Transversal spreads

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Abstract

Knarr shows that given a derivable affine plane, every line not in the associated derivable net produces a spread which is a dual spread in some $PG(3, K)$, for K a skewfield. More generally, if a derivable net has a transversal T, there is also a spread in $PG(3, K)$. This article generalizes results of Knarr by an investigation of spreads in three-dimensional projective spaces realized as transversals to derivable nets. As an application of the ideas, the finite derivable affine planes which are 'partially flag-transitive' are determined.

1 Introduction.

The author's work on derivable nets shows that every derivable net is combinatorially equivalent to a three-dimensional projective space over a skewfield K. More precisely, the points and lines of the net become the lines and points skew to a fixed line N. Recently, Knarr [18] proved that, for any derivable affine plane π, every line ℓ not belonging to the derivable net embeds to a set of lines in the projective space such this set union N becomes a spread $S(\ell)$ of $PG(3, K)$ which is also a dual spread.

Hence, it is of interest to ask what sorts of spreads arise from a given derivable affine plane. We first point out that it is not actually the existence of the derivable plane which provides the spread nor that of the stronger condition that there is an affine plane containing a derivable net but simply the existence of a 'transversal' to a derivable net. That is, given a transversal to a derivable net, there is a corresponding spread of the projective space associated combinatorially or 'geometrically' by the embedding process. The nature of the transversal determines whether the spread constructed is also a dual spread.

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Definition 1. *(1) A spread of a three-dimensional projective space over a skewfield shall be said to be a 'transversal spread' if and only if it arises from a transversal to a derivable net by the embedding process mentioned above.*

(2) A transversal spread shall be said to be a 'planar transversal-spread' if and only if there is an affine plane π containing the derivable net such that the transversal to the derivable net is a line of π.

We show that every spread is a transversal spread and, in fact, every spread is a planar transversal-spread. But more can be said concerning transversal spreads by applying some ideas of Ostrom regarding extension of derivable nets. We show how the extension techniques of Ostrom can be related to the question of transversal spreads and show how to interconnect these ideas with the geometric embedding process to determine that all spreads are planar transversal-spreads.

Using these ideas, we have a framework to be able to discuss the structure of derivable affine planes which admit various collineation groups. In particular, we show that a finite derivable affine plane of order q^2 which admits a (linear) collineation group leaving the derivable net $\mathcal{D}$ invariant and acting flag-transitively on flags on lines not in $\mathcal{D}$ must always be a semi-translation plane which admits either an elation group of order q or a Baer group of order q. When the semi-translation plane is a translation plane, the structure of the spread is more-or-less determined in this situation.

The assumption on partial flag-transitivity may be relaxed to assumptions on orders of certain groups. In this case, we are able to provide some structure theory for finite affine planes as follows:

Theorem 2. *(1) If a derivable affine plane π of order q^2 admits a linear p-group of order q^5 if q is odd or $2q^5$ if q is even that fixes the derivable net $\mathcal{D}$ then π contains a group which acts transitively on the affine points.*

(2) Furthermore, the group contains either an elation group of order q or a Baer group of order q with axis a subplane of $\mathcal{D}$. If the order of the stabilizer of a point H is at least $2q$ then the order is $2q$ and H is generated by an elation group of order q and a Baer involution with axis in $\mathcal{D}$ or by a Baer group of order q and an elation.

(3) π is a nonstrict semi-translation plane of order q^2 admitting a translation group of order q^3p^γ. Furthermore, either π is a translation plane or there is a unique $((\infty), \ell_\infty)$-transitivity and the remaining infinite points are centers for translation group of orders qp^γ.

(4) If π is non-Desarguesian in the elation case above then π admits a set of q derivable nets sharing the axis of the elation group of order q.

2 Extensions of Derivable Nets.

As is well known, the concept of the derivation of a finite affine plane was conceived by Ostrom in the 1960's. During this period, one of the associated problems that Ostrom considered concerned the extension of the so-called derivable nets to either a supernet or to an affine plane. At that time, coordinate geometry was the primary model in which to consider extension questions. With a particular vector-space structure assumed for a derivable net, Ostrom ([19]) was able to show that any transversal to such a finite derivable net allowed its embedding into a dual translation plane. The author was able to extend this to the arbitrary or infinite case. Before we proceed with other models, we review some of the definitions and recall some of the results of the coordinate or algebraic method.

Definition 3. *Let K be a skewfield and V a right two-dimensional vector space over F. A 'vector-space derivable net' $\mathcal{D}$ is a set of 'points' (x, y) $\forall x, y \in V$ and a set of 'lines' given by the following equations:*

$$x = c, \; y = x\alpha + b \; \forall c, b \in V, \, \forall \alpha \in F.$$

Definition 4. *A 'transversal' T to a net $\mathcal{N}$ is a set of net points with the property that each line of the net intersects T in a unique point and each point of T lies on a line of each parallel class of $\mathcal{N}$.*

A 'transversal function' f to a vector-space derivable net is a bijective function on V with the following properties:

(i) $\forall c, d$, $c \neq d$ of V, $f(c) - f(d)$ and $c - d$ are linearly independent,

(ii) $\forall \alpha \in F$ and $\forall b \in V$, there exists a $c \in V$ such that $f(c) = c\alpha + b$.

It follows from Johnson [8] that transversals and transversal functions to vector-space derivable nets are equivalent, each giving rise to the other. It should be noted that everything can be phrased over the 'left' side as well. That is, a 'right vector-space net' over a skewfield F is naturally a 'left vector-space net' over the associated skewfield F^{opp} where multiplication $\bullet$ in F^{opp} is defined by $a \bullet b = ba$ where juxtaposition denotes multiplication in F.

Theorem 5. *(Johnson [8] (1.7)) Let $\mathcal{D}$ be a vector-space derivable net and let T be a transversal. Then there is a transversal function f on the associated vector space V such that $\mathcal{D}$ may be extended to a dual translation plane with lines given as follows:*

$$x = c, y = f(x)\alpha + x\beta + b \; \forall \alpha, \beta \in K \text{ and } \forall b, c \in V.$$

Conversely, any dual translation plane whose associated translation plane has its spread in $PG(3, K)$ may be constructed from a transversal function as above.

Proof. The proof of the converse is not properly given in [8], however a dual translation plane arising from a translation plane with spread in $PG(3, K)$ has components of the general form

$$x = 0, \; y = x * m + b \; \forall m, b \in K \oplus K,$$

where $(K \oplus K, +, *)$ is a coordinatizing right quasifield. We generally consider a left vector space setting $x * m = m \circ x$ where the multiplication $m \circ x$ arises from the translation plane associated with the spread. In this instance, the coordinate structure for the dual translation plane becomes a right two-dimensional vector space over K and $m = e * \alpha + \beta$ where $\{1, e\}$ is a right basis. Then $y = x * m + b = (x * e) * \alpha + x * \beta + b$ and with $f(x) = x * e$, we obtain the form demanded of the extension process. The reader might note that these ideas will be considered more completely in the next section. ∎

We will be recalling a more geometric approach shortly and to distinguish between the two, we formulate the following definition:

Definition 6. *Let $\mathcal{D}$ be a (right) vector-space derivable net with transversal function f then the dual translation plane with lines given by*

$$x = c, y = f(x)\alpha + x\beta + b \ \forall \alpha, \beta \in K \ \text{ and } \forall b, c \in V$$

shall be called the 'algebraic extension' of $\mathcal{D}$ by f and the set of such shall be termed the set of 'algebraic extensions of $\mathcal{D}$'.

3 Geometric Extension.

Historically, perhaps the most important question left open by the coordinate approach was whether 'derivation' could be considered a geometric construction and the text of the author 'Subplane Covered Nets' [15] examines this question in detail. In particular, the structure of a derivable net $\mathcal{D}$ has been determined in Johnson [11] and [12]. The points and lines can be embedded as the lines and points of a 3-dimensional projective space Σ isomorphic to $PG(3, K)$, K a skewfield, which are skew or non-incident with a fixed line N. The parallel classes of the net become the planes of Σ containing N and the Baer subplanes of the net become the planes of Σ which do not contain N. It then turns out that every derivable net is a 'right' vector-space net over a skewfield F and, of course, may be considered a 'left vector-space net' over F^{opp}. In particular, when the net is considered a 'left net' the embedding into the projective space is determined by a left 4-dimensional vector space over the associated skewfield.

When one has a derivable affine plane π, [18], Knarr asked of the general nature of the lines of an affine plane π containing $\mathcal{D}$ in terms of the embedding. Knarr showed that every line of $\pi - \mathcal{D}$ produces a spread of lines of Σ that contain N. Also, this spread is a dual spread.

Since we are interested in the more general situation, we assume only that there is a transversal T to the derivable net $\mathcal{D}$ which defines a simple net extension $\mathcal{D}^{+T}$. In Johnson [14], it is pointed out that it is possible to embed any derivable net into an affine plane where the affine plane may not be derivable itself. Hence, we distinguish between having a net extension and having a 'derivable-extension' by which we mean that each Baer subplane of the net remains Baer when considered within the extension net; each point is on a line of each subplane, taken projectively (the subplane structure is 'point-Baer') and each line is incident with a point of each subplane, taken projectively (the subplane structure is 'line-Baer'). In essence, we would merely require that T intersect each Baer subplane.

Theorem 7. *(see Knarr [18]) Let $\mathcal{D}$ be a derivable net and assume that T is a transversal to $\mathcal{D}$ defining a extension net $\mathcal{D}^{+T}$.*

(1) Then the points of T determine a spread $S(T)$ of lines in the projective space Σ associated with $\mathcal{D}$ that contains the special line N.

(2) If the net extension is a derivable-extension then $S(T)$ is a dual spread.

(3) Conversely, if $S(T)$ is a dual spread, for each line of $T - \mathcal{D}$, then the net extension is a derivable extension.

Proof. Knarr's proof generalizes to cover this situation but we have changed the assumptions so much that we repeat it here.

A point of $\Sigma - N$ is a line of the net $\mathcal{D}$ which must intersect T in a unique point as $\mathcal{D}^{+T}$ is a net. Hence, every point of $\Sigma - N$ is incident with a unique line of $S(T) - N$ and it thus follows that every point of Σ is incident with a unique line of $S(T)$. This proves (1).

Now assume that the net extension is a derivable extension. To show that $S(T)$ is a dual spread, we need to show that every plane contains a unique line of $S(T)$. Since the planes not containing N correspond to Baer subplanes of $\mathcal{D}$, the question becomes whether each Baer subplane shares a unique net-point of T. Since each line of the net which is not in $\mathcal{D}$ shares a point, taken projectively, with each Baer subplane, it is immediate that this point is affine; i.e. an actual point of T. Hence, each plane of Σ contains exactly one line of $S(T)$; $S(T)$ is a dual spread.

It also follows that if $S(T)$ is a dual spread then the line T must share a net-point with each Baer subplane. Furthermore, since $\mathcal{D}$ is a derivable net, every point is incident with a line of each Baer subplane. Hence, it is now immediate that $S(T)$ is a dual spread for each line of T exterior to the derivable net $\mathcal{D}$ if and only if the net extension is a derivable-extension. ∎

Now we see that any derivable net is, in fact, a vector-space derivable net so the two approaches merge.

Definition 8. *Let $\mathcal{D}$ be any derivable net. Then $\mathcal{D}$ may be considered a 'left' vector-space net over a skewfield K. Let Σ denote the three-dimensional projective space $PG(3,K)$ for K a skewfield, with special line N defined combinatorially by $\mathcal{D}$ and so that $\mathcal{D}$ may be embedded in Σ. Let T be any transversal to $\mathcal{D}$ and let $S(T)$ denote the spread of Σ defined by the net-points of T as lines of Σ together with the line N. Let $\pi_{S(T)}$ denote the associated translation plane and let $\pi^D_{S(T)}$ denote any affine dual translation plane whose projective extension dualizes to $\pi_{S(T)}$, taken projectively. Then $\pi^D_{S(T)}$ contains a derivable net isomorphic to $\mathcal{D}$ but considered as a 'right' vector-space net over the skewfield K^{opp}.*

We shall call $\pi^D_{S(T)}$ a 'geometric extension of $\mathcal{D}$' by $S(T)$.

Hence, given a derivable net $\mathcal{D}$ with transversal T, we may consider two possible situations. First of all, we know that $\mathcal{D}$ may be considered a right vector-space net over a skewfield F and there is an associated transversal function which we may use to extend $\mathcal{D}$ to a dual translation plane π^D_f (the algebraic extension). On the other hand, we may consider $\mathcal{D}$ as a left vector-space net over $F^{opp} = K$, embed the net combinatorially into a (left) three-dimensional projective space Σ isomorphic to $PG(3,K)$, with distinguished line N and then realize that the transversal T, as a set of points of $\mathcal{D}$, is a set of lines whose union with N, is a spread of Σ which defines a translation plane with an associated dual translation plane $\pi^D_{S(T)}$ (the geometric extension).

Hence, we arrive at the following fundamental question:

Given a derivable net D with transversal T and associated transversal function f, is the algebraic extension of D by f isomorphic to a geometric extension of D by $S(T)$?

Before we consider this question, we note the following connection with spreads in three-dimensional projective space and transversals to derivable nets. However, the reader might note that there are possible left and right transversal functions depending on whether the vector-space derivable net is taken as a left or a right vector space and this will play a part in our discussions.

Theorem 9. *The set of spreads in* 3*-dimensional projective spaces is equivalent to the set of transversals to the set of derivable nets; every spread is a transversal spread.*

Proof. Let K be any skewfield and let Σ be isomorphic to $PG(3,K)$ and let S be a spread of Σ. Choose any line N of S and form the corresponding derivable net $\mathcal{D}$ defined combinatorially with lines the points of $\Sigma - N$. Then $S - N$ is a set of lines of Σ and hence points of $\mathcal{D}$ such each point of $\Sigma - N$ 'line of $\mathcal{D}$' is incident with a unique line 'point of $\mathcal{D}$'. Hence, $S - N$, as a set of net ponts, is a transversal to the derivable net $\mathcal{D}$. ∎

In the next sections, we show that the algebraic and geometric extensions processes are equivalent and examine the nature of the transversal extensions.

4 Planar Transversal Extensions

Theorem 10. *(see Knarr [18] (2.7)) Let $\mathcal{P}$ be any spread in $PG(3,K)$, for K a skewfield. Let $\mathcal{D}$ be a derivable net and T a transversal to it which geometrically constructs $\mathcal{P}$ by the embedding process.*

Then there is a dual translation plane π_D^T constructed by the algebraic extension process. For any line of $\pi_D^T - \mathcal{D}$, the spread in the associated three-dimensional projective geometry obtained by the geometric embedding process produces a translation plane with spread $S(T)$ isomorphic to $\mathcal{P}$ and whose dual is isomorphic to π_D^T.

Hence, all spreads are planar transversal spreads.

Proof. We refer the reader to Johnson [15] for any background information not given explicitly. The reader might note that when a translation plane is defined from a spread in $PG(3,K)$, it is usually most convenient to consider the vector space as a left vector space and spread components left 2-dimensional vector spaces of the general form $x = 0$, $y = xM$ where M is a 2×2 matrix over K. Ultimately, we shall be constructing a derivable net from $K \oplus K$ considered as a right K-space. On the other hand, following the structure theory of the embedding of the derivable net into $PG(3,K)$ considered as a left vector space, we obtain a derivable net with components $x = 0, y = \alpha x$ as opposed to $x = 0, y = x\alpha$. Furthermore, we may define $x \diamond \alpha = \alpha x$ which forces the vector-space derivable net to be defined over K^{opp} as opposed to K. However, we shall see that the derivable net arising from a spread and, hence, a translation plane, is a right two-dimensional vector space over K. To be clear, a derivable net written as $x = 0, y = x\alpha$ for α in K embeds within $PG(3,K)$ considering the associated vector space as a 'right' vector space. If we start with a 'left' vector space to facilitate the spread and the translation plane, we end up with a derivable net contained in a dual translation plane written as $x = 0, y = x\alpha$ with α in K which then embeds within $PG(3,K)$ as a 'right' vector space which may be taken as a 'left' K^{opp} space $PG(3,K^{opp})$.

Given a derivable net $\mathcal{D}$, there is a skewfield K such that the set of points of the net is $K \oplus K \oplus K \oplus K$ and there is an associated four-dimensional K-vector space V, which we fix as a left space, such that the corresponding three-dimensional projective space Σ isomorphic to $PG(3, K)$ has a fixed line N generated as $\langle (1,0,0,0), (0,1,0,0) \rangle$ and such that the points of $\mathcal{D}$ (d_1, d_2, d_3, d_4) correspond to $\langle (d_1, d_2, 1, 0), (d_3, d_4, 0, 1) \rangle$. In this context, the derivable net will have components written in the form $x = 0, y = \alpha x$ for all $\alpha \in K$. We note that the set of vectors of $y = \alpha x$ is not necessarily always a left K-subspace, although each point of the net may be considered a left K-subspace embedded in the associated projective space.

When there is a transversal T to $\mathcal{D}$, we may form the algebraic extension process, dualize, and construct a 'left' spread in $PG(3, K)$. We will be taking the spread in $PG(3, K)$, more properly a 'spread set' and forming the associated translation plane. We choose the spread set as follows: We choose a particular set of three lines and vectorially denote these by $x = 0, y = 0, y = x$. Then, any other line (including $y = 0$ and $y = x$) has the following form:

$$y = x \begin{bmatrix} g(t,u) & f(t,u) \\ t & u \end{bmatrix} \forall u, t \in K$$

where g and f are functions from $K \times K$ to K, and x and y are denoted by row 2-vectors over K.

We define a multiplication

$$x \circ (t, u) = x \begin{bmatrix} g(t,u) & f(t,u) \\ t & u \end{bmatrix}.$$

Note that we assume when $t = 0$ and $u = 1$, we obtain $y = x$ so that $g(0, 1) = 1$ and $f(0, 1) = 0$.

To define an associated dual translation plane, we define

$$x * m = m \circ x.$$

However, when $x \in K$, we see that the coordinate structure for the dual translation plane contains K^{opp} instead of K and is a 'right' two-dimensional K^{opp} vector space. Furthermore, we may take lines to have the following equations:

$$x = c, \ y = x * m + b \ \forall c, m, b \in K^{opp} \oplus K^{opp}.$$

We note that allowing $(0, \alpha) = \alpha \in K$, we have a dual translation plane containing a vector-space derivable net defined by lines

$$x = c, y = x * \alpha + b \ \forall \alpha \in K^{opp} and \ \forall b, c \in K^{opp} \oplus K^{opp}$$

which is isomorphic to the original net $\mathcal{D}$. Now let $\{e, 1\}$ be a right K^{opp}-basis for $K^{opp} \oplus K^{opp}$ so that a general element $m = \alpha * e + \beta = (\alpha, \beta)$. Since $x * m = (x * e) * \alpha + x * \beta$, we have the representation of the lines of the dual translation plane as given in the introduction. Note that basically all that we have done is return to the 'right' vector-space derivable net over K^{opp} from which we started. The transversal T is simply a line not in the derivable net.

Now we combine the two concepts and consider the spread $S(T)$ geometrically constructed and arising from a line of the form $y = x * (t, u) + (b_1, b_2)$ (i.e. the transversal T).

We note that $y = x * (t, u) + (b_1, b_2) = (t, u) \circ (x_1, x_2) + (b_1 b_2)$, where

$$\begin{aligned} (t,u)\circ(x_1,x_2) &= (t,u)\begin{bmatrix} g(x_1,x_2) & f(x_1,x_2) \\ x_1 & x_2 \end{bmatrix} \\ &= (tg(x_1,x_2)+ux_1, tf(x_1,x_2)+ux_2), \end{aligned}$$

is the following set of points:

$$\{(x_1, x_2, tg(x_1,x_2)+ux_1+b_1, tf(x_1,x_2)+ux_2+b_2); b_1, b_2 \in K\}.$$

Now we have a delicate issue. In order to consider this set of points as 'left' vectors so as to apply the appropriate embedding as 'left' 2-dimensional vector spaces or perhaps 'left lines', we need to consider the vector subspace as a left space over K.

Now each of these points embeds as a 'left' line (a two-dimensional left K subspace) as

$$\begin{aligned} & \langle (x_1,x_2,1,0), (tg(x_1,x_2)+ux_1+b_1, tf(x_1,x_2)+ux_2+b_2,0,1)\rangle \text{ which is} \\ & \alpha(x_1,x_2,1,0)+\beta(tg(x_1,x_2)+ux_1+b_1, tf(x_1,x_2)+ux_2+b_2,0,1) \\ = & (\alpha x_1+\beta(tg(x_1,x_2)+ux_1+b_1), \alpha x_2+\beta(tf(x_1,x_2)+ux_2+b_2), \alpha, \beta) \\ \forall \alpha,\beta \in & K. \end{aligned}$$

To reconstruct a spread, we choose to reconstruct a spread set, hence, letting

$$\begin{aligned} \widehat{x_2} &= \alpha x_1+\beta(tg(x_1,x_2)+ux_1+b_1), \\ \widehat{x_1} &= \alpha x_2+\beta(tf(x_1,x_2)+ux_2+b_2), \end{aligned}$$

it follows that when there is an inverse, we have:

$$(\widehat{x_1},\widehat{x_2})\begin{bmatrix} x_2 & x_1 \\ tf(x_1,x_2)+ux_2+b_2 & tg(x_2,x_2)+ux_1+b_1 \end{bmatrix}^{-1} = (\alpha,\beta).$$

When $x_1 = x_2 = 0$ then we obtain the subspace generated by

$$(\widehat{x_1},\widehat{x_2}) = (b_2, b_1).$$

Now translate by adding $-(b_2, b_1)$. We note that, in this form, N has equation $y = 0$.

Now change the spread set by applying the mapping $(x, y) \longmapsto (y, x)$ so that now N has the form $x = 0$, and, generally, we have the spread represented as

$$x=0, y=0, y = x\begin{bmatrix} v & s \\ tf(s,v)+uv & tg(s,v)+us \end{bmatrix} \ \forall v,s \in K.$$

Now change bases by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

to obtain the form of the spread as:

$$x = 0, y = 0, y = x \begin{bmatrix} tg\,(s,v) + us & tf(s,v) + uv \\ s & v \end{bmatrix} \forall v, s \in K.$$

For fixed elements $t \neq 0$ (as we must have a proper transversal to the derivable net) and u in K, we have

$$\begin{bmatrix} tg\,(s,v) + us & tf(s,v) + uv & v \\ s & v & \end{bmatrix} = \begin{bmatrix} t & u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g(s,v) & f(s,v) \\ s & v \end{bmatrix},$$

a basis change again by

$$\begin{bmatrix} A & 0_2 \\ 0_2 & I_2 \end{bmatrix}$$

where $A = \begin{bmatrix} t & u \\ 0 & 1 \end{bmatrix}$ transforms the spread into the form:

$$x = 0, y = x \begin{bmatrix} g(s,v) & f(s,v) \\ s & v \end{bmatrix} \forall s, v \ \in K.$$

Hence, the geometric extension process produces (by dualization) the original spread constructed from the algebraic extension process (by dualization). ∎

5 Planar Transversal-Spreads and Dual Spreads.

We have not yet dealt with the possibility that a planar transversal-spread may not actually arise from a derivable affine plane, that it may be possible that the spread is not a dual spread.

In Johnson [14], similar constructions to the following are given and the reader is referred to this article for additional details.

Theorem 11. *Let $\mathcal{D}$ be a derivable net and let Σ be isomorphic to $PG(3,K)$ and correspond to $\mathcal{D}$ with special line N. If K is infinite then there exists a dual translation plane π extending $\mathcal{D}$ and a line T of $\pi - \mathcal{D}$ such that $S(T)$ is a spread of $PG(3,K)$ which is not a dual spread. In particular, if K is a field then $S(T)$ is non-Pappian.*

There exist planar transversal-spreads which are not dual spreads.

Proof. As noted above, if K is infinite, we may embed $\mathcal{D}$ into a non-derivable dual translation plane. Hence, there exists a line T such that there is some Baer subplane which does not intersect T in an affine point. Therefore, $S(T)$ is not a dual spread. If K is a field then any Pappian spread in $PG(3,K)$ is a dual spread (see e.g. Johnson [15]). Thus, $S(T)$ is non-Pappian. ∎

6 Translation Extension-Nets.

Suppose that $\mathcal{D}$ is a derivable net and there exists a transversal T and construct the spread $S(T)$. In Knarr [18], the question was raised when $S(T)$ is Pappian or what happens when $\mathcal{D}$ is contained in a translation plane. By the previous sections, we may apply the algebraic extension process to consider such questions. In particular, there are lines $x = c$, $y = f(x)\alpha + x\beta + b$ which define any affine plane containing the derivable net $\mathcal{D}$. Hence, if $\mathcal{D}$ is contained in a translation plane then the dual translation plane containing $\mathcal{D}$ is a translation plane which implies that $S(T)$ is a semifield spread. Knarr observes this fact by noting that the 'translation' collineation group of the derivable net would then act on the net extended by the transversal implying a collineation group fixing a component N and transitive on the remaining components of the spread (points of T). Hence, we obtain:

Theorem 12. *(see Knarr [18], also see Johnson [8]) Let $\mathcal{D}$ be a derivable net and let T be a transversal. If the extension net $\langle \mathcal{D} \cup \{T\} \rangle$ defined by $\mathcal{D} \cup \{T\}$ is a translation net then the spread $S(T)$ defines a semifield plane.*

Furthermore, all semifield spreads in $PG(3, K)$, for K a skewfield, are 'semifield planar transversal-spreads' (arising from semifield planes).

Proof. Apply the main result of the section on planar transversal-spreads. ∎

Remark 13. *Let $\mathcal{P}$ be any non-Desarguesian semifield spread in $PG(3, K)$, for K a skewfield. If we choose the axis of the affine elation group to be N and view the spread as a transversal to a derivable net with the embedding in $PG(3, K) - N$, the corresponding dual translation plane will be a semifield plane. On the other hand, if any other line of $\mathcal{P}$ is chosen as N in the embedding, the affine dual translation plane will not be an affine semifield plane. So, a semifield spread in $PG(3, K)$ could arise as a planar transversal-spread without the affine plane containing the derivable net being an affine semifield plane.*

With the above remark in mind, we now examine the semifield spreads which can be obtained when $\langle \mathcal{D} \cup \{T\} \rangle$ is a translation extension-net.

From the section on algebraic and geometric extensions, the affine plane containing the derivable net will correspond to the dual translation plane side where the components are left subspaces over the skewfield K^{opp}, provided the geometric embedding is in the left projective space $PG(3, K)$. By the arguments of Johnson [13], there is a vector space V over a prime field $\mathcal{P}$ of the form $W \oplus W$ such that points are the vectors (x, y) for $x, y \in W$ and we may choose a basis so that $x = 0, y = 0, y = x$ belong to the derivable net $\mathcal{D}$. We want to consider the derivable net as a right vector-space net over a skewfield at the same time we are considering the vector space and the components of the derivable net as left spaces over the same skewfield. Furthermore, there is a skewfield K^{opp} such that $W = K^{opp} \oplus K^{opp}$ as a left K^{opp}-vector space and components of $\mathcal{D}$ may be represented as follows:

$$x = 0, y = x \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}; B \in K^{opp}.$$

We again note that the components of $\mathcal{D}$ are not necessarily all right K^{opp}-subspaces, although we say that $\mathcal{D}$ is a 'right' vector-space net over K^{opp}. We note that following the ideas in the section on algebraic and geometric extensions, we are working in the dual translation plane side which contains the derivable net. The translation plane obtained by dualization has its spread in $PG(3,K)$. Recall that this means that the so-called 'right-nucleus' of the semifield in question is K^{opp}.

It also follows that any translation net has components of the general form $y = xT$ where T is a $\mathcal{P}$-linear bijection of W. If W is decomposed as $K^{opp} \oplus K^{opp}$ over the prime field $\mathcal{P}$, choose any basis $\mathcal{B}$ for K^{opp} over $\mathcal{P}$. Then, we may regard V as (x_1, x_2, y_1, y_2) where x_i, y_i are in K for $i = 1, 2$ and also may be represented as vectors over $\mathcal{B}$. That is, for example, $x_j = (x_{j,i}; i \in \lambda)$, for $j = 1, 2$, with respect to $\mathcal{B}$ for $x_{j,i} \in \mathcal{P}$ for some index set λ. With this choice of basis, we may represent T as follows:

$$T = (y = x \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix})$$

where T_i are linear transformations over $\mathcal{P}$ represented in the basis $\mathcal{B}$. Note that we are not trying to claim that the $T_i's$ are K^{opp}-linear transformations, merely $\mathcal{P}$-linear.

The action is then

$$(y_1, y_2) = (x_1, x_2) \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = (x_1T_1 + x_2T_3, x_1T_2 + x_2T_4).$$

where the x_i and y_j terms are considered as $\mathcal{P}$-vectors.

Hence, the $T_i's$ are merely additive mappings on K^{opp} but not necessarily K^{opp}-linear.

In reading the next theorem, it might be kept in mind that a derivable net may always be considered algebraically a pseudo-regulus net with spread in $PG(3, K^{opp})$ when the geometric embedding is in $PG(3, K)$. When K is a field, this is not to say that these two projective spaces are the same as $\mathcal{D}$ can be a regulus in a three-dimensional projective space while being embedded in another and both projective spaces are isomorphic.

Theorem 14. *If $\mathcal{D}$ is a derivable net and $\langle \mathcal{D} \cup \{T\} \rangle$ a translation net regarded as a left vector space net over the associated prime field $\mathcal{P}$, and $\mathcal{D}$ regarded as a right vector-space net over K^{opp} then the geometric embedding of $\mathcal{D}$ into Σ isomorphic to $PG(3, K)$ is considered as a 'left' space embedding.*

Representing $\mathcal{D}$ as

$$x = 0, y = x \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}; B \in K^{opp},$$

we may represent T as

$$(y = x \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix})$$

where the T_i are additive mappings of K^{opp} and $\mathcal{P}$-linear transformations.

(1) The line $y = x \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$ determines a semifield spread admitting an affine homology group with axis $y = 0$ and coaxis $x = 0$ isomorphic to $K^{opp} - \{0\}$ (the dual semifield plane has its spread in $PG(3, K)$ and is $S(T)$).

The semifield spread has the following form:

$$x=0, y=x\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}+\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}\right) \ \forall A, B \in K^{opp}.$$

(2) The semifield spread with spread in $PG(3, K)$ (the $S(T)$) has the following form (where here the T_i's are considered additive mappings of K):

$$x=0, y=x\begin{bmatrix} sT_1+vT_3 & sT_2+vT_4 \\ s & v \end{bmatrix} \forall v, s \in K.$$

(3) The semifield spread in $PG(3, K)$ is a skew-Desarguesian spread if and only if the T_i's are all K-linear transformations (i.e. multiplication by elements of K) if and only if $\mathcal{D}^{+T}$ is a partial spread in $PG(3, K^{opp})$; considering $\mathcal{D}$ as a pseudo-regulus in $PG(3, K^{opp})$, T is a subspace in the projective spread $PG(3, K^{opp})$.

(4) (see Knarr [18]) If K is a field then the semifield spread in $PG(3, K)$ is Pappian if and only if the T_i's are all K-linear transformations (i.e. multiplication by elements of K) if and only if $\langle \mathcal{D} \cup \{T\} \rangle$ is a partial spread in $PG(3, K)$; T is a subspace in the projective spread $PG(3, K)$ wherein $\mathcal{D}$ is considered a regulus.

Proof. Although some of the following has been previously presented in the section on algebraic and geometric extension, we revisit these ideas here. Part (1) follows immediately from the algebraic extension process considering T as a transversal function.

We consider a point $(x_1, x_2, x_1T_1 + x_2T_3, x_1T_2 + x_2T_4)$. By Johnson [12], we may represent N by $\langle(1, 0, 0, 0), (0, 1, 0, 0)\rangle$, the zero vector $(0, 0, 0, 0)$ by $\langle(0, 0, 1, 0), (0, 0, 0, 1)\rangle$ and a general point (d_1, d_2, d_3, d_4) by $\langle(d_1, d_2, 1, 0), (d_3, d_4, 0, 1)\rangle$ where the 2-dimensional K-subspaces are considered right spaces and lines in $PG(3, K)$.

Hence, the lines associated with the net-points $(x_1, x_2, x_1T_1 + x_2T_3, x_1T_2 + x_2T_4)$ are

$$\begin{aligned} & \langle(x_1, x_2, 1, 0), (x_1T_1 + x_2T_3, x_1T_2 + x_2T_4, 0, 1)\rangle \\ = \ & \alpha(x_1, x_2, 1, 0) + \beta(x_1T_1 + x_2T_3, x_1T_2 + x_2T_4, 0, 1) \end{aligned}$$

for all $\alpha, \beta \in K$.

Let $x_2^* = (\alpha x_1 + \beta(x_1T_1 + x_2T_3))$, and $x_1^* = (\alpha x_2 + \beta(x_1T_2 + x_2T_4))$.

Then

$$(x_1^*, x_2^*)\begin{bmatrix} x_2 & x_1 \\ x_1T_2 + x_2T_4 & x_1T_1 + x_2T_3 \end{bmatrix}^{-1} = (\alpha, \beta).$$

Now change bases by interchanging $x = 0$ and $y = 0$ to obtain the spread as

$$x=0, y=x\begin{bmatrix} v & s \\ sT_2+vT_4 & sT_1+vT_3 \end{bmatrix} \forall v, s \in K.$$

Change bases by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

to change the form into:

$$x = 0, y = x \begin{bmatrix} sT_1 + vT_3 & sT_2 + vT_4 \\ s & v \end{bmatrix} \forall v, s \in K.$$

This proves (2). The proofs to (3) and (4) and then immediate. (The reader is referred to Johnson [15] for a definition of skew-Desarguesian planes.) ■

We noted in part (4) that the associated spreads in $PG(3, K)$ are Pappian if a derivable net is a regulus net in $PG(3, K)$ and the transversal is a subspace within the same $PG(3, K)$. We might inquire as to the nature of the semifield spreads if we assume initially that the transversal is a subspace in $PG(3, K)$ but the derivable net is not necessarily a K-regulus.

The following background result is required for our analysis.

Lemma 15. *(see Johnson [9] for the finite case and Jha-Johnson [5], for the infinite case) A derivable net $\mathcal{D}$ with partial spread in $PG(3, K)$, for K a field, may be represented in the following form:*

$$x = 0, y = x \begin{bmatrix} u & A(u) \\ 0 & u^{\sigma} \end{bmatrix} \forall u \in K$$

and where σ is an automorphism of K, and where x and y are 2-vectors over K and A is a function on K such that

$$\begin{bmatrix} u & A(u) \\ 0 & u^{\sigma} \end{bmatrix} \forall u \in K$$

is a field isomorphic to K.

Furthermore, $A \equiv 0$ in the finite case, or when there are at least two Baer subplanes incident with the zero vector which are K-subspaces. When there is exactly one K-subspace Baer subplane, the characteristic is two, $\sigma = 1$ and $A(u) = Wu + uW$ for some linear transformation W of K over the prime field.

First consider the situation when there are two Baer subplanes which are K-subspaces so that A is identically zero. Suppose that we have $y = x \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ a transversal to the derivable net for a, b, c, d in K. We need to re-coordinatize so as to realize the derivable net as a regulus in an associated projective space.

We consider the mapping:

$$\tau : (x_1, x_2, y_1, y_2) \longmapsto (x_1, x_2^{\sigma^{-1}}, y_1, y_2^{\sigma^{-1}}).$$

It follows that τ maps

$$(x_1, x_2, x_1 u, x_2 u^{\sigma}) \longmapsto (x_1, x_2^{\sigma^{-1}}, x_1 u, x_2^{\sigma^{-1}} u).$$

It then follows that the derivable net has the general form

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall u \in K$$

in the associated projective space. Now we consider the τ-image of $y = x\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. As a linear transformation over the prime field, consider $x^{\sigma^{-1}} = xM$. Then, the image of $y = x\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be written in the following form:

$$y = x\begin{bmatrix} a & Mb^{\sigma^{-1}} \\ M^{-1}c & d^{\sigma^{-1}} \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

in the notation of the previous section, and where $x = (x_1, x_2^{\sigma^{-1}} = x_2M)$ and $y = (y_1, y_2^{\sigma^{-1}} = y_2M)$. Hence, we obtain the semifield spread in $PG(3, K)$ in the form

$$x = 0, y = x\begin{bmatrix} sa + v^{\sigma^{-1}}c & s^{\sigma^{-1}}b^{\sigma^{-1}} + vd^{\sigma^{-1}} \\ s & v \end{bmatrix} \forall v, s \in K.$$

Now change bases by

$$\begin{bmatrix} c & b^{\sigma^{-1}} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

to transform the spread set into the form:

$$x = 0, y = x\begin{bmatrix} v^{\sigma} + s(ac^{-1} + c^{-1}d^{\sigma^{-1}}) & s^{\sigma^{-1}}c^{-1}b^{\sigma^{-1}} \\ s & v \end{bmatrix}.$$

Hence, we obtain:

Theorem 16. *A derivable net $\mathcal{D}$ with transversal extension T giving a partial spread $\langle \mathcal{D} \cup \{T\} \rangle$ that is in $PG(3, K)$, for K a field, and such that there are at least two Baer subplanes which are K-subspaces constructs a semifield spread in $PG(3, K)$ of the following form:*

$$x = 0, y = x\begin{bmatrix} v^{\sigma} + sk & s^{\sigma^{-1}}l \\ s & v \end{bmatrix} \forall v, s \in K,$$

for σ an automorphism of K, and constants $k, l \in K \in K$.

Remark 17. *The spreads mentioned above are considered in Johnson [10] and are generalization of spreads originally defined by Knuth and hence, perhaps, these should be called 'generalized Knuth spreads'.*

We now consider the possibility that the function A is not identically zero.
for all $u, v \in K$. With $A(u) = Wu + uW$, a basis change by

$$\begin{bmatrix} I & W & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & W \\ 0 & 0 & 0 & I \end{bmatrix}$$

will change the form of the derivable net into a regulus in $PG(3,K)$. We note that the regulus will have the form:

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall u \in K.$$

Moreover, a transversal in $PG(3,K)$ of the net D (in the original form) of the form

$$y = x \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for a, b, c, d becomes a transversal to the standard regulus (note that the transversal is no longer then in $PG(3,K)$) and has the form:

$$y = x \begin{bmatrix} a + Wc & (a + Wc)W + b + Wd \\ c & cW + d \end{bmatrix}.$$

Hence, we obtain in $PG(3,K)$, the spread:

$$x = 0, y = x \begin{bmatrix} sT_1 + vT_3 & sT_2 + vT_4 \\ s & v \end{bmatrix} \forall v, s \in K$$

where $a + Wc = T_1, (a + Wc)W + b + Wd = T_2, c = T_3$ and $cW + d = T_4$. It is important to note that the vector space is now of the form $(x_1, x_1W + x_2, y_1, y_1W + y_2)$ which we identify with (x_1, x_2, y_1, y_2). With this identification, it follows that T_i 's are additive mappings of K.

Hence, we obtain:

Theorem 18. *Let a derivable net $\mathcal{D}$ with transversal extension T giving a partial spread $\langle \mathcal{D} \cup \{T\} \rangle$ that is in $PG(3,K)$, for K a field, such that there is exactly one Baer subplane which is a K-subspace. Then there is an associated semifield spread in $PG(3,K)$ of the following form:*

$$x = 0, y = x \begin{bmatrix} s(a + Wc) + vc & s((a + Wc)W + b + Wd) + v(cW + d) \\ s & v \end{bmatrix} \forall v, s \in K$$

where W is some prime field linear transformation of K, a, b, c, d constants in K.

7 Vector Space Transversals

Now assume that we have a derivable net and a vector space transversal. We represent the derivable net as in the previous section as

$$x = 0, y = x \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \forall B \in K^{opp}$$

where $K^{opp} = F$ is a skewfield and we represent the transversal in the form

$$(y = x \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}) = xT$$

where the $T_i's$ are prime field linear transformations and additive F-mappings.

First let F be isomorphic to $GF(q)$ for $q = p^r$, p a prime. Let $H_{p,f,g,u}$ be a group of order $q-1$ whose elements are defined as follows:

$$\tau_u = \begin{bmatrix} p(u) & 0 & 0 & 0 \\ 0 & f(u) & 0 & 0 \\ 0 & 0 & p(u)\lambda(u) & 0 \\ 0 & 0 & 0 & f(u)\lambda(u) \end{bmatrix} \text{ for } u \in F - \{0\}$$

and p, f, λ are functions on F. We require that the derivable net is left invariant. For this, we must have

$$\begin{bmatrix} p(u)^{-1}p(u)\lambda(u) = \lambda(u) & 0 \\ 0 & f(u)^{-1}f(u)\lambda(u) = \lambda(u) \end{bmatrix}$$

for some function v of u.

We consider situations under which

$$x = 0, y = x \begin{bmatrix} p(u)^{-1} & 0 \\ 0 & f(u)^{-1} \end{bmatrix} T \begin{bmatrix} p(u)\lambda(u) & 0 \\ 0 & f(u)\lambda(u) \end{bmatrix} + wI$$

for all $u \neq 0, w \in F$ defines a spread. We note that there is an associated elation group E of order q and the spread fixes $x = 0$.

Definition 19. *When the above set defines a spread, we denote the spread by $\pi_{T,H_{p,f,p\lambda,f\lambda}}$ and call the spread a '$(T, EH_{p,f,p\lambda,f\lambda})$-spread'.*

More generally, it might be possible to have a group containing an elation group E of order q and a group H such that EH acts transitively on the components of the spread not in the derivable net but H may not be diagonal. In the more general case, we refer to the spread as a 'partially transitive elation group spread'.

Remark 20. *Any $(T, H_{p,f,p\lambda,f\lambda})$-spread is derivable and the derived plane admits a collineation group fixing the spread and acting transitively on the components not in the derivable net. The elation group is turned into a Baer group B and the group $H_{p,f,p\lambda,f\lambda}$ is turned into the group $H_{p,p\lambda,f,f\lambda}$.*

We call such a spread a '$(T^*, BH_{p,p\lambda,f,f\lambda})$-spread'. Also, more generally, if we have a partially transitive elation group spread, it derives to a so-called 'partially transitive Baer group spread'.

Proof. Change bases relative to F by the mapping that takes (x_1, x_2, y_1, y_2) to $(x_1, y_1, x_2, y_{2)}$. Then, the group $H_{p,f,g,h}$ is changed to the group $H_{p,g,f,h}$. ∎

Example 21. *Examples of partially transitive elation group spreads are as follows:*

(1) Any semifield plane of order q^2 whose semifield is of dimension two over its middle nucleus. This plane is a $(T, EH_{\lambda^{-1},\lambda^{-1},1,1})$-spread which derives to a $(T^*, BH_{\lambda^{-1},1,\lambda^{-1},1})$-spread that is also known as a 'generalized Hall spread' (i.e. of type 1).

Note that if $y = xT$ is a vector-space transversal to a finite derivable net then we may realize the derivable net as a left vector space net and automatically use the *'left' extension process to construct a dual translation plane which then becomes a semifield plane with spread:*

$$x = 0, y = \alpha T + \beta I, \forall \alpha, \beta \in F.$$

(2) Any semifield plane of order q^2 whose semifield is of dimension two over its right nucleus. This plane provides a $(T, EH_{1,1,\lambda,\lambda})$-spread which derives a $(T^*, BH_{1,\lambda,1,\lambda})$-spread that is also known as a 'generalized Hall spread of type 2'.

This is merely the situation with which we began, realizing the derivable net as a right vector space and constructing the dual translation plane which is then a semifield plane with spread:

$$x = 0, y = T\alpha + \beta I, \forall \alpha, \beta \in F.$$

(3) Other known examples all correspond to situations where the $y = xT$ is a line in the projective space wherein the derivable net is a regulus. The partially transitive elation spreads correspond to flocks of quadratic cones. When the order q^2 is odd, there is a classification of such partially transitive elation spreads in Hiramine and Johnson [4] and the possibility that the group H may not be diagonal is included in this study. However, the known examples are $(T, EH_{p,f,p\lambda,f\lambda})$-spreads. When $q = 2$, any partially transitive elation spread turns out to be a $(T, EH_{p,f,p\lambda,f\lambda})$-spread. Furthermore, combining this work with a study of Penttila and Storme [20] shows that the known examples are the only possible examples. For details and additional information, the reader is referred to the survey paper of Johnson and Payne [16].

To list but one of these examples in the odd order case, we consider the spread:

$$x = 0, y = x \begin{bmatrix} u & \gamma t^\sigma \\ t & u \end{bmatrix}; \forall u, t \in GF(q)$$

where γ is a nonsquare, q is odd and σ is an automorphism of $GF(q)$.

We note that the elements of the elation group E have the following form:

$$\begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \forall u \in GF(q).$$

Furthermore, the group is $H_{1,f,\lambda,f\lambda}$ where $f(u) = u^\sigma, \lambda(u) = u^{\sigma+1}$.

One of the early results using group theory to determine the structure of finite affine planes is the result of Wagner [21] who proved that finite affine planes that admit a collineation group acting transitive on the flags of the plane are always translation planes.

In a later section, we take up the following problem:

Problem 22. *Let π be a finite derivable affine plane and let $\mathcal{D}$ denote the derivable net.*

(1) If π admits a collineation group G leaving $\mathcal{D}$ invariant that acts transitively on the flags of π on lines not in $\mathcal{D}$, is π a translation plane?

(2) If π is a translation plane, is it always either a partially transitive elation plane or a partially transitive Baer plane?

Definition 23. *We shall call a derivable affine plane that admits a collineation group leaving the derivable net invariant and acting flag-transitively on the flags on lines not in the derivable net a 'partially flag-transitive plane'.*

7.1 Dual Translation Planes.

Let π be a translation plane with spread in $PG(3,q)$. Choose any component $x = 0$ and let (∞) denote the parallel class containing the component. Take any 1-dimensional $GF(q)$-subspace X on any component $y = 0$. Then the lines $x(\infty)$ union (∞), for all x in X, form a derivable net in the dual translation plane obtained by taking (∞) as the line at infinity in the dual plane. Notice that the dual translation plane admits a collineation group of order $q^3(q-1)$ which leaves the derivable net invariant. This group consists of the translation group of order q^3 generated by the subgroups with center (∞) of order q^2 and the group of order q with center (0) which leaves X invariant. The kernel homology group H of order $q-1$ leaves X invariant. Hence, the product of these two groups is a group W of the dual translation plane π^*. Furthermore, there is an infinite point $(\infty)^*$ of the derivable net $\mathcal{D}^*$ which is fixed by W. The group of order q mentioned above becomes an elation group E of order q. Note that HE acts transitively on the lines of (∞) not in $(\infty)X$ since E fixes no lines and has q orbits of length q and H fixes exactly one line and has orbits of length $q-1$. Note that E is normal in EH so that H can fix exactly one orbit of E. Hence, HE has orbits of length q and $q(q-1)$.

Thus, EH acting on the dual translation plane is transitive on the infinite points of the plane which are not in the derivable net $\mathcal{D}^*$.

Now assume that the associated translation plane admits a collineation group of order q^2 that is in the linear translation complement (i.e. in $GL(4,q)$), fixes (∞) and acts transitively on the components other than $x = 0$.

It follows from the work of Johnson and Wilke [17] that the group of order q^2 may be represented so that acting on $x = 0$, the group is a subgroup of

$$\left\langle \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}; a \in GF(q) \right\rangle .$$

Hence, it follows that, representing the vector space by 4-vectors (x_1, x_2, y_1, y_2), the lines $x = (0, \alpha)$ $\forall$ $\alpha \in GF(q)$ are fixed. Thus, if we take

$$X = \{(0, \alpha, 0, 0); \alpha \in GF(q)\}$$

as defining the derivable net $\mathcal{D}^*$, we have a group S of order q^2 acting on the dual translation plane. Furthermore, we then obtain a collineation group of order $q^5(q-1)$ which fixes the derivable net and acts partially flag-transitively on the affine dual translation plane. In order to see this, we note that the group of order $q(q-1)$ mentioned previously is transitive on the parallel classes. The translation group with center (∞) becomes a translation group of the dual translation plane with center $(\infty)^*$ and this group acts transitively on the affine lines of any parallel class not equal to $(\infty)^*$. Take any point P of the translation plane such that $P(\infty)$ is not a line of $(\infty)X$. Then, using the translation group, there is a group conjugate to S by a translation which acts transitively on the lines incident with P other than $P(\infty)$.

Hence, we obtain:

Theorem 24. *Let π be any translation plane of order q^2 with spread in $PG(3,q)$. Assume that there exists a collineation group in the linear translation complement of order q^2 that fixes a component and acts transitively on the remaining components of π.*

Then the dual translation plane is a partially flag-transitive derivable affine plane admitting a collineation group of order $q^5(q-1)$.

The dual translation plane is a translation plane (and hence a semifield plane) if and only if the group of order q^2 mentioned above is an elation group of the associated translation plane.

Corollary 25. *Any semi-translation plane obtained by the derivation of a partially flag-transitive dual translation plane is also partially flag-transitive. The dual translation plane is of 'elation' type whereas the semi-translation plane is of 'Baer' type (the dual translation plane admits an elation group of order q and the semi-translation plane admits a Baer group of order q).*

The previous examples all involve solvable groups. Are there 'nonsolvable' partially flag-transitive affine planes?

Remark 26. *The Hall and Desarguesian planes are the only translation planes of order q^2 that admit $SL(2,q)$ and are partially flag-transitive affine planes*

Proof. The translation planes admitting $SL(2,q)$ as a collineation group are determined in Foulser and Johnson [2] and [1]. The only derivable planes that admit a collineation group transitive on the components exterior to the derivable net are the Hall and Desarguesian. Any translation plane of this sort is partially flag-transitive. ∎

8 Transposed Spreads.

In this section, we ask the following question:

Let $\mathcal{D}$ be a derivable net and T a transversal such that $\mathcal{D}\cup\{T\}$ is a derivable-extension. Let $\widehat{\mathcal{D}}\cup\{T\}$ denote the corresponding derivable-extension where $\widehat{\mathcal{D}}$ is the derived net of $\mathcal{D}$. When is the spread $S(T)$ with respect to $\mathcal{D}\cup\{T\}$, $S(T)_{\mathcal{D}^{+T}}$, isomorphic to the spread $S(T)_{\widehat{\mathcal{D}}^{+T}}$ with respect to $\widehat{\mathcal{D}}\cup\{T\}$?

First of all, we note that if the corresponding projective space is $PG(3,K)$ of $\mathcal{D}^{+T}$ then $S(T)_{\mathcal{D}^{+T}}$ is in $PG(3,K)$, whereas $S(T)_{\widehat{\mathcal{D}}^{+T}}$ is in $PG(3,K^{opp})$. However, the points and (Baer) subplanes of $PG(3,K)$ are subplanes and points of $PG(3,K^{opp})$ respectively. Hence, the question only makes genuine sense when K is a field.

Theorem 27. *Let $\mathcal{D}$ be a derivable net embedded in $PG(3,K)$, where K is a field, and let T be a transversal to $\mathcal{D}$ which is also a transversal to the derived net $\widehat{\mathcal{D}}$. Then the spread $S(T)_{\mathcal{D}^{+T}}$ corresponding to $\mathcal{D}^{+T}$ is isomorphic to the spread $S(T)_{\widehat{\mathcal{D}}^{+T}}$ corresponding to $\widehat{\mathcal{D}}^{+T}$ is and only if there is a duality of $PG(3,K)$ which maps one spread to the other.*

Proof. By Johnson [15], we may interpret derivation in terms of a duality of the associated projective space. Here we have merely extended these notions to the corresponding constructed spreads. ∎

9 Reconstruction.

We have noted that transversals to derivable nets are basically equivalent to spreads in $PG(3, K)$. However, although transversals are then used to construct dual translation planes, there are other affine planes containing a derivable net which are not dual translation planes. The question is whether there is a way to use various sets of spreads to construct or reconstruct an affine plane containing a derivable net.

Definition 28. *A 'skew parallelism' of $PG(3, K) - N$ is a set of spreads each containing N which forms a disjoint cover of the lines of $PG(3, K)$ skew to N.*

Let $\mathcal{S}$ be a skew parallelism of $PG(3, K) - N$ and let $\mathcal{P}$ be any spread containing N. We shall say that $\mathcal{P}$ is 'orthogonal' to $\mathcal{S}$ if and only if $\mathcal{P}$ intersects each spread of $\mathcal{S}$ in a unique line $\neq N$.

A set of skew parallelisms of $PG(3, K) - N$ is 'orthogonal' if and only if each spread of any one skew parallelism is orthogonal to each of the remaining skew parallelisms.

A set $\mathcal{A}$ of skew parallelisms of $PG(3, K) - N$ is said to be 'planar' if and only if, given any two lines ℓ_1 and ℓ_2 of $PG(3, K)$ which are skew to N, there is a skew parallelism of $\mathcal{A}$ containing a spread sharing ℓ_1 and ℓ_2.

If the spreads of a set of skew parallelisms are all dual spreads, we shall say that the set is a 'derivable' set of skew parallelisms.

Theorem 29. *Given an orthogonal and planar set $\mathcal{A}$ of skew parallelisms of $PG(3, K) - N$, then there is a unique affine plane $\pi_{\mathcal{A}}$ containing a derivable net such that the set of transversals to the derivable net are the spreads of the set $\mathcal{A}$.*

Conversely, any affine plane containing a derivable net corresponds to a uniquely defined orthogonal and planar set of skew parallelisms.

Hence, the set of derivable affine planes is equivalent to the set of derivable, orthogonal and planar sets of skew parallelisms.

Proof. We have seen that any spread $\mathcal{P}$ containing N may be considered a transversal to a derivable net. A skew parallelism S containing $\mathcal{P}$ consists of a set of mutually disjoint transversals with the property that each point of the derivable net is incident with exactly one transversal. (Since the 'points' of the derivable net are lines of the projective space, each net-point is a line of exactly one spread of S. Hence, each net-point is incident with exactly one transversal.) So, a skew parallelism corresponds exactly to a parallel class external to the derivable net. Two skew parallelisms which are orthogonal then correspond to two distinct parallel classes of a net extension of a derivable net and a planar and orthogonal set of skew parallelisms is such that any two distinct points of the net as lines of the projective space are incident with exactly one spread of some skew parallelism; two distinct points are incident with a unique transversal to the derivable net. Hence, an affine plane is constructed from a planar and orthogonal set of skew parallelisms. In order that the affine plane actually be derivable, it follows that each transversal spread must actually be a dual spread. Hence, a derivable, planar and orthogonal set of skew parallelisms constructs a derivable affine plane. ■

Now we ask the nature of a 'transitive' skew parallelism.

Definition 30. *A skew parallelism of $PG(3,K)-N$ is 'transitive' if and only if there exists a subgroup of $P\Gamma L(4,K)_N$ that acts transitively on the spreads of the skew parallelism.*

A planar and orthogonal set of skew parallelisms is 'transitive' if and only if there exists a subgroup of $P\Gamma L(4,K)_N$ which acts transitively on the set.

We shall say that the set is 'line-transitive' if and only if the stabilizer of a spread is transitive on the lines not equal to N of the spread, for each spread of the skew parallelism.

We have seen the following in a previous section, we re-introduce the ideas again here.

Remark 31. *Let π^D be a dual translation plane with transversal function $f(x)$ to a right vector-space derivable net so that lines have the equations:*

$$x=0, y=f(x)\alpha+x\beta+b$$

for all $\alpha,\beta\in K$ and for all $b\in V$ (see the notation of introductory section).

Then, there is a collineation group of π^D which leaves invariant the derivable net and acts transitively on the lines not in the derivable net and of the form $y=f(x)\alpha+x\beta+b$ where $\alpha\neq 0$.

The 'translation group' T is transitive on the lines of each such parallel class and represented by the mappings:

$$(x,y)\longmapsto(x,y+b)\ \forall b\in V.$$

The affine elation group E is represented by mappings of the form

$$(x,y)\longmapsto(x,x\beta+y)\ \forall\beta\in K$$

and the affine homology group H is represented by mappings of the form:

$$(x,y)\longmapsto(x,x\alpha)\ \forall\alpha\in K-\{0\}.$$

Notice that T and E correspond to certain translation subgroups with fixed centers of the corresponding translation plane and the group H corresponds to the kernel homology group of the associated translation plane.

(1) It also follows that any derivable affine plane coordinatized by a cartesian group will admit a group isomorphic to T and hence corresponds to a transitive skew parallelism.

(2) Any such dual translation plane will produce a transitive planar and orthogonal set of transitive skew parallelisms.

(3) Any semifield spread which contains a derivable net as above will admit a translation group with center $(f(x))$ which fixes $f(x)$ and acts transitively on the points of $f(x)$ which implies that the transversal-spread is transitive.

Hence, any semifield spread produces a line-transitive planar and orthogonal set of skew parallelisms.

So, we would ask whether a line-transitive planar and orthogonal set of skew parallelisms corresponds either to a translation plane, a dual translation plane or a semi-translation plane. Hence, not all such sets of skew parallelisms in three-dimensional projective spaces can force the affine plane containing the derivable net to be a semifield plane or even a translation plane. We formulate a fundamention question?

Is a finite derivable partially flag-transitive plane of order q^2 a translation plane, a dual translation plane or a semi-translation plane and if so, is there either a Baer group or elation group of order q?

10 Partially Flag-transitive Affine Planes.

Assume that π is a finite partially flag-transitive affine plane. The derivable net D is combinatorially equivalent to a projective space $PG(3,K)$, where K is isomorphic to $GF(q)$, relative to a fixed line N of $PG(2,K)$. Furthermore, the full collineation group of the net D is $P\Gamma L(4,K)_N$. Assume now that the given collineation group G of π is linear; i.e. in $PGL(4,K)_N$. It follows that the linear subgroup which fixes an affine point and the derivable net (which is now a regulus net) is a subgroup of $GL(2,q)GL(2,q)$ where the product is a central product with common group the center of order $q-1$. Note we are not trying to say that the derivable net corresponds to a regulus in the particular $PG(3,q)$ wherein lives the skew parallelisms, merely that the derivable net can be realized as arising from a regulus in some three-dimensional projective space.

Let $p^r = q$. Then, there must be a group of order divisible by $q^2(q^2)(q^2-q)$ by the assumed transitive action. Hence, the p-groups have orders divisible by q^5 and note that the full linear p-group of the derivable net has order q^6. Any such p-group S_p must leave invariant an infinite point (∞) of the derivable net $\mathcal{D}$ as the derivable net consists of $q+1$ parallel classes.

So, again let G denote the full collineation group of the associated affine plane π, under the assumption that the group is 'linear' with respect to the derivable net and let T denote the translation group with center (∞) of S_p. We note that T is normal in S_p. Let ℓ be any transversal line to the derivable net. Then there exists a collineation group G_ℓ which acts transitively on the points of ℓ.

Lemma 32. *The p-groups of G_ℓ have orders q^2, or $2q^2$ and q is even.*

Proof. There is a spread in $PG(3,q)$, $S(\ell)$. Furthermore, there is a group which fixes a line N of the spread and acts transitively on the remaining lines of the spread. We have assumed that the group is a subgroup of $PGL(4,q)_N$. Now consider the associated translation plane and realize that the collineation group, as a translation complement, acting here is a subgroup of $GL(4,q)$. Since the group fixes a component and is linear, the group induced on that component is a subgroup of $GL(2,q)$. Since the group is transitive on $S(\ell)-N$, let the order of the p-group be $p^\alpha q^2$. Hence, the elation group with axis N as order at least $p^\alpha q$. On the other hand, since the group is linear and transitive on the components not equal to N, the stabilizer of a second line is a Baer group of order p^α. By Johnson and Wilke [17] (2.7), if $p^\alpha > 1$ then $p^\alpha = 2$. This completes the proof of the lemma. ∎

Lemma 33. *The order of a Sylow p-group of G is either q^5, or $2q^5$ and q is even.*

Proof. Since the group G acts transitively on components of the derivable affine plane $\pi - \mathcal{D}$, it follows that S_p permutes $q(q-1)$ points on the line at infinity and hence there must be an orbit of S_p of length q as otherwise pq would divide $q(q-1)$. Since this is, in fact, an orbit, the stabilizer of an infinite point (β) has order $|S_p|/q$. Since the lines of (β) are also in an orbit then the stabilizer of a line ℓ has order $|S_p|/q^3$ which is either q^2 or $2q^2$. This completes the proof. ∎

Lemma 34. *Let S_p be a Sylow p-subgroup. Hence, S_p leaves invariant an infinite point of $\mathcal{D}$, say (∞). The stabilizer of a second point fixes all infinite points of $\mathcal{D}$.*

Proof. The group G is a subgroup of $GL(2,q)GL(2,q)T$ acting on the derivable net $\mathcal{D}$. A Sylow p-group is a subgroup of a group S_p^+ of order q^6 consisting of a Sylow p-group of each of the two $GL(2,q)'s$ and T of order q^4. The subgroup of S_p^+ that fixes two infinite points has order q^5 and consists of a Baer p-group together with T and hence fixes all infinite points. ∎

Lemma 35. *Let σ be an nontrivial elation with axis $x = 0$ in $\mathcal{D}$ and let τ be a nontrivial Baer collineation of $\mathcal{D}$, also fixing $x = 0$ such that $\sigma\tau$ is in S_p (acts as a collineation of π). Then $\sigma\tau$ is a Baer collineation of π such that $Fix\sigma\tau$ has q parallel classes exterior to $\mathcal{D}$.*

Proof. Coordinatize so that

$$\sigma = \begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\tau = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for a and $u \in F \simeq GF(q)$.

Then $\sigma\tau$ fixes $\{(0, y_1au^{-1}, y_1, y_2); y_i \in F\}$ pointwise and fixes no component of the derivable net except $x = 0$. Moreover, σ and τ share fixed points on $x = 0$. Hence, $\sigma\tau$ fixes exactly q^2 points, fixes $x = 0$, fixes exactly q points on $x = 0$ and fixes any line of the plane which contains points of $Fix\sigma\tau$. Suppose that $\sigma\tau$ fixes lines $x = 0$ and ℓ concurrent with a particular affine point P. So, $\sigma\tau$ fixes a Baer subplane π_o incident with P of $\mathcal{D}$ and induces an elation on π_o. Let (α) denote the parallel class containing ℓ. Then, $\sigma\tau$ fixes each of the q affine lines of π_o incident with (∞), fixes (α) so fixes each of the q affine lines incident with (α), as $\sigma\tau$ induces an elation on π_o. It follows that $\sigma\tau$ fixes q points on each of the lines fixed by $\sigma\tau$ incident with (∞). This argument shows that the set of fixed points belongs to a Baer subplane. ∎

Lemma 36. *(1) Assuming that S_p fixes (∞), the stabilizer of a line $x = 0$ in S_p has order either q^3 or $2q^3$.*

(2) Further, the stabilizer H of an affine point has order at least q in the q^3 case or at least $2q$ in the $2q^3$ case above.

(3) H contains either an elation group of order q, or a Baer group of order q with axis a subplane of $\mathcal{D}$.

(4) If the order of H is at least $2q$ then the order is $2q$ and H is generated by an elation group of order q and a Baer involution with axis in $\mathcal{D}$ or by a Baer group of order q and an elation.

Proof. (1) is clear since the group acts transitively on the points of ℓ and fixes (∞) also acts transitively on the affine lines of (∞). Since $x = 0$ has q^2 points, the stabilizer of one of these has order at least q or $2q$.

Now H must actually fix a line of some Baer subplane π_o pointwise and permute the parallel class of subplanes to which π_o belongs. Since this parallel class has q^2 total members, H must leave invariant another subplane π_1 disjoint to π_o (the parallel class to which π_o belongs is the set of all subplanes of $\mathcal{D}$ disjoint from π_o union π_o). It follows from Johnson [11] that π_o and π_1 share a parallel class of lines.

If the shared parallel class is not (∞) then the group H fixes a second infinite point so it fixes all infinite points of $\mathcal{D}$ and hence fixes π_o pointwise. However, in this case, the order of H must, in fact be q.

Hence, assume that π_o and π_1 share a parallel class of lines of (∞). It follows that the elements of H are in $GL(2, q)GL(2, q)$ so that each element is the product of an elation σ and a Baer p-element τ. However, this implies that either σ or τ is 1 or there is an 'external' Baer p-element by a previous lemma. If this is so, then $p = 2$ and the order of H is at least $2q$. Note that each external Baer subplane contributes exactly q components outside of the derivable net. If there is an overlap, we may assume the overlap occurs at least on ℓ so there is a group generated by Baer elements of order at least 4 and fixing ℓ. But, this means that within the translation plane of the associated spread corresponding to ℓ, we have a group of order at least $4q^2$, a contradiction. Hence, there are at most $q-1$ 'external' Baer involutions (axes external to the derivable net). Let the external Baer involutions be denoted by $\sigma_i\tau_i$ where the σ_i are elations (not necessarily distinct and possibly trivial) and the τ_i are Baer involutions (not necessarily distinct but nontrivial) for $i = 1, 2, ..., b_o$ where $b_o \leq q - 1$ and let the elements of H be denoted by $\sigma_i\tau_i = \rho_i$ for $i = 1, 2, ..., \geq 2q$. Note that if $\sigma_i\tau_i$ is a Baer involution and σ_k is an elation not equal to σ_i then $\sigma_k\sigma_i\tau_i$ is an external Baer involution. Since the plane can be derived, it follows that if τ_s is a Baer involution (internal) not equal to τ_i then $\sigma_i\tau_i\tau_s$ is also a Baer involution. So, consider a given external Baer involution $\sigma_1\tau_1$. Then there are at least q elements which are non-identity elations or non-identity internal Baer involutions in H and at least $q-2$ of these are neither equal to σ_1 nor τ_1. Hence, multiplication of a given $\sigma_1\tau_1$ by elements of H result in $q - 2$ distinct other external Baer involutions. That is, if $\sigma_k\sigma_1\tau_1$ is $\sigma_s\sigma_1\tau_1$ or $\sigma_1\tau_1\tau_j$ then either σ_k is σ_s or τ_j. So, there are exactly $q - 1$ distinct external Baer involutions, assuming that there is one. The remaining two non-identity elements of H then cannot produce external Baer involutions. Hence, it must be that the remaining two elements are either σ_1 or τ_1. So, there are exactly $q-1$ external Baer involutions, implying there are exactly $q-1$ non-identity elations or internal Baer involutions.

Assume that there are s $\sigma_j's$ and hence $q-s$ $\tau_k's$. Note that the $\sigma_j\tau_k$'s are $s(q-s)$ distinct external Baer involutions. Hence, we must have

$$s(q-s) \leq q-1.$$

So, we have

$$(s-1)q \leq s^2-1$$

so that if $s-1$ is not zero then $q \leq s+1$. But, $q-1 \leq s \leq q-1$, implying $s = q-1$.

Thus, $s = 1$ or $q-1$ so that there is either an elation group of order q in H or an internal Baer group of order q in H. This implies that there is either an elation group of order q and an extra internal Baer involution that generates H of there is an internal Baer group of order q and an extra elation that generates H.

If the order of H is q then it follows that in all cases, we can have only elations types or only Baer p-element types for otherwise we generate external Baer p-elements and a larger p-group than possible. That is, either H is a Baer p-group of order q with fixed axis in the derivable net or H is an elation group of order q.

Thus, the result is completed unless the order of H is at least $2q$. In this case, the stabilizer of ℓ has order at least $2q^2$ and hence, exactly $2q^2$ by Johnson [9] (section 6). So, the order of H is either $2q$ or q and our above argument completes the proof. ∎

Hence, in any case, by derivation, we may assume that the group H contains an elation group E of order q which then acts transitively on the infinite points of $\mathcal{D}-(\infty)$. Then, the group Z which fixes a second infinite point and hence all infinite points has order q^4 or $2q^4$. If this group is not fixed-point-free then there is a Baer p-element with axis in $\mathcal{D}$. We may set up our argument so that such an element is in H and commutes with H and hence generates with E exactly $q-1$ external Baer p-elements and $p = 2$. (Since the group acts transitively on the lines of (∞), the axis of E may be chosen to correspond to the fixed point of the element of Z in question). Hence, for each affine point P, there is a unique internal Baer involution τ_P with axis in $\mathcal{D}$. Thus, there is a set of q^2 mutually disjoint Baer subplanes in an orbit under Z. Therefore, Z is transitive on the set of Baer subplanes of some parallel class (as a parallel class of the derived net).

So, in any case, there are at least $2q^4 - q^4 = q^4$ elements of Z which are products of Baer p-elements and translations on the net $\mathcal{D}$ and which are fixed-point-free. We now require some information on how a group or set can be fixed-point-free. But, the Baer p-group part comes from a group of order q and we have a group of order q^4 or $2q^4$, and hence, a fixed-point-free set of cardinality at least q^4.

Lemma 37. *Represent a p-group fixing all infinite points of $\mathcal{D}$ as a subgroup of*

$$\left\langle \sigma_a = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}; a \in F \right\rangle T$$

where T has the following form acting on the points of the net $\mathcal{D}$:
$\left\langle \tau_{(c_1,c_2,c_3,c_4)} : (x_1,x_2,y_1,y_2) \longmapsto (x_1+c_1, x_2+c_2, y_1+c_3, y_2+c_4); c_i \in F \right\rangle$.

Then the subset which acts transitively on the points of $x = 0$ *contains elements* $\sigma_a \tau_{(0,0,c_3,c_4)}$*, for various values of* a *in* F*, and, for all* $c_3, c_4 \in F$.

Similarly, the subset which acts transitively on the points of $y = 0$ *contains elements* $\sigma_b \tau_{(c_1,c_2,0,0)}$*, for various values of* b *in* F*, and, for all* $c_1, c_2 \in F$.

Proof. We note that the stabilizer of $x = 0$ in Z contains a set of cardinality q^2 which acts transitively and fixed-point-free on the affine points of $x = 0$. Hence, $(0, 0, 0, 0)$ is fixed by any element of the form σ_a so in order to obtain a transitive action, we require all translations with center (∞) acting on $\mathcal{D}$. The proof for the action on $y = 0$ is analogous. ∎

Lemma 38. *Referring to the lemma above, if there is a fixed-point-free transitive action on* $x = 0$ *then the group acting contains a translation group of order* q *with center* (∞).

Proof. We note that $(0, 0, y_1, y_2) \longmapsto (0, 0, y_1 + c_3, y_1 a + y_2 + c_4)$ for various values of a and c_3, c_4. Since we require all c_3 and $c_4' s$ for the transitive action, assume that $c_3 = 0$ and $c_4 \neq 0$. Then, if a is non-zero, we have the fixed points $(0, 0, -y_2 c_4 a^{-1}, y_2)$ for all y_2. Hence, all such values a must be zero. If $c_3 = 0$ then c_4 must be nonzero since otherwise the collineation is a Baer p-element. Therefore, we obtain that an element $\sigma_a \tau_{(0,0,0,c_4)}$ forces a to be zero. ∎

Since we may repeat the above argument for any infinite point, we have a translation group of order q with centers (∞) and (0).

More precisely,

Lemma 39. *The derivable affine plane is a semi-translation plane.*

Proof. The plane of order q^2 admits a translation group of order q^2 at least one of whose orbits is a subplane π_o of order q. The group leaving the infinite points of $\mathcal{D}$ pointwise fixed is transitive on the affine points and is transitive on the set of subplanes of the parallel class containing π_o. To see this, note that the elements of the group Z fixing the infinite points are products of Baer collineations fixing π_o pointwise times translations. Hence, Z permutes the parallel class of Baer subplanes containing π_o. The images of π_o by elements of Z admit the same translation group of order q^2 as π_o. Hence, the plane is a semi-translation plane. ∎

Lemma 40. *The plane* π *admits a translation group of order at least* q^3.

Proof. The elements of Z are products of Baer p-elements and translations. For any Baer p-element σ_a, assume that there are fewer than q^3 translations τ such that $\sigma_a \tau$ is a collineation. Since the translations are formally normal in the group of the net, it follows that each element $\sigma_a \tau$ has a unique representation of this kind. Therefore, the group order is strictly less than qq^3. Hence, there exists a collineation $\sigma_a \tau$, for which there is a group S_a of at least q^3 translations of the net such that $\sigma_a \tau g$ is a collineation for all $g \in S_a$. Fix g and consider $\sigma_a \tau g (\sigma_a \tau h)^{-1}$. This is a collineation which is also a translation which is not 1 if and only if $g \neq h$. Hence, there are at least $q^3 - 1$ non-identity collineations which are translations. The translation group has order p^β so $p^\beta \geq q^3 - 1$ which implies that there is a translation group of order at least q^3.

Lemma 41. *If (∞) is moved by a collineation of G then π is Desarguesian.*

■

Proof. We have an elation group of order q with axis $x = 0$. If (∞) is moved, we may assume that we have an elation group with axis $y = 0$ by the existing transitivity. Hence, the group generated by the elations acts on a regulus net and thus generated a collineation group isomorphic to $SL(2, q)$. Since the group has orbits of lengths $q + 1$ and $q^2 - q$, the translation group of order $q^3 p^\gamma$ has a decomposition into groups with fixed centers of orders qp^α and p^β such that

$$(q + 1)(qp^\alpha - 1) + (q^2 - q)(p^\beta - 1) + 1 = q^3 p^\gamma.$$

This equation reduces to:

$$qp^\alpha + p^\alpha + qp^\beta - p^\beta - q = q^2 p^\gamma.$$

Hence it follows that $p^\alpha - p^\beta$ is divisible by q. First assume that $p^\alpha = p^\beta$. Then

$$2p^\alpha - 1 = qp^\gamma$$

which is a contradiction. If $p^\alpha \neq p^\beta$ then $\min(\alpha, \beta) = r$ if $q = p^r$ which implies that $p^\alpha = q$. If $p^\alpha = q$ then there are at least $q+1$ centers of translation group of order q^2 so that plane is a translation plane. Thus, the plane is a translation plane of order q^2 which admits a collineation group isomorphic to $SL(2, q)$ generated by elations. Therefore, the plane is Desarguesian by Foulser, Johnson and Ostrom [3].

Lemma 42. *The translation group contains a $((\infty), \ell_\infty)$-transitivity. Furthermore, the translation group has order $q^3 p^\gamma$ and all infinite points not equal to (∞) are centers for translation groups of order qp^γ.*

■

Proof. We may assume that the group fixes (∞). Let the translation group with center (∞) have order qp^δ.

Then,

$$qp^\delta - 1 + q(qp^\alpha - 1) + (q^2 - q)(p^\beta - 1) + 1 = q^3 p^\gamma.$$

This equation is equivalent to:

$$p^\delta + qp^\alpha + qp^\beta - p^\beta - q = q^2 p^\gamma.$$

So that $p^\delta - p^\beta$ is divisible by q. If $p^\delta \neq p^\beta$ then q must divide p^δ so that we have a $((\infty), \ell_\infty)$-transitivity. Thus, we are finished or we obtain:

$$p^\alpha + p^\beta - 1 = qp^\gamma$$

which is a contradiction.

If $p^\delta = q$ then $p^\beta = qp^\rho$

$$p^\alpha + p^\beta - p^\rho = qp^\gamma.$$

For this equation, it clearly follows that $p^\alpha = p^\rho = p^\gamma$. ■

Hence, we obtain:

Theorem 43. *(1) A finite derivable partially flag-transitive affine plane of order q^2 with linear group is a nonstrict semitranslation plane with a translation group of order q^3p^γ.*

(2) The plane admits either an elation group or a Baer group of order q.

(3) Furthermore, in the elation case, either the plane is Desarguesian or the plane admits a $((\infty), \ell_\infty)$-transitivity, the point (∞) is invariant and the infinite points not equal to (∞) are centers for translation groups of order qp^γ.

Corollary 44. *A finite derivable partially flag-transitive affine plane of order q^2 where $(p, r) = 1$ for $p^r = q$, is a nonstrict semitranslation plane with a translation group of order q^3p^γ.*

We have seen that dual translation planes arising from translation planes with spreads in $PG(3, q)$ that admit a collineation group of order q^2 in the translation complement and transitive on the components other than a fixed component admit collineation groups of order $q^5(q-1)$ fixing a derivable net $\mathcal{D}$. Since the plane is a dual translation plane, there is an elation group of order q^2. Hence, there is a collineation group of order $q^6(q-1)$. We may ask if partially flag-transitive planes admitting the larger group must be dual translation planes. We first establish some additional general properties.

Theorem 45. *Let π be a finite derivable affine plane of order q^2 admitting an elation group H of order q leaving invariant a derivable net. Then the H-orbits of infinite points union the center of H define a set of q derivable nets of π.*

Proof. Let $\mathcal{D}$ denote the derivable net in question and choose (∞) to be the center of H. Let T be any line of $\pi - \mathcal{D}$. Then there is a dual translation plane defined by the algebraic extension process admitting H as a collineation group. But, it is now clear that any dual translation plane has the property stated in the theorem and one of the derivable nets is defined by the image of T under H union the center of H. But, the orbits of H in the dual translation plane share at least the orbit of π containing T. Since this argument is valid for any such line, it follows that the affine plane is a union of derivable nets whose infinite points share (∞). ∎

Lemma 46. *If π is a non-Desarguesian partially flag-transitive affine plane of order q^2 with elation group of order q then H is normal in the subgroup of the stabilizer of the derivable net $\mathcal{D}$ that fixes the axis of H.*

Proof. We have seen that (∞) is invariant. Since H is the maximal elation group with axis $x = 0$ fixing the derivable net, it follows that the stabilizer of $x = 0$ must normalize H. ∎

Hence, we see that

Theorem 47. *Let π be a non-Desarguesian partially flag-transitive affine plane of order q^2 with linear group and of elation type. Then the corresponding group G fixes one derivable net containing the axis of (∞) and acts transitively on the remaining $q-1$ derivable nets sharing (∞).*

Theorem 48. *Let π be a non-Desarguesian partially flag-transitive affine plane of order q^2 with linear group and of elation type. If $q = p^r$, assume that $(p, r) = 1$.*

(1) If π admits a p-group S of order q^6, or $2q^6$ if $q = 2$, then π admits a collineation group which fixes an infinite point (∞) and is transitive on the remaining infinite points.

(2) Furthermore, either the plane is a dual translation plane or the full group acts two-transitively on a set of q derivable nets sharing the infinite point (∞).

Proof. Let H^+ denote the full elation group with center (∞). Since H^+ is elementary Abelian, H^+ permutes the orbits of H on the line at infinity each of which defines a derivable net. It follows that the full collineation group of π which fixes the axis of H^+ must normalize H^+. Let S be a p-group of order q^6. Then S must fix (∞). Within S, there is a subgroup of order at least q^4 which fixes $x = 0$ and there is a subgroup S^- of order at least q^2 which fixes a point 0 on $x = 0$. Suppose that some element g of S^- fixes a line ℓ incident with 0. So, g leaves invariant the H^+-orbit containing ℓ.

Now assume that there exists another elation group of order q, H_1 of H^+, of order q, whose orbits define derivable nets. Then, either H_1 is H or $H_1 \cap H = \langle 1 \rangle$. Hence, H is characteristic in H^+ and is, hence, normal or $\langle H_1, H \rangle$ has order q^2. That is, either the plane is a dual translation plane or H is normal in the full collineation group of the affine plane.

If g fixes ℓ then g fixes two derivable nets, one of which does not contain ℓ. Hence, the stabilizer of ℓ may be regarded as a subgroup of $P\Gamma L(4, q)_{N^*}$ for some line N^* as indicated in the embedding. However, if $(p, r) = 1$ then the group is linear with respect to this group. In this setting, we have seen that p can only be 2 and the order of the stabilizer can only be 2. So, we have a group which acts transitively on the infinite points not equal to (∞) and two-transitive on the derivable nets sharing (∞). This completes the proof of the theorem. ∎

Corollary 49. *Let π be a finite derivable partially flag-transitive affine plane with linear group which is a translation plane and assume that the order is even. Then π is either a $(T, EH_{p,f,p\lambda,f\lambda})$-plane or a $(T^*, BH_{p,p\lambda,f,f\lambda})$-plane.*

Proof. We may assume that we have an elation group of order q and that the plane is a translation plane.

When q is even and we have a group of order divisible by $q-1$ acting on the derivable net then, since $(q-1, q+1) = 1$, it follows that any element of order dividing $q-1$ must leave at least two Baer subplanes invariant. We now assume that the groups in question are in the translation complement. Now there is a subgroup of $GL(2, q)GL(2, q)$ which normalizes the elation group E of order q. Assuming that E is a subgroup of the first $GL(2, q)$, it follows that we have a subgroup H^* of the first $GL(2, q)$ of order $(q-1)^2$ which normalizes E so that our group is a subgroup of $H^*GL(2, q)$. We note that an element of H^* will fix either exactly two or all Baer subplanes incident with the zero vector of $\mathcal{D}$. Let g be an element of order dividing $q-1$. Let $g = g^*g_*$ where g^* is in H^* and g_* is in the second $GL(2, q)$ and which is generated by Baer collineations of the net. It also follows that g_* also fixes exactly two or all Baer subplanes of the net. Since g^* and g_* commute and $q-1$ is odd, it follows that an element g will fix either exactly two or all Baer subplanes incident with the zero vector. Hence, we may assume that there is a set of at least $q-1$ group elements that fix two Baer subplanes incident with the zero vector. With the appropriate coordinate change, we notice that these group elements will act as distinct diagonal elements. These group elements will generate a group of order dividing $(q-1)^4$ which is the direct sum of four cyclic groups of order $q-1$. Let $q-1 = \Pi p_i^{\alpha_i}$ be the prime decomposition. Then there is a p-subgroup of order $p_i^{\beta_i}$ for $\beta_i \geq \alpha_i$. Hence, there is a subgroup of order $p_i^{\alpha_i}$. Since the group is Abelian, it follows that there is a subgroup of order $q-1$. Since the group leaves two Baer subplanes invariant, we clearly have a $H_{p,f,p\lambda,f\lambda}$ group which gives a $(T, EH_{p,f,p\lambda,f\lambda})$-plane. ■

10.1 p-Groups.

Most of what we developed for partially flag-transitive planes could be generalized assuming only that there is a linear p-group S of order q^5 (or $2q^5$ if q is even) fixing the derivable net $\mathcal{D}$.

In this case, S must fix an infinite point (∞) of $\mathcal{D}$. Since S acts on the remaining q^2-q infinite points of $\pi - \mathcal{D}$, it follows that the S-orbits cannot all has lengths strictly larger than q. Hence, there exists an orbit Γ of length $\leq q$ and, for $\alpha \in \Gamma$, S_α has order divisible by q^4 or $2q^4$ if q is even. Then, for ℓ a line of α, $S_{\alpha,\ell}$ has order divisible by q^2 or $2q^2$. Then the spread $S(\ell)$ admits a linear group of order q^2 or $2q^2$ that fixes the line N using the associated embedding in $PG(3,q)$. Since the group is linear, we must have an elation group of order divisible by q with axis N of the translation plane associated with $S(\ell)$.

When q is odd, this implies that there can be no Baer p-elements which implies that the group of order q^2 acts regularly on the components of $S(\ell)$ and this is the largest possible stabilizer. So, when q is odd, S has an orbit of length exactly q, S_α is transitive on the affine lines of α, and $S_{\alpha,\ell}$ is transitive on the points of ℓ. Thus, S_α acts transitively and regularly on the affine points.

When q is even assume we have a linear group of at least order $2q^2$ acting on the translation associated with the spread $S(\ell)$. In Johnson [9], the action of linear groups of order q^2 acting on spreads in $PG(3,q)$ is analyzed. It is shown that the group is transitive or non-Abelian and in the latter case, if not transitive, then has two orbits of components of length $q^2/2$ and there is an elation group E of order q which is in the center of the group. Each of the orbits of length $q^2/2$ are $q/2$ orbits of length q of E. Choose a group of order $2q^2$ containing a given group of order q^2. Hence, the two orbits of length $q^2/2$ are inverted or both fixed. The center E is characteristic in the group of order q^2 and hence normal in the group of order $2q^2$. If the two orbits are not inverted then there is a Baer group B of order 4 which fixes an E-orbit.

We may choose the elements of B to have the following general form:

$$\begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix}$$

where T is a 2×2 matrix over $GF(q)$. At the same time, we may choose the elements of group E to have the following form:

$$\begin{bmatrix} 1 & 0 & u & m(u) \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \forall u \in GF(q) \text{ and } m \text{ is a function on } GF(q).$$

Furthermore, the lines of a net defined by the E orbit of $y = 0$ and incident with the zero vector are of the form:

$$y = x \begin{bmatrix} u & m(u) \\ 0 & u \end{bmatrix} \forall u \in GF(q).$$

Note that B leaves invariant $x = 0$ and $y = 0$ and must leave the orbit of E containing $y = 0$ invariant. Since B has order $2^s \geq 4$, it follows that B must fix another component of this E-orbit which we may take as $y = x$ without loss of generality. This implies that $T = 0$ so that B commutes with E. But, by the incompatibility results of Jha and Johnson [6], B has order less than or equal 2. Hence, the group of order divisible by $2q^2$ is transitive on the components of $S(\ell)$ and the stabilizer of a second component other than N has order exactly 2. Thus, when q is even and there is a 2-group of order $2q^5$, there is a subgroup which is transitive on the affine points.

Hence, in both cases, we obtain a group of order q^4 or $2q^4$ which acts transitive on the affine points of the derivable affine plane. The argument given in the section on partially flag-transitive affine planes applies directly to show:

Theorem 50. *(1) If a derivable affine plane π of order q^2 admits a linear p-group of order q^5 if q is odd or $2q^5$ if q is even then π contains a group which acts transitively on the affine points.*

(2) Furthermore, the group contains either an elation group of order q or a Baer group of order q with axis a subplane of $\mathcal{D}$. If the order of the stabilizer of a point H is at least $2q$ then the order is $2q$ and H is generated by an elation group of order q and a Baer involution with axis in $\mathcal{D}$ or by a Baer group of order q and an elation.

(3) π is a nonstrict semi-translation plane of order q^2 admitting a translation group of order q^3p^γ. Furthermore, either π is a translation plane or there is a unique $((\infty), \ell_\infty)$-transitivity and the remaining infinite points are centers for translation group of orders qp^γ.

(4) If π is non-Desarguesian in the elation case above then π admits a set of q derivable nets sharing the axis of the elation group of order q.

Proof. Consider part (4). By Johnson [7], every infinite point not in $\mathcal{D}$ must be the center for a translation group of order exactly qp^γ. The union of the groups with centers in $\mathcal{D}$ has cardinality $q^2 + q(qp^\gamma) - q$. On the elation side (either the plane or the derived plane), we have an orbit of length q. Hence,

$$(qp^\alpha - 1) + q(qp^\beta - 1) + 1 = q^2 + q^2p^\gamma - q$$

so that

$$p^\alpha + qp^\beta = q + qp^\gamma.$$

Thus, it follows that $p^{\alpha} = q$ and $p^{\beta} = p^{\gamma}$. So, this establishes the structure of the translation groups on the elation side. Moreover, we have that the elation side plane certainly is a nonstrict semi-translation plane of the required type. But, there is a $((\infty), l_{\infty})$-transitivity so the derived plane becomes a nonstrict semitranslation plane with exactly the same properties since the translations fix the derivable net and induce translations in the derived plane. In the derived plane, the natural coordinate structure induced from the original plane shows that there is a $(0), \ell_{\infty})$-transitivity in the Baer side. Hence, the derived plane is also a non-strict semi-translation plane with the same properties on translation subgroups.

In case (4), we have an elation group H of order q and the argument given previously in the partially flag-transitive case shows that there is a set of derivable nets defined by the orbits of H. ■

Corollary 51. *Let π be a derivable affine plane of order q^2 that admits a p-group of order q^6 if q is odd or $2q^6$ if q is even containing a linear subgroup of order q^5 or $2q^5$ leaving invariant the derivable net $\mathcal{D}$ invariant. Assume that when $q = p^r$ then $(r, p) = 1$ and assume that π admits an elation group H of order q.*

(1) Then either the plane is Desarguesian or the center of H is invariant.

(2) Then π admits a collineation group fixing an affine point of order q^2 or $2q^2$ that fixes an infinite point (∞) of $\mathcal{D}$ and acts transitively on the remaining infinite points.

(3) Either the plane is a dual translation plane or the group acts transitively on a set of q derivable nets sharing (∞).

Proof. Reread the proof of the corresponding proof in the partially flag-transitive case to realize that the proof provides the proofs to the statements listed. ■

11 Subplane Covered Nets.

All of these ideas can be generalized by replacing the word 'derivable net' by 'subplane covered net' in any of the definitions. Given a subplane covered net $\mathcal{S}$, there is a projective geometry Σ and a codimension two subspace N of Σ such that the points, lines, parallel classes, subplanes of $\mathcal{S}$ are, respectively, the lines skew to N, points of $\Sigma - N$, hyperplanes of Σ containing N and planes of Σ each of which intersects N in a point (see Johnson [15]).

Theorem 52. *Let $\mathcal{S}$ be a subplane covered net and let T be a transversal to $\mathcal{S}$. Then, using the embedding of $\mathcal{S}$ into the projective space Σ with distinguished codimension two subspace N, it follows that T, as a set of lines of Σ, is a partial line spread of Σ which covers $\Sigma - N$.*

We consider the generalization of the problem on partially flag-transitive affine planes, this time with respect to a subplane covered net. We leave the following as an open problem.

Problem: Let π be a finite affine plane containing a subplane covered net S. If there exists a collineation group G which leaves S invariant and acts flag-transitively on the flags on lines not in S, can π be determined?

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