A note on an hyperbolic differential inclusion in Banach spaces

Mouffak Benchohra

Abstract

In this paper we investigate the existence of solutions on a compact domain to an hyperbolic differential inclusion in Banach spaces. We shall rely on a fixed point theorem for condensing maps due to of Martelli.

1 Introduction

This note deals with the existence of solutions defined on a compact domain for the following hyperbolic differential inclusion (Darboux problem):

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} \in F(x,y,u(x,y)), \quad (x,y) \in J \times J = [0,T] \times [0,T] \tag{1}$$

$$u(x,0) = f(x), \; u(0,y) = g(y) \tag{2}$$

where $F : J \times J \times E \longrightarrow 2^E$ is a multivalued map with nonempty compact and convex values, $f, \; g : J \to E$ and $(E, |.|)$ a separable Banach space.

The single and multivalued finite dimensional versions of the problem (1)-(2) were considered by DeBlasi and Myjak [2], [3] who established the topological regularity of the solutions set. Kubiaczyk [6] considers the single-valued infinite dimensional version of the problem where a Kneser-type theorem is proved for the solutions set. By using a compactness type condition involving the measure of noncompactness, Papageorgiou gives in [9] existence results for the problem (1)-(2).

Received by the editors November 2000.
Communicated by F. Bastin.
1991 *Mathematics Subject Classification :* 35L70, 35L20, 35R70.
Key words and phrases : Hyperbolic differential inclusion, convex multivalued map, existence, condensing map, fixed point .

In this note we shall give an existence result for the problem (1)-(2). The method we are going to use is to reduce the existence of solutions to problem (1)-(2) to the search for fixed points of a suitable multivalued map on the Banach space $C(J \times J, E)$. In order to prove the existence of fixed points, we shall rely on a fixed point theorem for condensing maps due to Martelli [8]. Our result complements the few existence results on the problem.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper. In the sequel we will note $\mathcal{J} = J \times J$.

$C(\mathcal{J}, E)$ is the Banach space of continuous functions from $\mathcal{J}$ into E with the norm

$$\|z\|_\infty := \sup\{|z(x,y)| : (x,y) \in \mathcal{J}\}, \quad \text{for each } z \in C(\mathcal{J}, E).$$

A measurable function $z : \mathcal{J} \longrightarrow E$ is Bochner integrable if and only if $|z|$ is Lebesgue integrable. (For properties of the Bochner integral see Yosida [10]).
$L^1(\mathcal{J}, E)$ denotes the Banach space of measurable functions $z : \mathcal{J} \longrightarrow E$ which are Bochner integrable.

Let $(X, \|\cdot\|)$ be a Banach space. A multivalued map $G : X \longrightarrow 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called upper semicontinuous (u.s.c.) on X if for each $x_* \in X$ the set $G(x_*)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(x_*)$, there exists an open neighbourhood V of x_* such that $G(V) \subseteq B$.

G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

In the following $CC(X)$ denotes the set of all nonempty compact and convex subsets of X.

A multivalued map $G : \mathcal{J} \longrightarrow CC(E)$ is said to be measurable if for each $w \in E$ the function $Y : \mathcal{J} \longrightarrow \mathbb{R}$ defined by

$$Y(x,y) = d(w, G(x,y)) = \inf\{|w - v| : v \in G(x,y)\}$$

is measurable.

Definition 2.1. *A multivalued map $F : \mathcal{J} \times E \longrightarrow 2^E$ is said to be an L^1-Carathéodory if*

(i) $(x,y) \longmapsto F(x,y,u)$ *is measurable for each* $u \in E$;

(ii) $u \longmapsto F(x,y,u)$ *is upper semicontinuous for almost all* $(x,y) \in \mathcal{J}$;

(iii) For each $k > 0$, there exists $h_k \in L^1(\mathcal{J}, \mathbb{R}_+)$ such that

$$\|F(x,y,u)\| = \sup\{\|v\| : v \in F(x,y,u)\} \leq h_k(t) \quad \textit{for all} \ \ |u| \leq k$$
$$\textit{and for almost all} \ \ (x,y) \in \mathcal{J}.$$

An upper semi-continuous map $G : X \longrightarrow 2^X$ is said to be condensing if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompactness. For properties of the Kuratowski measure, we refer to Banas and Goebel [1].

We remark that a completely continuous multivalued map is the easiest example of a condensing map. For more details on multivalued maps see the books of Deimling [4] and Hu and Papageorgiou [5].

We will need the following hypotheses:

(H1) $F : \mathcal{J} \times E \longrightarrow CC(E)$ is an L^1- Carathéodory multivalued map and for each fixed $u \in C(\mathcal{J}, E)$ the set

$$S_{F,u} = \Big\{ v \in L^1(\mathcal{J}, E) : \ v(x,y) \in F(x,y,u(x,y)) \text{ for a.e. } (x,y) \in \mathcal{J} \Big\}$$

is nonempty;

(H2) There exists $H \in L^1(\mathcal{J}, \mathbb{R}^+)$ such that

$$\|F(x,y,u)\| := \sup\{\|v\| : v \in F(x,y,u)\} \leq H(x,y)$$

for almost all $(x,y) \in \mathcal{J}$ and all $u \in E$;

(H3) The functions $f, \ g : J \to E$ are continuous with $f(0) = g(0)$;

(H4) For each bounded set $B \subseteq C(\mathcal{J}, E)$ and for each $(x,y) \in \mathcal{J}$ the set

$$\{f(x) + g(y) - f(0) + \int_0^x \int_0^y v(t,s)dtds : v \in S_{F,B}\}$$

is relatively compact in E, where $S_{F,B} = \cup\{S_{F,u} : u \in B\}$.

Remark 2.2. *(i) If $dimE < \infty$, then for each $u \in C(\mathcal{J}, E)$ the set $S_{F,u}$ is nonempty (see Lasota and Opial [7]).*
(ii) If $dimE = \infty$ then $S_{F,u}$ is nonempty if and only if the function $Y : \mathcal{J} \longrightarrow \mathbb{R}^+$ defined by

$$Y(x,y) := \inf\{|v| : v(x,y) \in F(x,y,u(x,y))\}$$

is measurable (see Hu and Papageorgiou [5]).

Definition 2.3. *By a solution of (1)-(2) we mean a function $u(.,.) \in C(\mathcal{J}, E)$ such that there exists $v \in L^1(\mathcal{J}, E)$ for with we have*

$$u(x,y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v(t,s)dtds \quad \textit{for each } (x,y) \in \mathcal{J}$$

and with $v(t,s) \in F(t,s,u(t,s))$ a.e. on $\mathcal{J}$.

Our considerations are based on the following lemmas.

Lemma 2.4. *[7]. Let F be a multivalued map satisfying (H1) and let Γ be a linear continuous mapping from $L^1(\mathcal{J}, E)$ to $C(\mathcal{J}, E)$, then the operator*

$$\Gamma \circ S_F : C(\mathcal{J}, E) \longrightarrow CC(C(\mathcal{J}, E)),\ u \longmapsto (\Gamma \circ S_F)(u) := \Gamma(S_{F,u})$$

is a closed graph operator in $C(\mathcal{J}, E) \times C(\mathcal{J}, E)$.

Lemma 2.5. *[8]. Let X be a Banach space and $N : X \longrightarrow CC(X)$ be a condensing map. If the set*

$$\Omega := \{u \in X : \lambda u \in N(u) \ \ \textit{for some} \ \ \lambda > 1\}$$

is bounded, then N has a fixed point.

3 Main Result

Now, we are able to state and prove our main theorem.

Theorem 3.1. *Assume that hypotheses (H1)-(H4) hold. Then the problem (1)- (2) has at least one solution on $\mathcal{J}$.*

Proof. Let $C(\mathcal{J}, E)$ be the Banach space provided with the norm

$$\|u\|_\infty := \sup\{|u(x, y)| : (x, y) \in \mathcal{J}\},\ \text{for } u \in C(\mathcal{J}, E).$$

Transform the problem into a fixed point problem. Consider the multivalued map, $N : C(\mathcal{J}, E) \longrightarrow 2^{C(\mathcal{J},E)}$ defined by:

$$N(u) = \left\{h \in C(\mathcal{J}, E) : h(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v(t, s)dtds\right\}$$

where

$$v \in S_{F,u} = \left\{v \in L^1(\mathcal{J}, E) : v(t, s) \in F(t, s, u(t, s)) \ \text{ for a.e. } \ (t, s) \in \mathcal{J}\right\}.$$

Remark 3.2. *It is clear that the fixed points of N are solutions to (1)-(2).*

We shall show that N satisfies the assumptions of Lemma 2.5. The proof will be given in several steps.

Step 1: *$N(u)$ is convex for each $u \in C(\mathcal{J}, E)$.*

Indeed, if $h_1,\ h_2$ belong to $N(u)$, then there exist $v_1, v_2 \in S_{F,u}$ such that for each $(x, y) \in \mathcal{J}$ we have

$$h_i(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v_i(t, s)dtds, \quad i = 1, 2.$$

Let $0 \leq \alpha \leq 1$. Then for each $(x, y) \in \mathcal{J}$ we have

$$(\alpha h_1 + (1-\alpha)h_2)(x, y) \quad = f(x) + g(y) - f(0) + \int_0^x \int_0^y [\alpha v_1(t, s) + (1-\alpha)v_2(t, s)]ds.$$

Since $S_{F,u}$ is convex (because F has convex values) then

$$\alpha h_1 + (1-\alpha)h_2 \in N(u).$$

Step 2: *N is bounded on bounded sets of $C(\mathcal{J}, E)$.*

Indeed, it is enough to show that there exists a positive constant c such that for each $h \in N(u), u \in B_r = \{u \in C(\mathcal{J}, E) : \|u\|_\infty \leq r\}$ one has $\|h\|_\infty \leq c$.

If $h \in N(u)$, then there exists $v \in S_{F,u}$ such that for each $(x, y) \in \mathcal{J}$ we have

$$h(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v(t, s)dtds.$$

By (H1) we have for each $(x, y) \in \mathcal{J}$ that

$$\|h(x, y)\| \leq |f(x)| + |g(y)| + |f(0)| + \int_0^x \int_0^y h_r(t, s)dtds.$$

Then

$$\|h\|_\infty \leq \|f\|_\infty + \|g\|_\infty + |f(0)| + \int_0^T \int_0^T h_r(t, s)dtds = c.$$

Step 3: *N sends bounded sets of $C(\mathcal{J}, E)$ into equicontinuous sets.*

Let $(x_1, y_1), (x_2, y_2) \in \mathcal{J},\ x_1 < x_2,\ y_1 < y_2$ and B_r be a bounded set of $C(\mathcal{J}, E)$. For each $u \in B_r$ and $h \in N(u)$, there exists $v \in S_{F,u}$ such that

$$h(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v(t, s)dtds.$$

Thus we obtain

$$\begin{aligned} \|h(x_2, y_2) - h(x_1, y_1)\| &\leq |f(x_2) - f(x_1)| + |g(y_2) - g(y_1)| + \int_{x_1}^{x_2} \int_{y_1}^{y_2} |v(t, s)|dtds \\ &\leq |f(x_2) - f(x_1)| + |g(y_2) - g(y_1)| + \int_{x_1}^{x_2} \int_{y_1}^{y_2} h_r(t, s)dtds. \end{aligned}$$

As $(x_2, y_2) \longrightarrow (x_1, y_1)$ the right-hand side of the above inequality tends to zero.

As a consequence of Step 2, Step 3 and (H4) together with the Arzela-Ascoli theorem we can conclude that N is completely continuous and therefore a condensing map.

Step 4: *N has a closed graph.*

Let $u_n \longrightarrow u_*,\ h_n \in N(u_n)$, and $h_n \longrightarrow h_*$. We shall prove that $h_* \in N(u_*)$.

$h_n \in N(u_n)$ means that there exists $v_n \in S_{F,u_n}$ such that

$$h_n(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v_n(t, s)dtds,\ (x, y) \in \mathcal{J}.$$

We must prove that there exists $g_* \in S_{F,u_*}$ such that

$$h_*(x,y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v_*(t,s)dtds,\ (x,y) \in \mathcal{J}.$$

Now, we consider the linear continuous operator

$$\Gamma : L^1(\mathcal{J}, E) \longrightarrow C(\mathcal{J}, E)$$

$$v \longmapsto \Gamma(v)(x,y) = \int_0^x \int_0^y v(t,s)dtds,\ (x,y) \in \mathcal{J}.$$

From Lemma 2.4, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Clearly we have

$$\|(h_n(x,y) - f(x) - g(y) + f(0)) - (h_*(x,y) - f(x) - g(y) + f(0))\|_\infty \to 0 \text{ as } n \to \infty.$$

Moreover from the definition of Γ we have

$$(h_n(x,y) - f(x) - g(y) + f(0)) \in \Gamma(S_{F,u_n}).$$

Since $u_n \longrightarrow u_*$, it follows from Lemma 2.4 that

$$h_*(x,y) - f(x) - g(y) + f(0) = \int_0^x \int_0^y v_*(t,s)dtds,\ (x,y) \in \mathcal{J}$$

for some $v_* \in S_{F,u_*}$.

Step 5: *The set*

$$\Omega := \{u \in C(\mathcal{J}, E) : \lambda u \in N(u) \ \ \textit{for some} \ \ \lambda > 1\}$$

is bounded.

Let $u \in \Omega$. Then $\lambda u \in N(u)$ for some $\lambda > 1$. Thus there exists $v \in S_{F,u}$ such that

$$u(x,y) = \lambda^{-1} f(x) + \lambda^{-1} g(y) - \lambda^{-1} f(0) + \lambda^{-1} \int_0^x \int_0^y v(t,s)dtds,\ \ (x,y) \in \mathcal{J}.$$

This implies by (H2) that for each $(x,y) \in \mathcal{J}$ we have

$$\|u(x,y)\| \leq |f(x)| + |g(y)| + |f(0)| + \int_0^x \int_0^y H(t,s)dtds.$$

Thus

$$\|u\|_\infty \leq \|f\|_\infty + \|g\|_\infty + |f(0)| + \int_0^T \int_0^T H(t,s)dtds = K.$$

This shows that Ω is bounded.

Set $X := C(\mathcal{J}, E)$. As a consequence of Lemma 2.5 we deduce that N has a fixed point which is a solution of (1)-(2) on $\mathcal{J}$.

References

[1] J. Banas and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel-Dekker, New York, 1980.

[2] F. De Blasi and J. Myjak, On the structure of the set of solutions of the Darboux problem for hyperbolic equations, *Proc. Edinburgh Math. Soc.* **29** (1986), 7-14.

[3] F. De Blasi and J. Myjak, On the set of solutions of a differential inclusion, *Bull. Inst. Math., Acad. Sin.* **14** (1986), 271-275.

[4] K. Deimling, *Multivalued Differential Equations*, Walter de Gruyter, Berlin-New York, 1992.

[5] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis*, Kluwer, Dordrecht, Boston, London, 1997.

[6] I. Kubiaczyk, Kneser's theorem for hyperbolic equations, *Fonctiones et Approximatio* **14** (1984), 183-196.

[7] A. Lasota and Z. Opial, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **13** (1965), 781-786.

[8] M. Martelli, A Rothe's type theorem for noncompact acyclic-valued map, *Boll. Un. Math. Ital.* **4** (1975), 70-76.

[9] N.S. Papageorgiou, Existence of solutions for hyperbolic differential inclusions in Banach spaces, *Arch. Math.* (Brno) **28** (1992), 205-213.

[10] K. Yosida, *Functional Analysis*, 6^{th} edn. Springer-Verlag, Berlin, 1980.

Department of Mathematics
University of Sidi Bel Abbès
BP 89, 22000 Sidi Bel Abbès, Algeria
e-mail: benchohra@yahoo.com