Weighted eigenfunctions and Gauss curvature of conical revolution surfaces

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Abstract

We give a description of Gauss curvatures in revolution surfaces with conical singularities at the extreme opposite points thanks to positive eigenfunctions of an eigenvalue problem in dimension one with a prescribed singular weight.

1 Introduction

Given a revolution surface

$$S = \{(\alpha(v)\cos u, \alpha(v)\sin u, \beta(v)); 0 < u < 2\pi,\ a < v < b\} \tag{1}$$

where $\alpha(v) > 0, \alpha, \beta$ regular functions and supposing the generating curve $\gamma = (\alpha(v), 0, \beta(v))$ parametrized by arc-length, that is

$$\alpha'^2 + \beta'^2 = 1 \text{ in }]a, b[$$

Then the Gauss curvature K of S is given by

$$K = \frac{-\alpha''(v)}{\alpha(v)},\ v \in]a, b[\tag{2}$$

[DC, p. 162].

If α, β are regular up to $[a, b]$ and

$$\begin{cases} \alpha(a) = \alpha(b) = 0,\ 0 < \alpha'(a) \leq 1,\ -1 \leq \alpha'(b) < 0 \\ \beta(a) < \beta(b) \end{cases} \tag{3}$$

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the surface S will have conical singularities at $P_a = (0, 0, \beta(a))$ and $P_b = (0, 0, \beta(b))$ with angles θ_a and θ_b determined by

$$(\cos\theta_a, \sin\theta_a) = (\beta'(a), \alpha'(a)), (\cos\theta_b, \sin\theta_b) = (-\beta'(b), -\alpha'(b))$$

where θ_a, θ_b are in $]0, \pi[$.

We describe the family of K's looking for solutions of the following boundary value problem

$$\begin{cases} \alpha'' + \lambda g(v)\alpha = 0 \text{ in }]a, b[\\ \alpha(a) = \alpha(b) = 0 \\ \alpha(v) > 0 \text{ in }]a, b[\end{cases} \tag{4}$$

which are in $C^2(]a, b[) \cap C^1([a, b])$ and satisfy $\alpha'(a) > 0 > \alpha'(b)$. We do so because for a given weight $g(v)$, in the half-line $\{tg; t > 0\}$ there will be at most one $K = \lambda g$. Functions g will be allowed to have singularities at a and b like simple poles (if they were analytic) by considering natural examples. The main result on (4) for such a g is supplementary to those on the subject found in [DF, M-M].

In [T], M. Troyanov fixes a Riemann surface with a metric ds_0^2 having prescribed conical singularities at a prescribed finite numbers of points and gives rather complete results on the Gauss curvatures on metrics ds^2 conformal to ds_0^2, (i.e. $ds^2 = e^{2f} ds_0^2$). Here we describe curvatures associated to warped singular metrics $\alpha^2(v)du^2 + dv^2$ on $]a, b[\times S^1$ which are not conformally equivalent.

In §2 we give a pointwise necessary condition on K, additional to the integral-ones given in [E-T] and examples motivating the conditions on g which appear in the result on (4) in §3. Finally, building surfaces S having the same curvature $K = \lambda g$ associated to $\{s\alpha_g;\ 0 < s \leq 1\}$ where $||\alpha_g'||_\infty = 1$ is indicated.

2 Necessary condition, examples

The area element of S is $dA = \alpha(v)\, du\, dv$. A curvature K given by (2) satisfies [cf E-T],

$$\int_S K dA = 2\pi(\alpha'(a) - \alpha'(b)) > 0 \tag{5}$$

and

$$\int_S K' dA = -2\pi(\alpha'(a) + \alpha'(b))(\alpha'(a) - \alpha'(b)) \tag{6}$$

(5) implies that K is positive somewhere. A pointwise necessary condition, independent of α, is given by Barta's inequality [B]

$$\sup_{]a,b[} K \geq \left(\frac{\pi}{b-a}\right)^2 \geq \inf_{]a,b[} K \tag{7}$$

For $\varphi_1 = \sin\dfrac{\pi v}{b-a}$, we have

$$\int_S [K - \left(\frac{\pi}{b-a}\right)^2] \varphi_1\, dA = 0 \tag{8}$$

integrating $K\alpha\varphi_1 = -\alpha''\varphi_1$ by parts. If we have one equality in (7), we deduce from (8) that $K \equiv \left(\dfrac{\pi}{b-a}\right)^2$ and $\alpha = \varphi_1$ is a solution which is unique modulo a normalization. We remark that $\left(\frac{\pi}{b-a}\right)^2$ is the only positive constant curvature. This uniqueness of K in $\{tK; t>0\}$ will also hold for non constant K's.

The following examples are characteristic of the type of curvatures we will prescribe.

Example 1. (Small circle). The curve

$$\gamma_s = (\sin v - \sin\delta, 0, -\cos v + \cos\delta),\ v \in [\delta, \pi-\delta]$$

where $0 < \delta < \frac{\pi}{2}$, describes a circular arc of length $\pi - 2\delta$ parametrized by arc-length. The corresponding surface S_s has conical singularities with same angle at $(0,0,0)$ and $(0,0,2\cos\delta)$. The curvature

$$K_s = \frac{\sin v}{\sin v - \sin\delta}, \quad]\delta, \pi-\delta[$$

satisfies $K_p(v) > 0$ and has simple poles at δ and $\pi - \delta$.

Example 2. (Big circle)

$$\gamma_b = (\sin v + \sin\delta, 0, \cos\delta - \cos v),\ v \in [-\delta, \pi+\delta]$$

describes the complementary circular arc to γ_s. The surface S_b has singularities at the same points than S_s with complementary angles to those of S_s. The Gauss curvature of S_b is

$$K_b = \frac{\sin v}{\sin v + \sin\delta},\ v \in]-\delta, \pi+S[.$$

K_p changes sign at $v = 0$ and $v = \pi$ and has also simple poles at $-\delta$ and $\pi + \delta$.

3 A sufficient condition

Taking into account the examples in §2 we introduce a condition on g to obtain a positive eigenfunction α of (4) in the Sobolev space $H_0^1(]a,b[)$. We proced as in [M-M,DF], consequently we only detail the differences in our proof.

Theorem 3.1. *Let $g \in C(]a,b[)$ be such that*

$$d_a = \lim_{v\to a^+}(v-a)g(v),\ d_b = \lim_{v\to b^-}(b-v)g(v) \tag{9}$$

exist. If g is positive at one point, then there is a unique positive λ such that (4) has a solution $\alpha \in H_0^1(]a,b[)$. Moreover $\alpha \in C^1([a,b])$ and if $d_\alpha d_b \neq 0$ we have $\alpha'(a) > 0 > \alpha'(b)$.

Proof. We may suppose $[a,b] = [0,L]$. Let $\varphi \in C_c^1(]0,L[)$ and $\psi \in C_0^1([a,b])$, i.e. $\psi \in C^1([a,b])$, $\psi(0) = \psi(L) = 0$. From

$$\begin{aligned}\int_0^L g\psi\varphi\,dv &= \int_0^{L/2} vg(v)\left(\frac{1}{v}\int_0^v \psi'(t)\,dt\right)\varphi(v)\,dv \\ &+ \int_0^{L/2}(L-v)g(v)\left(\frac{1}{L-v}\int_{L-v}^L \psi'(t)\,dt\right)\varphi(v)\,dv.\end{aligned} \tag{10}$$

and from Hardy's inequality : $||\frac{1}{v}\int_0^v w(t)\,dt||_{L^2(\mathbb{R}_+)} \leq 2||w||_{L^2(\mathbb{R}_+)}$ applied to ψ extended by 0 out of [0,L] and to $\tilde{\varphi}(v) = \psi(L-v)$ on [0,L] also extended by 0 to $\mathbb{R}_+$, we deduce

$$|\int_0^L g\psi\varphi dv| \leq ||vg(v)\varphi(v)||_{L^2} 2||\psi'||_{L^2} + ||(L-v)g(v)\varphi(v)||_{L^2} 2||\psi'||_{L^2}$$

From (10) we obtain

$$|\int_0^L g\psi\varphi\, dv| \leq M||\varphi||_{L^2}||\psi'||_{L^2},\ \varphi \in C_c^1(]0,L[),\ \psi \in C_0^1([0,L]) \tag{11}$$

where $M = 2(||vg(v)||_\infty + ||(L-v)g(v)||_\infty)$.

As in [M-M,DF] the map $\varphi \to T\varphi :\ H_0^1(]0,L[) \to H_0^1(]0,L[)$ defined by

$$\int_0^L (T\varphi)'\psi'\, dv = \int_0^L g\varphi\psi\, dv,\ \psi \in H_0^1(]0,L[)$$

is then linear, compact and symmetric for the scalar product $\int_0^L \varphi'\psi'\, dv$ in $H_0^1(]0,L[)$. Non zero eigenvalues μ of T correspond to eigenvalues $\lambda = \mu^{-1}$ of (4). The hypothesis $g(v_0) > 0$ for some $v_0 \in]0,L[$ gives that the eigenvalues $\lambda \geq 0$ of (4) form a sequence $0 < \lambda_k < \lambda_{k+1},\ k = 1,2,\ldots$ with $\lim_{k\to\infty} \lambda_k = \infty$. The first one λ_1 called principal eigenvalue is simple and is the only λ_k with a positive eigenfunction $\varphi_1(v) \underset{\neq}{>} 0$ on $]0,L[$. Besides $\varphi_1(v) > 0$ on $]0,L[$ and $\varphi_1 \in C_0([0,L])$.

The type of singularity of $K(v) \equiv \lambda_1 g(v)$ at $v = 0$ implies for $\alpha = \varphi_1$ that $\lim_{v\to 0^+} v\alpha''(v) = -\lim K(v)\alpha(v) = 0$, so $v\alpha''(v) \in C([0,\frac{L}{2}]$ and $va''(v) = h'(v)$ on $]0,L[$, where $h(v) = v\alpha'(v) - \alpha(v)$. Hence $\lim_{v\to 0^+}\frac{h(v)}{v} = \lim_{v\to 0^+}\alpha'(v) - \frac{\alpha(v)}{v} = 0$ also, so $\alpha'(0^+) \equiv \alpha'(0)$ exists. Analogously $\alpha'(L^-) \equiv \alpha'(L)$ exists and $\alpha \in C_0^1([a,b])$. Finally $\lambda = \lambda_1$, $\alpha = \varphi_1$ is our solution.

If $d_a \equiv \lim_{v\to 0^+} vg(v) \neq 0$, we have a $\delta > 0$ such that $g(v) > 0$ on $]0,\delta[$ and $vg(v)$ is continuous and bounded on $]0,\delta[$ and $\alpha'' + \lambda g\alpha = 0$ on $]0,\delta[$ with $-\alpha$ having a as maximum value attained at 0. These four properties and a well adapted maximum principle for (9) [P-W, Th. 4, p. 7] insure $\alpha'(0) > 0$. Also $\alpha'(L) < 0$ follows. Q. E. D.

Remark 3.2. The existence of λ and α holds if $(v-a)g(v)(b-v)$ is bounded in $]a,b[$. Conditions (9) with $d_a d_b \neq 0$ are meaningful for $g(v)$ unbounded. If $g(v)$ is bounded (so $d_a = d_b = 0$), from $g = g^+ - g^-$ and $-\alpha'' - \lambda g^+\alpha = \lambda g^-\alpha$ we have $\alpha'(a) > 0 > \alpha'(b)$. [DF, Th. 1.17].

4 Building S

Given g fulfilling the hypothesis of the preceding theorem, there is only one $\alpha = \alpha_g \in C^2(]a,b[) \cap C^1([a,b])$ such that $||\alpha_g'||_\infty = 1$, $\alpha(v) > 0$.

If $g(v) > 0$ on $]a, b[$, then $K(v) = \lambda g(v) > 0$ and $-\alpha''(v) = K(v)\alpha(v) > 0$ also. Hence α is concave on $[a, b]$ and

$$\lim_{v\to a^+} K(v)\alpha(v) = \lim_{v\to a^+} K(v)(v-a)\frac{a(v)}{v-a} = \lambda d_a \alpha'(a)$$

implies $\alpha \in C^2([a, b])$ and α' strictly decreasing in $[a, b]$ with $||\alpha'||_\infty = 1$. If $0 < \alpha'(a) < 1$ we deduce $\alpha'(b) = -1$. Defining

$$\beta(x) = \int_a^v (1 - \alpha'(t)^2)^{1/2}\, dt, \tag{12}$$

the surface generated by $(\alpha, 0, \beta)$ will have a conical singularity at $(0, 0, 0)$ with angle $\theta_a \in]0, \frac{\pi}{2}[$ and of angle $\theta_b = \frac{\pi}{2}$ at $(0, 0, \beta(b))$ *i.e.* no singularity. If we consider $\rho\alpha$, $0 < \rho < 1$, (12) gives β_ρ and we obtain a family of surfaces S_ρ with conical singularities with the same curvature $K(v) = \lambda g(v)$. If $\alpha'(a) = -\alpha'(b) = 1$, S will have no singularities, however S_ρ will have.

Two examples illustrating this case are $g \equiv 1$ on $[c, b] = [0, L]$, then $\lambda = \left(\frac{\pi}{L}\right)^2$, $K = \left(\frac{\pi}{L}\right)^2$ and $\alpha_1 = \frac{L}{\pi}\sin\frac{\pi}{L}v$ satisfies $\alpha_1'(0) = -\alpha'(L) = 1$. The surface S is a sphere of radius $\frac{L}{\pi}$. The other example is $g(v) = [v(L-v)]^{-1}$ on $]0, L[$, then $\lambda = 2$ and $\alpha_g = \frac{1}{L}v(L-v)$.

Finally, if g changes sign a finite number of times (hence K, as in the "big circle") that is if g has a finite number of zeros in $[a, b[$ and at each zero v_0, we have $g'(v_0) \neq 0$, then $-\alpha_g'' = \lambda g \alpha_g$ has the same zeros, so $|\alpha_g'(v)| = 1$ has a finite number of solutions. For the associated partition of $]a, b[$, α will be successively convex then concave or vice-versa on contiguous subintervals. A convenient choice of the sign in $\pm(1 - \alpha'(t)^2)^{1/2}$ at each subinterval and (12) define β and we obtain $S = S_g$ of class C^1.

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