On copies of c_0 and ℓ_∞ in $L_{w^*}(X^*, Y)$

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Abstract

The aim of this paper is to prove that (a) $L_{w^*}(X^*, Y)$ contains a copy of c_0 if and only if either X or Y contains a copy of c_0, or $L_{w^*}(X^*, Y)$ contains a copy of ℓ_∞, and (b) If both X and Y contain a copy of c_0, then $L_{w^*}(X^*, Y)$ contains a copy of ℓ_∞. From these facts we extract some consequences.

1 Preliminaries

If X and Y are two Banach spaces over the same field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), we denote by $L(X, Y)$ the Banach space of all bounded linear operators from X into Y equipped with the operator norm and by $L_{w^*}(X^*, Y)$ the closed linear subspace of $L(X^*, Y)$ formed by all weak*-weakly continuous linear operators. The closed subspace of $L_{w^*}(X^*, Y)$ consisting of all those compact operators will be designed by $K_{w^*}(X^*, Y)$, whereas $W(X, Y)$ will stand for the closed linear subspace of $L(X, Y)$ consisting of all weakly compact operators. If (Ω, Σ, μ) is a non-trivial positive finite measure space and X a Banach space, we will denote by $\mathcal{P}_1(\mu, X)$ the linear space over the field K of all X-valued [classes of scalarly equivalent] weakly μ-measurable Pettis integrable functions f defined on Ω, equipped with the norm

$$\|f\|_1 = \sup\left\{\int_\Omega |x^* f(\omega)|\, d\mu(\omega) : x^* \in X^*,\ \|x^*\| \leq 1\right\}.$$

In what follow we will shorten the sentence 'weakly unconditionally Cauchy' by 'wuC'. Three relevant results concerning copies of c_0 and ℓ_∞ in $L_{w^*}(X^*, Y)$ and $K_{w^*}(X^*, Y)$ are in order.

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Theorem 1.1. ([1, main Thm.]) *$K_{w^*}(X^*, Y)$ contains a copy of ℓ_∞ if and only if either X contains a copy of ℓ_∞ or Y contains a copy of ℓ_∞.*

Theorem 1.2. ([2, main Thm.]) *Assuming $L_{w^*}(X^*, Y)$ contains a complemented copy of c_0, then either X contains a copy of c_0 or Y contains a copy of c_0.*

Theorem 1.3. ([3, main Thm.]) *If c_0 embeds into $K_{w^*}(X^*, Y)$, either $K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$ or $K_{w^*}(X^*, Y)$ is uncomplemented in $L_{w^*}(X^*, Y)$.*

The aim of this paper is to complete the study of copies of c_0 in $L_{w^*}(X^*, Y)$ by proving the two theorems below, from which we will obtain several consequences; among them, Theorem 1.2.

Theorem 1.4. *$L_{w^*}(X^*, Y)$ contains a copy of c_0 if and only if either (a) X or Y contains a copy of c_0, or (b) $L_{w^*}(X^*, Y)$ contains a copy of ℓ_∞.*

Theorem 1.5. *If both X and Y contain a copy of c_0, then $L_{w^*}(X^*, Y)$ contains a copy of ℓ_∞.*

2 Proof of Theorem 1.4

We will show the nontrivial 'only if' part, which is essentially contained in [2]. So assume that c_0 embeds into $L_{w^*}(X^*, Y)$ but neither X nor Y contain a copy of c_0. Let $\{T_n\}$ be a normalized sequence in $L_{w^*}(X^*, Y)$ equivalent to the unit vector basis $\{e_n\}$ of c_0, and let J be a topological isomorphism from c_0 into $L_{w^*}(X^*, Y)$ such that $Je_n = T_n$ for each $n \in \mathbb{N}$. Since the formal series $\sum_{n=1}^\infty T_n$ is wuC and the linear form on $L_{w^*}(X^*, Y)$ given by $T \to y^* T x^*$ is continuous for each $x^* \in X^*$ and $y^* \in Y^*$, it follows that $\sum_{n=1}^\infty |y^* T_n x^*| < \infty$. Hence, the series $\sum_{n=1}^\infty T_n x^*$ in Y is wuC for each $x^* \in X^*$ and, as Y contains no copy of c_0, this implies that the series $\sum_{n=1}^\infty \xi_n T_n x^*$ converges [in norm] in Y for each $\xi \in \ell_\infty$ and $x^* \in X^*$. This fact allows us to consider the linear map $\varphi : \ell_\infty \to L(X^*, Y)$ defined by

$$(\varphi\xi)\, x^* = \sum_{n=1}^{\infty} \xi_n T_n x^*$$

for each $x^* \in X^*$. This linear operator is well-defined and bounded. Indeed, choosing $C > 0$ such that

$$\sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{n} \xi_i T_i x^* \right\|_Y \leq C \|x^*\| \|\xi\|_\infty$$

for each $\xi \in \ell_\infty$ and $x^* \in X^*$, for each fixed pair $(\xi, x^*) \in \ell_\infty \times X^*$ there exists $n_0 \in \mathbb{N}$ with $\left\| \sum_{i=n_0+1}^\infty \xi_i T_i x^* \right\|_Y < \epsilon$, which implies that

$$\|(\varphi\xi)\, x^*\|_Y \leq C \|x^*\| \|\xi\|_\infty + \epsilon.$$

This shows at the same time that $\varphi\xi \in L(X^*, Y)$ and that φ is bounded. Now let us prove that $\varphi(\ell_\infty) \subseteq L_{w^*}(X^*, Y)$.

Let ξ be a fixed non null element of ℓ_∞. We are going to see that $\varphi\xi$ is weak*-weakly continuous. Since $T \to x^*(T^* y^*) = y^* T x^*$ is a continuous linear form on $L(X^*, Y)$, then $\sum_{n=1}^\infty |x^*(T_n^* y^*)| < \infty$ and thus the series $\sum_{n=1}^\infty T_n^* y^*$ in X is wuC for each $y^* \in Y^*$. Given that c_0 is not embedded into X, then $\sum_{n=1}^{\infty} T_n^* y^*$ is unconditionally convergent in norm for each $y^* \in Y^*$. In particular, $\sum_{n=1}^{\infty} \xi_n T_n^* y^*$ converges in X for each $y^* \in Y^*$. Let $\{x_d^* : d \in D\}$ be a net in X^* which converges to some $x^* \in X^*$ under the weak* topology of X^*. Working from now onwards with some concrete $y^* \in Y^*$ and given $\epsilon > 0$, there is $k \in D$ such that

$$\left|\left\langle x_d^* - x^*, \sum_{n=1}^{\infty} \xi_n T_n^* y^* \right\rangle\right| < \epsilon$$

for each $d > k$. On the other hand, since $\sum_{n=1}^{\infty} \xi_n T_n^* y^*$ converges in norm in X, then

$$\lim_{n\to\infty} \sum_{i=1}^{n} \xi_i \left(x_d^* - x^*\right) T_i^* y^* = \left|\left\langle x_d^* - x^*, \sum_{n=1}^{\infty} \xi_n T_n^* y^* \right\rangle\right|$$

for each $d \in D$, and due to the fact that $\sum_{n=1}^{\infty} \xi_n T_n \left(x_d^* - x^*\right)$ converges in norm in Y to $(\varphi\xi)\left(x_d^* - x^*\right)$ for each $d \in D$, one has that

$$\begin{aligned} y^* (\varphi\xi)\left(x_d^* - x^*\right) &= \lim_{n\to\infty} \sum_{i=1}^{n} \xi_i y^* T_i \left(x_d^* - x^*\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \xi_i \left(x_d^* - x^*\right) T_i^* y^* \\ &= \left|\left\langle x_d^* - x^*, \sum_{n=1}^{\infty} \xi_n T_n^* y^* \right\rangle\right| \end{aligned}$$

for each $d \in D$. Therefore,

$$\left|y^* (\varphi\xi)\left(x_d^* - x^*\right)\right| < \epsilon$$

for each $d > k$, i.e. $y^* (\varphi\xi)\, x_d^* \to y^* (\varphi\xi)\, x^*$. Since this is true for every $y^* \in Y^*$, it follows that $(\varphi\xi)\, x_d^* \to (\varphi\xi)\, x^*$ under the weak topology of Y. Consequently, we have that $\varphi(\ell_\infty) \subseteq L_{w^*}(X^*Y)$ as stated. Finally, since

$$\begin{aligned} \|\varphi e_n\| &= \sup\left\{\|(\varphi e_n)\, x^*\|_Y : x^* \in X^*, \|x^*\| \le 1\right\} \\ &= \sup\left\{\|T_n x^*\|_Y : x^* \in X^*, \|x^*\| \le 1\right\} = \|T_n\| = 1 \end{aligned}$$

for each $n \in \mathbb{N}$, Rosenthal's ℓ_∞ theorem implies that ℓ_∞ embeds into $L_{w^*}(X^*Y)$.

3 Proof of Theorem 1.5

Let $\{x_n\}$ and $\{y_n\}$ be two normalized basic sequences in X and Y, respectively, equivalent to the unit vector basis $\{e_n\}$ of c_0. Since the formal series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are wuC, it follows that $x_n \to 0$ under the weak topology of X and $y_n \to 0$ under the weak topology of Y. Consider the linear mapping $\psi : \ell_\infty \to L(X^*, Y)$ defined by

$$(\psi\xi)\, x^* = \sum_{n=1}^{\infty} \xi_n x^* x_n \cdot y_n$$

for each $x^* \in X^*$. This linear operator is well-defined and bounded. Indeed, first note that $\xi_n x^* x_n \to 0$, so $\sum_{n=1}^{\infty} \xi_n x^* x_n \cdot y_n$ converges in Y in norm. On the other hand, if $C > 0$ satisfies that

$$\sup_{n\in\mathbb{N}} \left\|\sum_{i=1}^{n} \xi_i x^* x_i \cdot y_i\right\|_Y \le C \sup_{n\in\mathbb{N}} |\xi_n x^* x_n| \le C\, \|\xi\|_\infty \|x^*\|$$

for each $\xi \in \ell_\infty$ and $x^* \in X^*$, then

$$\|(\psi\xi)\, x^*\|_Y = \left\| \sum_{i=1}^{\infty} \xi_i x^* x_i \cdot y_i \right\|_Y \leq C \, \|\xi\|_\infty \, \|x^*\|$$

for each $\xi \in \ell_\infty$ and $x^* \in X^*$. Hence, $\psi\xi \in L(X^*, Y)$ for each $\xi \in \ell_\infty$ and ψ is bounded. Now, let us show that $\psi(\ell_\infty) \subseteq L_{w^*}(X^*, Y)$. So, choose some fixed $\xi \in \ell_\infty$ and consider a net $\{x_d^* : d \in D\}$ in X^* converging to some $x^* \in X^*$ under the weak* topology of X^*. We have to prove that $y^*(\psi\xi)\, x_d^* \to y^*(\psi\xi)\, x^*$ for each $y^* \in Y^*$. Thus, let us work with some concrete $y^* \in Y^*$. Since $\sum_{n=1}^{\infty} \xi_n y^* y_n \cdot x_n \in X$, we have

$$x_d^* \left(\sum_{n=1}^{\infty} \xi_n y^* y_n \cdot x_n \right) \to x^* \left(\sum_{n=1}^{\infty} \xi_n y^* y_n \cdot x_n \right)$$

But, since $\sum_{n=1}^{\infty} \xi_n y^* y_n \cdot x_d^* x_n = x_d^* \left(\sum_{n=1}^{\infty} \xi_n y^* y_n \cdot x_n \right)$ for each $d \in D$ and

$$\sum_{n=1}^{\infty} \xi_n y^* y_n \cdot x^* x_n = x^* \left(\sum_{n=1}^{\infty} \xi_n y^* y_n \cdot x_n \right),$$

we obtain that

$$\sum_{n=1}^{\infty} \xi_n y^* y_n \cdot x_d^* x_n \to \sum_{n=1}^{\infty} \xi_n y^* y_n \cdot x^* x_n. \tag{3.1}$$

On the other hand, since $\sum_{n=1}^{\infty} \xi_n x^* x_n \cdot y_n$ converges in Y, we have

$$\sum_{n=1}^{\infty} \xi_n x_d^* x_n \cdot y^* y_n = y^* \left(\sum_{n=1}^{\infty} \xi_n x_d^* x_n \cdot y_n \right) = y^*(\psi\xi)\, x_d^* \tag{3.2}$$

and

$$\sum_{n=1}^{\infty} \xi_n x^* x_n \cdot y^* y_n = y^* \left(\sum_{n=1}^{\infty} \xi_n x^* x_n \cdot y_n \right) = y^*(\psi\xi)\, x^* \tag{3.3}$$

Therefore, from (3.1), (3.2) and (3.3), we conclude that

$$y^*(\psi\xi)\, x_d^* \to y^*(\psi\xi)\, x^*$$

as required.

Finally, since $\|\psi e_n\| = \sup\{\|x^* x_n \cdot y_n\| : \|x^*\| \leq 1\} = \|x_n\| \, \|y_n\| = 1$ for each $n \in \mathbb{N}$, Rosenthal's ℓ_∞ theorem guarantees that ℓ_∞ is embedded into $L_{w^*}(X^*, Y)$.

4 Some consequences

Corollary 4.1. *If $\mathcal{P}_1(\mu, X)$ contains a copy of c_0, then either X contains a copy of c_0 or $L_{w^*}(X^*, L_1(\mu))$ contains a copy of ℓ_∞.*

Proof. According to a result of Huff [4], $\mathcal{P}_1(\mu, X)$ is embedded into $L_{w^*}(X^*, L_1(\mu))$. Since $L_1(\mu)$ contains no copy of c_0, the statement of the corollary is an obvious consequence of Theorem 1.4. ∎

Corollary 4.2. (Theorem 1.2) *Assuming $L_{w^*}(X^*, Y)$ contains a complemented copy of c_0, then either X contains a copy of c_0 or Y contains a copy of c_0.*

Proof. Looking at the proof of Theorem 1.4, assuming by contradiction that neither X or Y contains a copy of c_0 and denoting by P a bounded projection operator from $L_{w^*}(X^*, Y)$ onto $J(c_0)$, then $J^{-1} \circ P \circ \varphi$ is a bounded quotient map from ℓ_∞ onto c_0, a contradiction. ∎

Corollary 4.3. *Assume that $W(X, Y)$ contains a copy of c_0. If c_0 is not embedded into X^* or Y, then $W(X, Y)$ contains a copy of ℓ_∞.*

Proof. This is due to Theorem 1.4 and to the fact that $W(X, Y)$ is isomorphic to $L_{w^*}(X^{**}, Y)$. ∎

Corollary 4.4. *Assume that $L_{w^*}(X^*, Y)$ contains a copy of c_0. If $K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$, then either X contains a copy of c_0 or Y contains a copy of c_0.*

Proof. Assume $L_{w^*}(X^*, Y)$ contains a copy of c_0. If neither X or Y contains a copy of c_0, according to Theorem 1.4, $L_{w^*}(X^*, Y)$ must contain a copy of ℓ_∞. Since $K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$, applying Theorem 1.1, either X or Y contains a copy of ℓ_∞, a contradiction. ∎

Corollary 4.5. *Assume that both X and Y contain a copy of c_0. If neither X nor Y contain a copy of ℓ_∞, then $K_{w^*}(X^*, Y)$ is not complemented in $L_{w^*}(X^*, Y)$.*

Proof. According to Theorem 1.5, $L_{w^*}(X^*, Y)$ must contain a copy of ℓ_∞. But since neither X nor Y contain a copy of ℓ_∞, Theorem 1.1 implies that $K_{w^*}(X^*, Y) \neq L_{w^*}(X^*, Y)$. Since c_0 is embedded into $K_{w^*}(X^*, Y)$, Theorem 1.3 guarantees that $K_{w^*}(X^*, Y)$ is uncomplemented in $L_{w^*}(X^*, Y)$. ∎

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