha := \alpha_1, \tag{3}$$
we then get that
$$\alpha_i = \Delta_{1i}^{\beta}\alpha, \tag{4}$$
and the condition $\alpha_i = \Delta_{ji}^{\beta}\alpha_j\Delta_{ij}$ becomes
$$(\Delta_{j1}^{\beta}\Delta_{ij}^{\beta}\Delta_{1i}^{\beta})\alpha = \alpha\Delta_{ij}, \tag{5}$$
for all $i,j \in \{1,\ldots,1+st\}$. So, we have proved the following theorem.

Theorem 3.1 *If $\theta \in \mathcal{A}_S$, then we can define a permutation α of L_1 and a permutation β of $\{1, \dots, 1+st\}$ by (1), (2), (3) and these permutations satisfy equation (5). Conversely, if (5) is satisfied by a permutation α of L_1 and a permutation β of $\{1, \dots, 1+st\}$, then (2) and (4) define a permutation $\theta \in \mathcal{A}_S$.*

The following corollary of this theorem was already proved in [2].

Corollary 3.2 *If θ is an automorphism of $\mathcal{Q}$ fixing each line of S, then there exists a permutation α of L_1, which commutes with every element of G, and such that*

$$\theta[L_1, L_i](x) = [L_1, L_i]\alpha(x) \tag{6}$$

holds for all $i \in \{1, \dots, 1+st\}$ and all $x \in L_1$. Conversely, if a permutation α of L_1 commutes with every element of G, then (6) defines an automorphism of $\mathcal{Q}$ which fixes each line of S.

Let $\widehat{G}$ be the set of all permutations α of L_1 for which there exists a permutation β of $\{1, \dots, 1+st\}$ such that (5) is satisfied for all $i, j \in \{1, \dots, 1+st\}$.

Theorem 3.3 *If G is commutative, then $\widehat{G}$ is a group.*

Proof. Since $\widehat{G}$ is finite, it suffices to prove that $\mu_1\mu_2 \in \widehat{G}$ for every two elements $\mu_1, \mu_2 \in \widehat{G}$. Let β_1 and β_2 be the permutations such that

$$\Delta_{j1}^{\beta_1}\Delta_{ij}^{\beta_1}\Delta_{1i}^{\beta_1} = \mu_1\Delta_{ij}\mu_1^{-1},$$

and

$$\Delta_{j1}^{\beta_2}\Delta_{ij}^{\beta_2}\Delta_{1i}^{\beta_2} = \mu_2\Delta_{ij}\mu_2^{-1}$$

for all $i, j \in \{1, \dots, 1+st\}$. One calculates then that

$$\mu_1\mu_2\Delta_{ij}\mu_2^{-1}\mu_1^{-1} = \Delta_{\beta_2(1)1}^{\beta_1}\Delta_{j1}^{\beta_1\beta_2}\Delta_{ij}^{\beta_1\beta_2}\Delta_{1i}^{\beta_1\beta_2}\Delta_{1\beta_2(1)}^{\beta_1},$$

and this is equal to $\Delta_{j1}^{\beta_1\beta_2}\Delta_{ij}^{\beta_1\beta_2}\Delta_{1i}^{\beta_1\beta_2}$ if G is commutative. □

Example 1: the GQ $P(W(q), u)$

We use the AT-model for $P(W(q), u)$, $q = p^h$, given in Section 2.4; the associated spread consists of the lines $L_{(x,y)} = \{(z, x, y) | z \in \mathrm{GF}(q)\}$, $x, y \in \mathrm{GF}(q)$. Consider $L_1 = L_{(0,0)}$ as base line. We can identify the point set of L_1 with the elements of $\mathrm{GF}(q)$ and hence with the points of $\mathrm{AG}(1, q)$. The permutations Δ_{ij} of L_1 are then translations of $\mathrm{AG}(1, q)$: if $L_i = L_{(x_1,y_1)}$ and $L_j = L_{(x_2,y_2)}$, then one easily calculates that $\Delta_{ij} = T(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix})$, i.e. a translation over $x_1y_2 - x_2y_1$. Let $\theta \in \mathcal{A}_S$, then θ induces an automorphism of $AG(2, q)$. Such an automorphism looks like

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x^\sigma \\ y^\sigma \end{bmatrix} + \begin{bmatrix} b \\ c \end{bmatrix}, \tag{7}$$

where the map $x \mapsto x^\sigma$ is an automorphism of GF(q) and $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a non-singular (2×2)-matrix over GF(q). One calculates that $\Delta^\beta_{j1}\Delta^\beta_{ij}\Delta^\beta_{1i} = T(\det(A)k^\sigma)$ if $\Delta_{ij} = T(k)$. From equation (5), it follows then that

$$\alpha(k) = \det(A)k^\sigma + d,$$

where $d = \alpha(0)$. Hence

$$\alpha_{(x,y)}(k) = (ba_{21} - ca_{11})x^\sigma + (ba_{22} - ca_{12})y^\sigma + (a_{11}a_{22} - a_{12}a_{21})k^\sigma + d,$$

where $\alpha_{(x,y)} := \alpha_i$ with i such that $L_i = L_{(x,y)}$. The automorphisms of $\mathcal{A}_S$ are hence the following maps:

$$\begin{bmatrix} z \\ x \\ y \end{bmatrix}^\theta = \begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & ba_{21} - ca_{11} & ba_{22} - ca_{12} \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} z^\sigma \\ x^\sigma \\ y^\sigma \end{bmatrix} + \begin{bmatrix} d \\ b \\ c \end{bmatrix},$$

where σ is an arbitrary automorphism of GF(q) and $a_{11}, a_{12}, a_{21}, a_{22}, b, c, d$ are arbitrary elements of GF(q) satisfying $a_{11}a_{22} - a_{12}a_{21} \neq 0$. The knowledge of the orbit of S under $\mathcal{A}$ gives now a good idea of how close $\mathcal{A}$ is to $\mathcal{A}_S$. The spread S is a normal spread and need to be mapped to a normal spread. If $q \neq 3$ is odd, then S is the only normal spread; hence $\mathcal{A} = \mathcal{A}_S$. If $q = 3$, then $AS(3) \simeq Q(5,2)$ and $\mathcal{A}$ acts transitively on the set of all normal spreads. Suppose now that q is even and consider the model $T_2^*(O)$. If $q \neq 2$, then the elements of $\mathcal{A}$ are induced by automorphisms of PG$(3,q)$ which fix the hyperoval. If $q = 2$ or $q = 4$, then $\mathcal{A}$ acts transitively on the set of all normal spreads. If $q \geq 8$, then S corresponds with the unique point a of O such that $O \setminus \{a\}$ is a conic; hence $\mathcal{A} = \mathcal{A}_S$.

Example 2: the GQ $Q(5,q)$

We use the AT-model for $Q(5,q)$, $q = p^h$, given in Section 2.3 and suppose that $(\bar{a}, \bar{b}) = \bar{a}^T\bar{b}^q$ is the nonsingular Hermitian form in $V(3,q^2)$. The associated spread consists of the lines $L_r = \{(k,r) | k \in K\}$, $r \in U$. Consider $L_1 = L_{\langle \bar{a} \rangle}$ as base line. We can identify the points of L_1 with the elements of K. The permutations Δ_{ij} are then multiplications: if $L_i = L_r$ and $L_j = L_s$, then one easily verifies that $\Delta_{ij} = M(\Delta(r,s))$, i.e. a multiplication with $\Delta(r,s) \in K$. Let $\theta \in \mathcal{A}_S$, then θ induces an automorphism of the design related to U. This automorphism is an automorphism of $PG(2,q^2)$ fixing U, and hence is an element of $P\Gamma U(3,q^2)$ (see [7]). Such an automorphism looks like $\bar{a} \mapsto A\bar{a}^\sigma$, where σ is an automorphism of GF(q^2) and where A satisfies $A^TA^q = \alpha I$ with $\alpha^{q-1} = 1$. One calculates that $\Delta^\beta_{j1}\Delta^\beta_{ij}\Delta^\beta_{1i} = M(k^\sigma)$ if $\Delta_{ij} = M(k)$. From equation (5), it follows then that $\alpha(k) = lk^\sigma$ where $l = \alpha(1)$. $\mathcal{A}_S$ contains hence $|K| \times |P\Gamma U(3,q^2)| = 2hq^3(q+1)(q^2-1)(q^3+1)$ elements. Since $\mathcal{A}$ acts transitively on the set of $q^3(q-1)(q^2+1)$ normal spreads, $\mathcal{A}$ contains $2hq^6(q^2-1)(q^3+1)(q^4-1)$ elements. The group $P\Gamma U(4,q^2)$ has indeed this order.

Example 3: the GQ $T_2^*(O)$

Let O be a hyperoval of $\mathrm{PG}(2,q)$, $q=2^h$, and let S be the spread of $T_2^*(O)$ determined by $a \in O$. Let $\theta \in \mathcal{A}_S$, then θ induces an automorphism of the affine plane $\mathcal{D} = \mathrm{AG}(2,q)$ in the related AT-model (see Section 2). As before, let L_1 denote the base line of S; the points of L_1 can be identified with the elements of $\mathrm{GF}(q)$. For $1 \leq i,j \leq q^2$, one calculates that

$$\begin{aligned} \Delta_{ij} &= T(\Delta(L_1,L_i)+\Delta(L_i,L_j)+\Delta(L_j,L_1)), & (8)\\ \Delta^{\beta}_{j1}\Delta^{\beta}_{ij}\Delta^{\beta}_{1i} &= T(\Delta(L_{\beta(1)},L_{\beta(i)})+\Delta(L_{\beta(i)},L_{\beta(j)})+\Delta(L_{\beta(j)},L_{\beta(1)})). & (9)\end{aligned}$$

By equation (5), $\Delta^{\beta}_{j1}\Delta^{\beta}_{ij}\Delta^{\beta}_{1i}$ is only dependent of Δ_{ij}. Now, let L_{i_1} and L_{j_1} be such that L_1, L_{i_1}, L_{j_1} are three noncollinear points of $\mathrm{AG}(2,q)$. Let (i_k,j_k), $1 \leq k \leq q-1$, be the $q-1$ pairs such that

(a) $L_{i_k} \neq L_1 \neq L_{j_k}$,

(b) $L_1L_{i_k} = L_1L_{i_1}$ and $L_1L_{j_k} = L_1L_{j_1}$,

(c) $L_{i_k}L_{j_k} \| L_{i_1}L_{j_1}$.

If we put the $(q-1)$ pairs (i_k,j_k) in equations (8) and (9), then one finds that $\Delta^{\beta}_{j1}\Delta^{\beta}_{ij}\Delta^{\beta}_{1i} = T(lk^{\sigma})$ if $\Delta_{ij} = T(k)$. Here l is a fixed nonzero element of $\mathrm{GF}(q)$ and σ is the automorphism of $\mathrm{GF}(q)$ related (see equation (7)) to the above mentioned automorphism of $\mathrm{AG}(2,q)$. From equation (5), it follows then that $\alpha(k) = lk^{\sigma}+d$ with $d=\alpha(0)$. Hence $|\widehat{G}| \leq h(q-1)q$. Now, if β corresponds with a homothetic transformation of $\mathrm{AG}(2,q)$ with factor $l \neq 0$, then equation (5) has q solutions for α, namely the maps $\alpha(k) = lk+d$ with $d \in GF(q)$. Hence $|\widehat{G}| \geq (q-1)q$. Since $\widehat{G}$ is a group, $|\widehat{G}| = h'(q-1)q$ with $h' \mid h$. The exact value for $|\widehat{G}|$ depends on the kind of hyperoval we are working with.

4 Glued near hexagons

In [3], a new class of near hexagons was introduced. The construction needs the following objects:

(i) a GQ $\mathcal{Q}_1 = (\mathcal{P}_1,\mathcal{L}_1,\mathrm{I}_1)$ of order (s,t_1) with a spread $S_1 = \{L_1^{(1)},\ldots,L_{st_1+1}^{(1)}\}$,

(ii) a GQ $\mathcal{Q}_2 = (\mathcal{P}_2,\mathcal{L}_2,\mathrm{I}_2)$ of order (s,t_2) with a spread $S_2 = \{L_1^{(2)},\ldots,L_{st_2+1}^{(2)}\}$.

The lines $L_1^{(1)}$ and $L_1^{(2)}$ are called the base lines. Let G_i ($i \in \{1,2\}$) be the group of projectivities of $L_1^{(i)}$ with respect to S_i. To every bijection $\theta : L_1^{(1)} \to L_1^{(2)}$, we associate a graph $\Gamma(\mathcal{Q}_1,\mathcal{Q}_2,S_1,S_2,L_1^{(1)},L_1^{(2)},\theta)$ with vertex set $L_1^{(1)} \times S_1 \times S_2$. Two different vertices $(x,L_i^{(1)},L_j^{(2)})$ and $(y,L_k^{(1)},L_l^{(2)})$ are adjacent if and only if at least one of the following conditions is satisfied:

(a) $i=k$ and $[L_1^{(2)},L_j^{(2)}]\theta(x) \sim [L_1^{(2)},L_l^{(2)}]\theta(y)$,

(b) $j=l$ and $[L_1^{(1)},L_i^{(1)}](x) \sim [L_1^{(1)},L_k^{(1)}](y)$.

If $i = k$ and $j = l$, then both (a) and (b) are satisfied. One can prove that every two adjacent vertices are contained in a unique maximal clique and that this clique has size $s+1$. Taking these cliques as lines, one can define a partial linear space $S(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$. One easily verifies that $S(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta) \simeq S(\mathcal{Q}_2, \mathcal{Q}_1, S_2, S_1, L_1^{(2)}, L_1^{(1)}, \theta^{-1})$. The following theorem is proved in [3].

Theorem 4.1 *$S(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$ is a near hexagon if and only if*

$$[\theta^{-1}G_2\theta, G_1] = 0. \tag{10}$$

Here 0 denotes the trivial group and $[\theta^{-1}G_2\theta, G_1]$ is the group of permutations of $L_1^{(1)}$, generated by all commutators $[\theta^{-1}g_2\theta, g_1]$ with $g_2 \in G_2$ and $g_1 \in G_1$. The definition of the geometry $S(Q_1, Q_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$ looks to be dependent of the considered base lines $L_1^{(1)}$ and $L_1^{(2)}$. However, if the geometry is a near hexagon, then it can also be obtained by taking two other base lines (see [3]). Of course, the map θ must then be changed. Condition (10) implies the following properties (see [2]):

(a) S_i ($i \in \{1, 2\}$) is a spread of symmetry of $\mathcal{Q}_i$; hence there is an AT-model for Q_i with S_i as associated spread,

(b) if $\mathcal{Q}_i$ ($i \in \{1, 2\}$) is not a grid, then G_i has order $s + 1$.

In Section 2, all the GQ's with an AT-model known to the author are given. If $\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}$ and $L_1^{(2)}$ are fixed, then without any confusion one can talk about the graphs $\Gamma(\theta)$ and the geometries $S(\theta)$. We have now the following interesting problem: find all maps θ for which $S(\theta)$ is a near hexagon (such a map will be called a *good map*). The problem will be solved in the following section.

5 Determination of all good maps

Lemma 5.1 *Let X be a set of order $n \geq 1$. If G is a regular group of permutations on X, then there are precisely n permutations on X which commute with every element of G. These n permutations form a regular group $\widetilde{G}$ of permutations on X.*

Proof. Let $x \in X$ be fixed. If $h \in Sym(X)$ commutes with every element of G, then $hg(x) = gh(x)$ for all $g \in G$, hence the image of x under h determines h completely. Now, for every element $y \in X$, one can define the permutation h_y as follows:

$$h_y[g(x)] = g(y), \forall g \in G.$$

It is straightforward to check that h_y commutes with every element of G and that h_y is the trivial permutation if it has at least one fixpoint. □

Remark. The set X may be identified with G (with action the right multiplication); then $\tilde{G}$ is a copy of G acting on X by left multiplication.

Lemma 5.2 *Let X be a set of order $n \geq 1$ and let G_1 and G_2 be regular groups of permutations on X, then there exists a permutation $\theta \in Sym(X)$ such that $G_2 = \theta^{-1}G_1\theta$ if and only if G_1 is isomorphic to G_2. Moreover, if ϕ is an isomorphism from G_1 to G_2, then there exist n permutations θ such that $\phi(g_1) = \theta^{-1}g_1\theta$ for all $g_1 \in G_1$.*

Proof. If $G_2 = \theta^{-1}G_1\theta$, then the map ϕ defined by $\phi(g_1) = \theta^{-1}g_1\theta, \forall g_1 \in G_1$ defines an isomorphism from G_1 to G_2. Conversely, suppose that $\phi : G_1 \to G_2$ is an isomorphism. Let $x \in X$ be fixed. The condition $\theta[\phi(g_1)](x) = g_1\theta(x), \forall g_1 \in G_1$ implies that θ is completely determined as soon as $\theta(x)$ is known. For all $y, z \in X$, we define

$$\theta_y^\phi(z) = [\phi^{-1}(g_2)](y),$$

where g_2 is the unique element of G_2 such that $g_2(x) = z$. It is straightforward to check that θ_y^ϕ satisfies the required conditions. □

Remark. The set X can be identified with respectively G_1 and G_2 (with G_1 and G_2, respectively, acting by right multiplication) by choosing the identity element arbitrarily (giving rise to n permutations θ).

Lemma 5.3 *Let S be a spread of a generalized quadrangle Q, with Q not a grid, let H be the group of all automorphisms of $\mathcal{Q}$ fixing each line of S, let L_1 be the base line of S and let G be the group of projectivities of L_1 with respect to S_1. Then*

(a) $|H| \leq s + 1$,

(b) $|G| \geq s + 1$, *with equality if and only if G is a regular group of permutations on L_1.*

Proof. Properties (a) and (b) are respectively Theorem 5.1 and Theorem 6.2 of [2]. □

Let $\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}$ and $L_1^{(2)}$ be fixed and let $S(\theta) = S(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$. Above, we defined some groups related to $\mathcal{Q}_i$, S_i and $L_1^{(i)}$ ($i \in \{1, 2\}$), like G_i and H_i (we added a subscript i to distinguish between the two GQ's).

Theorem 5.4 *Suppose that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are not grids. The number of maps θ for which $S(\theta)$ is a near hexagon is equal to $(s + 1) \times |\mathrm{Isom}(G_1, H_2)|$. If G_2 is commutative, then $H_2 \simeq G_2$, hence this number is equal to $(s + 1) \times |\mathrm{Isom}(G_1, G_2)|$.*

Proof. If G_1 is not a regular group of permutations on $L_1^{(1)}$, then $|G_1| > s + 1$. In this case $S(\theta)$ is not a near hexagon for all choices of θ (see Section 4). Since $|H_2| \leq s + 1$, there are no isomorphisms between G_1 and H_2. Suppose now that G_1 is a regular group of permutations on $L_1^{(1)}$, then Lemma 5.1 implies that condition (10) is equivalent to $\theta^{-1}G_2\theta \subseteq \widetilde{G}_1$. Hence $G_2 = s + 1$ and $\widetilde{G}_2 = \theta G_1 \theta^{-1}$. From Corollary 3.2, it follows that $\widetilde{G}_2 \simeq H_2$. The result follows now by Lemma 5.2. □

Remark. If one of the $\mathcal{Q}_i$'s is a grid, then $S(\theta)$ is a near hexagon for all $(s + 1)!$ choices of θ.

6 Isomorphisms between glued near hexagons

A glued near hexagon $\mathcal{S} = S(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$ contains many GQ's as subgeometries, the so-called *quads* (see [9] for more details). There are three types of quads (see [3]): a quad of type (I) contains the points of $\mathcal{S}$ with fixed third coordinate and is isomorphic to $\mathcal{Q}_1$, a quad of type (II) contains the points of $\mathcal{S}$ with fixed second coordinate and is isomorphic to $\mathcal{Q}_2$, a quad of type (III) is an $(s+1)\times(s+1)$-grid. Let $\mathcal{S} = S(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$ and $\mathcal{S}' = S(\mathcal{Q}'_1, \mathcal{Q}'_2, S'_1, S'_2, l_1^{(1)}, l_1^{(2)}, \theta')$ be two glued near hexagons. Our aim is to determine under which conditions $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic. Since quads are mapped to quads, we may suppose that $\mathcal{Q}_1 \simeq \mathcal{Q}'_1$ and $\mathcal{Q}_2 \simeq \mathcal{Q}'_2$. If $\mathcal{Q}_1$ is a grid, then $\mathcal{S} \simeq \mathcal{S}'$ for all choices of S_2, S'_2, θ and θ'. Suppose therefore that none of the four GQ's is a grid. If we take a fixed quad of type (I) in $\mathcal{S}$, then the quads of type (II) will intersect this quad in lines that form a spread equivalent to S_1. Hence, we may suppose that S_i and S'_i $(i \in \{1,2\})$ are equivalent spreads. We can now state the problem that we will solve in this section.

Given two GQ's $\mathcal{Q}_1, \mathcal{Q}_2$ which are not grids, two spreads S_1, S_2 in the respective GQ's and two base lines $L_1^{(1)}$ and $L_1^{(2)}$, find the conditions that must be satisfied by θ_1 and θ_2 in order that $\mathcal{S}_1 = S(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta_1)$ and $\mathcal{S}_2 = S(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta_2)$ are two isomorphic near hexagons.

The fact that $\mathcal{S}_1$ and $\mathcal{S}_2$ are near hexagons already gives the conditions

$$[\theta_1^{-1} G_2 \theta_1, G_1] = 0, \tag{11}$$

and

$$[\theta_2^{-1} G_2 \theta_2, G_1] = 0. \tag{12}$$

Let Φ be a quad-type preserving isomorphism from $\mathcal{S}_1$ to $\mathcal{S}_2$. Then the following permutations exist:

(a) a permutation β of $\{1, \ldots, 1+st_1\}$,

(b) a permutation γ of $\{1, \ldots, 1+st_2\}$,

(c) permutations α_{ij} of $L_1^{(1)}$,

such that

$$\Phi[(x, L_i^{(1)}, L_j^{(2)})] = (\alpha_{ij}(x), L_{\beta(i)}^{(1)}, L_{\gamma(j)}^{(2)}),$$

for all $x \in L_1^{(1)}$, for all $i \in \{1, \ldots, 1+st_1\}$ and for all $j \in \{1, \ldots, 1+st_2\}$. We define

$$\alpha := \alpha_{11}.$$

Consider in $\mathcal{S}_1$ the adjacent points $(x, L_i^{(1)}, L_j^{(2)})$ and $(\delta_{ik}(x), L_k^{(1)}, L_j^{(2)})$ with $i \neq k$. Then $(\alpha_{ij}(x), L_{\beta(i)}^{(1)}, L_{\gamma(j)}^{(2)}) \sim (\alpha_{kj}\delta_{ik}(x), L_{\beta(k)}^{(1)}, L_{\gamma(j)}^{(2)})$ in $\mathcal{S}_2$ and hence

$$\alpha_{ij} = \delta_{ki}^{\beta} \alpha_{kj} \delta_{ik}. \tag{13}$$

Consider in $\mathcal{S}_1$ the adjacent points $(x, L_i^{(1)}, L_j^{(2)})$ and $(\theta_1^{-1}\Delta_{jk}\theta_1(x), L_i^{(1)}, L_k^{(2)})$ with $j \neq k$, then $(\alpha_{ij}(x), L_{\beta(i)}^{(1)}, L_{\gamma(j)}^{(2)}) \sim (\alpha_{ik}\theta_1^{-1}\Delta_{jk}\theta_1(x), L_{\beta(i)}^{(1)}, L_{\gamma(k)}^{(2)})$ in $\mathcal{S}_2$ and hence

$$\alpha_{ij} = \theta_2^{-1} \Delta_{kj}^{\gamma} \theta_2 \alpha_{ik} \theta_1^{-1} \Delta_{jk} \theta_1. \tag{14}$$

From (13) and (14), it follows that

$$\alpha_{ij} = \delta^{\beta}_{1i}\alpha_{1j} = \delta^{\beta}_{1i}\theta_2^{-1}\Delta^{\gamma}_{1j}\theta_2\alpha, \tag{15}$$

for all $i \in \{1, \ldots, 1 + st_1\}$ and for all $j \in \{1, \ldots, 1 + st_2\}$. From (13) and (15) it follows that

$$(\delta^{\beta}_{k1}\delta^{\beta}_{ik}\delta^{\beta}_{1i})\alpha = \alpha\delta_{ik}, \tag{16}$$

for all $i, j \in \{1, \ldots, 1 + st_1\}$. From (14) and (15) it follows that

$$(\Delta^{\gamma}_{k1}\Delta^{\gamma}_{jk}\Delta^{\gamma}_{1j})(\theta_2\alpha\theta_1^{-1}) = (\theta_2\alpha\theta_1^{-1})\Delta_{jk}, \tag{17}$$

for all $j, k \in \{1, \ldots, 1 + st_2\}$.

If Φ interchanges the quads of type (I) and (II), then we may suppose that $\mathcal{Q}_2 = \mathcal{Q}_1, S_2 = S_1$ and $L_1^{(2)} = L_1^{(1)}$; Φ defines then a quad-type preserving isomorphism from $S(\mathcal{Q}_1, \mathcal{Q}_1, S_1, S_1, L_1^{(1)}, L_1^{(1)}, \theta_1^{-1})$ to $\mathcal{S}_2$. Equation (17) becomes then

$$(\Delta^{\gamma}_{k1}\Delta^{\gamma}_{jk}\Delta^{\gamma}_{1j})(\theta_2\alpha\theta_1) = (\theta_2\alpha\theta_1)\Delta_{jk}, \tag{18}$$

for all $j, k \in \{1, \ldots, 1 + st_2\}$.

We have now proved the following theorem.

Theorem 6.1 *$\mathcal{S}_1$ and $\mathcal{S}_2$ are isomorphic glued near hexagons if and only if θ_1 and θ_2 satisfy (11),(12),(16),(17) or (11),(12),(16),(18) for some permutation α of L_1, some permutation β of $\{1, \ldots, 1 + st_1\}$ and some permutation γ of $\{1, \ldots, 1 + st_2\}$.*

Theorem 6.2 *Equations (11), (16) and (17) imply equation (12); similarly, equations (11), (16) and (18) imply equation (12).*

Proof. Suppose that (11), (16) and (17) hold, then

$$\alpha^{-1}[\theta_2^{-1}G_2\theta_2, G_1]\alpha \overset{(16)}{=} \alpha^{-1}[\theta_2^{-1}G_2\theta_2, \alpha G_1\alpha^{-1}]\alpha = [\alpha^{-1}\theta_2^{-1}G_2\theta_2\alpha, G_1]$$

$$\overset{(17)}{=} [\alpha^{-1}\theta_2^{-1}(\theta_2\alpha\theta_1^{-1})G_2(\theta_2\alpha\theta_1^{-1})^{-1}\theta_2\alpha, G_1] = [\theta_1^{-1}G_2\theta_1, G_1] \overset{(11)}{=} 0.$$

De rest of the statement is proved similarly. □

Remark. As in Section 3, one can define the sets $\widehat{G}_1$ and $\widehat{G}_2$; equations (16), (17) and (18) become then $\alpha \in \widehat{G}_1$, $\theta_2\alpha\theta_1^{-1} \in \widehat{G}_2$ and $\theta_2\alpha\theta_1 \in \widehat{G}_2$.

7 The number of nonisomorphic glued near hexagons

With the results obtained in Sections 3, 5 and 6, one can obtain estimates for the number of nonisomorphic glued near hexagons. For $i \in \{1, 2\}$, let $(\mathcal{D}_i, K_i, \Delta_i)$ be an AT yielding a generalized quadrangle $\mathcal{Q}_i$ of order (s, t_i) and let S_i be the associated spread. Put $S(\theta) = S(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_2^{(2)}, \theta)$ for every bijection θ between a line $L_1^{(1)} \in S_1$ and a line $L_1^{(2)} \in S_2$. There is a natural bijection between the points of $L_1^{(i)}$ and the elements of K_i; we hence may consider θ as a bijection from K_1 to K_2. For several examples for $(\mathcal{D}_1, K_1, \Delta_1)$ and $(\mathcal{D}_2, K_2, \Delta_2)$, we will determine all maps $\theta : K_1 \to K_2$, for which $S(\theta)$ is a near hexagon and we will make estimates for $N(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2)$, the total number of nonisomorphic glued near hexagons of the form $S(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$. In the previous sections, we defined several groups related to $\mathcal{Q}_i$, S_i and $L_1^{(i)}$, like G_i, H_i, $\tilde{G}_i$ and also $\widehat{G}_i$ if G_i is commutative. The following theorem gives a lower and an upper bound for $N(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2)$.

Theorem 7.1 *Suppose that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are not grids. If there exists an isomorphism from $\mathcal{Q}_1$ to $\mathcal{Q}_2$, mapping S_1 to S_2, then*

$$\frac{(s+1)|\mathrm{Isom}(G_1, H_2)|}{2|\widehat{G}_1||\widehat{G}_2|} \leq N(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2) \leq \frac{(s+1)|\mathrm{Isom}(G_1, H_2)|}{\max(|\widehat{G}_1|, |\widehat{G}_2|)}; \tag{19}$$

if there exists no such automorphism, then

$$\frac{(s+1)|\mathrm{Isom}(G_1, H_2)|}{|\widehat{G}_1||\widehat{G}_2|} \leq N(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2) \leq \frac{(s+1)|\mathrm{Isom}(G_1, H_2)|}{\max(|\widehat{G}_1|, |\widehat{G}_2|)}. \tag{20}$$

Proof. Let θ_1 be one of the $(s+1)|\mathrm{Isom}(G_1, H_2)|$ maps for which $\mathcal{S}(\theta_1)$ is a near hexagon. There are at most $|\widehat{G}_1||\widehat{G}_2|$ maps θ_2 satisfying equations (16) and (17), and a same remark holds for equations (16) and (18). This proves the lower bounds for $N(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2)$. Now, consider again equations (16) and (17). Let α and β be identical permutations, and let θ_2 be one of the $|\widehat{G}_2|$ maps for which $\theta_2\theta_1^{-1} \in \widehat{G}_2$, then $S(\theta_2)$ is a near hexagon by Theorem 6.2. Hence $N(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2) \leq \frac{(s+1)|\mathrm{Isom}(G_1,H_2)|}{|\widehat{G}_2|}$ and the upper bound follows by symmetry. □

7.1 Glued near hexagons arising from $P(W(q), x)$ or $T_2^*(O)$.

(A) Let $(\mathcal{D}_1, K_1, \Delta_1) = (\mathcal{D}_2, K_2, \Delta_2)$ be the AT from Section 2.4 with $\mathcal{Q} = P(W(q), x)$, $q = p^h$, as corresponding GQ and S as associated spread. The groups G_1 and G_2 are isomorphic to the additive group of $\mathrm{GF}(q)$. Hence the number of good maps θ is equal to $p^h|GL(h, p)|$ and we determine now how each such $\theta : K_1 \to K_2$ looks like. Let $x \in \mathrm{GF}(q)$ be fixed. For each automorphism ϕ of $\mathrm{GF}(q), +$ and each $y \in \mathrm{GF}(q)$, we have

$$\begin{aligned} \theta_y^\phi(z) &= \phi^{-1}(z - x) + y \\ &= \phi^{-1}(z) - \phi^{-1}(x) + y \end{aligned}$$

for all $z \in \mathrm{GF}(q)$. Hence the $p^h|GL(h, p)|$ good maps are the maps

$$z \mapsto \phi(z) + a,$$

where ϕ is an arbitrary automorphism of $\mathrm{GF}(q), +$ and a is an arbitrary element of $\mathrm{GF}(q)$. By Theorem 7.1, we have that

$$\frac{|GL(h, p)|}{2h^2(q-1)^2 q} \leq N(\mathcal{Q}, \mathcal{Q}, S, S) \leq \frac{|GL(h, p)|}{h(q-1)}.$$

The lower bound for $N(\mathcal{Q}, \mathcal{Q}, S, S)$ can easily be strengthened.

Theorem 7.2 *Let $\mathcal{Q} = P(W(q), x)$, $q = p^h$, and let S be the spread of $\mathcal{Q}$ defined by the lines of* $\mathrm{PG}(3, q)$ *through x; then*

$$1 + \frac{|GL(h,p)| - h(q-1)}{2h^2(q-1)^2 q} \leq A(q) := N(\mathcal{Q}, \mathcal{Q}, S, S) \leq \frac{|GL(h,p)|}{h(q-1)}.$$

If $q = 4$ or q is prime, then $A(q) = 1$; otherwise $A(q) > 1$.

Proof. If $\theta_1 \in \widehat{G}$, then $S(\theta_1)$ is a near hexagon. From equations (16), (17) and (18) it follows that $S(\theta_2) \simeq S(\theta_1)$ if and only if $\theta_2 \in \widehat{G}$; hence

$$1 + \frac{q|GL(h,p)| - |\widehat{G}|}{2h^2(q-1)^2 q^2} \leq A(q) \leq \frac{|GL(h,p)|}{h(q-1)}.$$

If $q = 4$ or if q is prime, then $A(q) = 1$ by the above inequalities. If $h \geq 2$ and $q \neq 4$, then $|GL(h,p)| \geq (p^h - 1)(p^h - p) > h(q-1)$; hence $A(q) > 1$. □

For fixed h, the lower bound in the previous theorem behaves like $\frac{1}{2h^2} p^{h^2 - 3h}$ for great values of p; hence many glued near hexagons arise. We treat now the case $h = 2$.

The case $h = 2$

For $p = 2$ there is exactly one glued near hexagon; suppose therefore that p is odd. The $p^2(p^2-1)(p^2-p)$ good maps θ are the maps

$$\theta : z \mapsto az + bz^p + c$$

with $a, b, c \in \mathrm{GF}(p^2)$ and $a^{p+1} \neq b^{p+1}$. If $z \mapsto a'z + b'z^p + c'$ is the inverse map, then one calculates that

$$a' = \frac{a^p}{a^{p+1} - b^{p+1}}, b' = -\frac{b}{a^{p+1} - b^{p+1}}, c' = \frac{bc^p - a^p c}{a^{p+1} - b^{p+1}}.$$

Let $\theta_1 : z \mapsto a_1 z + b_1 z^p + c_1$ and $\theta_2 : z \mapsto a_2 z + b_2 z^p + c_2$ be such maps. If $a_1 = 0$ or $b_1 = 0$, then $\mathcal{S}(\theta_1) \simeq \mathcal{S}(\theta_2)$ if and only if $a_2 = 0$ or $b_2 = 0$. If $a_1 \neq 0 \neq a_2$, then one calculates that $\mathcal{S}(\theta_1) \simeq \mathcal{S}(\theta_2)$ if and only if

(I) $a_2 \neq 0 \neq b_2$ and $(\frac{a_2}{b_2})^{p+1} = (\frac{a_1}{b_1})^{p+1}$; or

(II) $a_2 \neq 0 \neq b_2$ and $(\frac{a_2}{b_2})^{p+1} = (\frac{b_1}{a_1})^{p+1}$.

If θ is a good map, then (i) $a = 0$ or $b = 0$, (ii) $a \neq 0 \neq b$ and $(\frac{a}{b})^{p+1} = -1$, or (iii) $a \neq 0 \neq b$ and $(\frac{a}{b})^{p+1} \neq \pm 1$. Hence

$$A(p^2) = 1 + 1 + \frac{p-3}{2} = \frac{p+1}{2}.$$

(B) Let O be a hyperoval of $\mathrm{PG}(2, q)$, $q = 2^h$, and let $\alpha \in O$. Embed $\mathrm{PG}(2, q)$ as a hyperplane in $\mathrm{PG}(3, q)$; the lines of $\mathrm{PG}(3, q)$ intersecting O in α only form a spread S_α of $T_2^*(O)$. Let $(\mathcal{D}_1, K_1, \Delta_1) = (\mathcal{D}_2, K_2, \Delta_2)$ be the AT of Section 2.2 with $T_2^*(O)$ as corresponding GQ and S_α as associated spread. The groups G_1

and G_2 are isomorphic to $\mathrm{GF}(q),+$. The maps $\theta : K_1 \to K_2$ for which $S'(\theta) = S(T_2^*(O), T_2^*(O), S_\alpha, S_\alpha, L_1^{(1)}, L_1^{(2)}, \theta)$ is a near hexagon are again the maps $z \mapsto \phi(z)+a$, where ϕ is an arbitrary automorphism of $\mathrm{GF}(q),+$ and where a is an arbitrary element of $\mathrm{GF}(q)$. If θ_1 and θ_2 are such maps, then besides the near hexagons $S'(\theta_1)$ and $S'(\theta_2)$ defined here, we also have the near hexagons $S(\theta_1)$ and $S(\theta_2)$ defined in (A). From equations (16), (17), (18) and the treatment given in Section 3, it follows that $S'(\theta_1) \simeq S'(\theta_2)$ implies that $S(\theta_1) \simeq S(\theta_2)$. Since $q(q-1) \leq |\widehat{G}_1|$, we have then the following lower and upper bound:

$$A(q) \leq N(T_2^*(O), T_2^*(O), S_\alpha, S_\alpha) \leq \frac{|GL(h,2)|}{q-1}.$$

(C) Let O_1 and O_2 be two hyperovals of $\mathrm{PG}(2,q)$, $q = 2^h$. A point $\alpha_i \in O_i$ $(i \in \{1,2\})$ defines a spread S_i of $T_2^*(O_i)$. Suppose that there exists no automorphism of $\mathrm{PG}(2,q)$ mapping O_1 to O_2 and α_1 to α_2, then there exists no isomorphism from $T_2^*(O_1)$ to $T_2^*(O_2)$ mapping S_1 to S_2. For, $T_2^*(O_1) \simeq T_2^*(O_2)$ if and only if there exists an automorphism of $\mathrm{PG}(2,q)$ mapping O_1 to O_2 (see [4]) and every automorphism of $T_2^*(O_1)$ is induced by an automorphism of $\mathrm{PG}(3,q)$ fixing O_1. For $i \in \{1,2\}$, let $(\mathcal{D}_i, K_i, \Delta_i)$ be the AT of Section 2.2 with $T_2^*(O_i)$ as corresponding GQ and S_i as associated spread. Since $q(q-1) \leq |\widehat{G}_1|, |\widehat{G}_2| \leq hq(q-1)$, a similar reasoning as above yields

$$1 + \frac{|GL(h,p)| - h(q-1)}{h^2(q-1)^2 q} \leq N(T_2^*(O_1), T_2^*(O_2), S_{\alpha_1}, S_{\alpha_2}) \leq \frac{|GL(h,2)|}{q-1}.$$

7.2 Glued near hexagons arising from $Q(5,q)$

Let $(\mathcal{D}, K, \Delta)$ be the AT from Section 2.3 with $\mathcal{Q}_1 = \mathcal{Q}_2 = Q(5,q)$, $q = p^h$, as corresponding GQ and $S_1 = S_2$ as associated spread. The groups G_1 and G_2 are isomorphic to K. Hence the number of good maps is equal to $(q+1)\phi(q+1)$, with $\phi(q+1)$ the Euler indicator of $q+1$, i.e. the number of integers in the set $\{1,\ldots,q+1\}$ relative prime to $q+1$. We determine now how each good $\theta : K \to K$ looks like. Let $x \in K$ be fixed. For each automorphism χ of K and every $y \in K$, we have

$$\theta_y^\chi(z) = \chi^{-1}(zx^{-1})y = \chi^{-1}(z)\chi(x^{-1})y,$$

for all $z \in K$. Hence, the $(q+1)\phi(q+1)$ good maps are the maps $z \mapsto lz^n$ with $l \in K$ and $(n, q+1) = 1$; let $\theta_1(k) = l_1 k^{n_1}$ be such a map and we determine the number of maps $\theta_2 : k \mapsto l_2 k^{n_2}$, $(n_2, q+1) = 1$, satisfying (16),(17) or (16),(18). Put $\alpha(k) = mk^{p^{\sigma_1}}$ and $\mu(k) = nk^{p^{\sigma_2}}$. If $\mu = \theta_2 \alpha \theta_1^{-1}$, then $l_2 = n l_1^{p^{\sigma_2}} (m^{n_2})^{-1}$ and $n_2 = n_1 p^{\sigma_2 - \sigma_1}$; similarly, if $\mu = \theta_2 \alpha \theta_1$, then $n_1 n_2 = p^{\sigma_2 - \sigma_1}$. Let $X = \{x \in \mathbb{Z} | (x, q+1) = 1\}$, $Y = \{x \in \mathbb{Z} | x \equiv p^i \pmod{q+1} \text{ for some } i \in \mathbb{N}\}$, $X' = X \cap \{0,\ldots,q\}$ and $Y' = Y \cap \{0,\ldots,q\}$, then the number of maps θ_2 for which $S(\theta_2) \simeq S(\theta_1)$ is equal to

$$\begin{aligned}(q+1) \times |\{n \in X' | n_1 n \in Y \text{ or } n_1 n^{-1} \in Y\}| &= 4h(q+1) \text{ if } n_1^2 \notin Y,\\ &= 2h(q+1) \text{ if } n_1^2 \in Y.\end{aligned}$$

Let R denote the number of $n \in X'$ for which $n^2 \in Y$, then the number of nonisomorphic glued near hexagons is equal to

$$B(q) = \frac{(q+1)\phi(q+1) - (q+1)R}{4h(q+1)} + \frac{(q+1)R}{2h(q+1)} = \frac{\phi(q+1)+R}{4h}.$$

The tabel below gives the value of $B(q)$ for all prime powers $q \leq 19$.

q	$B(q)$	q	$B(q)$	q	$B(q)$
2	1	7	2	13	2
3	1	8	1	16	2
4	1	9	1	17	2
5	1	11	2	19	3

7.3 Four new glued near hexagons

Let p be a prime of the form $2^{2^n}+1$. The only known such primes are $p=3$, $p=5$, $p=17$, $p=257$ and $p=65537$. Let $(\mathcal{D}_1, K_1, \Delta_1)$, respectively $(\mathcal{D}_2, K_2, \Delta_2)$, be the AT's from Section 2 yielding $AS(p)$, respectively $Q(5,p-1)$. The groups G_1 and G_2 are isomorphic to the cyclic group of order p and $|\widehat{G}_1| = p(p-1)$, $|\widehat{G}_2| = p2^{n+1}$. From Theorem 7.1, it follows that

$$0 < N(AS(p), Q(5,p-1), S_1, S_2) \leq \frac{p(p-1)}{p(p-1)};$$

hence we proved the following theorem.

Theorem 7.3 *Let p be a prime of the form $2^{2^n}+1$; then there is a unique near hexagon of the form $S(AS(p), Q(5,p-1), S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$.*

If $p=3$, then $AS(p) \simeq Q(5,p-1)$ and we already met the corresponding glued near hexagon.

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