A Multiplicity Result for Nonlinear Second Order Periodic Equations with Nonsmooth Potential

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Abstract

In this paper we study a quasilinear scalar periodic problem with a nondifferentiable potential function. We only assume that, as a function of the state variable, the potential is locally Lipschitz. So the gradient is replaced by the generalized subdifferential in the sense of Clarke. Using a variational approach, based on the nonsmooth critical point theory of Chang (see [1]), we prove the existence of at least three distinct solutions for the periodic problem. An example is also presented, illustrating that our hypotheses on the potential function are realistic.

1 Introduction

Recently there have been some works dealing with the quasilinear scalar periodic problems driven by the one-dimensional p-Laplacian $\Delta_p x \stackrel{df}{=} (|x'|^{p-2}x')'$. We refer to the papers of Del Pino-Manasevich-Murua [5], Fabry-Fayyad [6], Dang-Oppenheimer [3] and Guo [8]. The study of the vector problem is less detailed and we refer to the papers of Halidias-Papageorgiou [10], Manasevich-Mawhin [14] and Papageorgiou-Yannakakis [17]. It should be mentioned, that from the above work, Dang-Oppenheimer [3] and Manasevich-Mawhin [14] use a more general nonlinear differential operator than the p-Laplacian, which is of the form $(\phi_p(x'))'$, with ϕ_p a strictly monotone, coercive function. In all these work the approach is based either

Received by the editors January 2001.

Communicated by F. Bastin.

1991 *Mathematics Subject Classification :* 47 J 30.

Key words and phrases : Scalar p-Laplacian, first eigenvalue, locally Lipschitz functional, generalized variational derivative, Clarke subdifferential, critical point, nonsmooth Palais-Smale condition, mountain pass theorem, maximal monotone and generalized pseudomonotone operators.

on degree theoretic techniques or the theory of nonlinear operators and fixed point arguments and deal with the existence problems. With the problem of existence of multiple periodic solutions, deal only the work of Del Pino-Manasevich-Murua [5], who assume a continuous right hand side function $f(t,\zeta)$ and their methods of proof is based on the interaction between the Fučik spectrum of the p-Laplacian and the nonlinearity of $f(t,\zeta)$. To our knowledge no other work addressed the problem of multiple periodic solutions. In this paper we study this problem for scalar equations with a nonsmooth, nonconvex potential, which incorporate equations with discontinuous right hand side. Since the potential is nondifferentiable, the gradient is replaced by the subdifferential and the resulting problem is a quasilinear second order periodic differential inclusion, known as hemivariational inequality. Hemivariational inequalities are a new kind of variational inequalities, which were motivated by problems in mechanics, in order to formulate variational principles for nonsmooth and nonconvex energy problems. The hemivariational inequalities formalism has been proved to be an efficient tool in the analysis of several complex mechanical structures, such as multilayered plates, von Karman plates in adhesive contact with rigid support, composite structures etc. For details in these mechanical and engineering applications we refer to the books of Naniewicz-Panagiotopoulos [15] and Panagiotopoulos [16].

2 Preliminaries

Our approach is variational and is based on the critical point theory of Chang [1] for locally Lipschitz energy functionals. The work of Chang uses the subdifferential theory of Clarke [2]. In this section we recall some basic definitions and facts from the theory, which we will use in the sequel. For details we refer to the books of Clarke [2] and Hu-Papageorgiou [10] and to the paper of Chang [1].

Let X be a Banach space and X^* its topological dual. By $\|\cdot\|$ we will denote the norm of X, by $\|\cdot\|_*$ the norm of X^* and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair (X,X^*). A function $\phi : X \longmapsto \mathbb{R}$ is said to be locally Lipschitz, if for every bounded set $U \subseteq X$, there exists $k_U > 0$, such that $|\phi(x)-\phi(y)| \leq k_U\|x-y\|$ for all $x,y \in U$.

Recall that, if $\psi : X \longrightarrow \overline{\mathbb{R}} \stackrel{df}{=} \mathbb{R}\cup\{+\infty\}$ is proper, convex and lower semicontinuous (i.e. $\psi \in \Gamma_0(X)$), then ψ is locally Lipschitz in the interior of its effective domain $\mathrm{dom}\psi \stackrel{df}{=} \{x \in X : \ \psi(x) < +\infty\}$. So, a coercive $\mathbb{R}$-valued function on X is locally Lipschitz. The function $\phi^0 : X \times X \longrightarrow \mathbb{R}$, defined by

$$\phi^0(x;h) \stackrel{df}{=} \limsup_{\substack{y \to x \\ t \searrow 0}} \frac{\phi(y+th)-\phi(y)}{t},$$

is called the generalized directional derivative of ϕ. For every $x \in X$, the function $\mathbb{R} \ni h \longrightarrow \phi^0(x;h) \in \mathbb{R}$ is sublinear continuous and so by the Hahn-Banach theorem is the support function of a nonempty, convex and w*-compact convex set

$$\partial\phi(x) \stackrel{df}{=} \{x^* \in X^* : \ \langle x^*,h\rangle \leq \phi^0(x;h) \ \ \forall h \in X\}.$$

So $\phi^0(x;h) = \sup\{\langle x^*, h\rangle : x^* \in \partial\phi(x)\}$ and the set-valued function $\partial\phi : X \longrightarrow 2^{X^*} \setminus \{\emptyset\}$ is known as the subdifferential (or Clarke subdifferential) of ϕ. This multifunction is sequentially closed in $X \times X^*_{w^*}$, i.e. if $x_n \to x$ in X, $x_n^* \to x^*$ weakly star in X^* and $x_n^* \in \partial\phi(x_n)$ for $n \geq 1$, then $x^* \in \partial\phi(x)$. Also, if $\phi, \psi : X \longmapsto \mathbb{R}$ are locally Lipschitz, then $\partial(\phi + \psi)(x) \subseteq \partial\phi(x) + \partial\psi(x)$ and $\partial(t\phi)(x) = t\partial\phi(x)$ for all $x \in X$ and $t \in \mathbb{R}$. If ϕ is also convex, then the subdifferential $\partial\phi$ coincides with the subdifferential in the sense of convex analysis (see Hu-Papageorgiou [10]). Finally, if ϕ is continuously Gateoux differentiable at x, then $\partial\phi(x) = \{\phi'(x)\}$.

Let $\phi : X \longmapsto \mathbb{R}$ be a given locally Lipschitz function. A point $x \in X$ is said to be a critical point of ϕ, if $0 \in \partial\phi(x)$. Then $c \stackrel{df}{=} \phi(x)$ is a critical value of ϕ. It is easy to check that, if $x \in X$ is a local extremum of ϕ, then x is a critical point of ϕ. It is well-known that the smooth critical point theory uses a compactness condition, known as the Palais-Smale condition (PS-condition). In the present nonsmooth setting, this condition takes the following form:

> A locally Lipschitz function $\phi : X \longmapsto \mathbb{R}$ satisfies the nonsmooth Palais-Smale condition at level c (nonsmooth PS_c-condition), if any sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\phi(x_n) \longrightarrow c$ and $m(x_n) \stackrel{df}{=} \inf\{\|x^*\|_* : x^* \in \partial\phi(x_n)\} \longrightarrow 0$ as $n \to +\infty$ has a strongly convergent subsequence.

If $\phi \in C^1(X)$, then since $\partial\phi(x) = \{\phi'(x)\}$ for all $x \in X$, we see that the above definition of the nonsmooth PS_c-condition coincides with the classical one. Using this notion, we have the following nonsmooth version of the well-known Saddle Point Theorem, due to Rabinowith [18].

Theorem 2.1. If

(i) X *is a reflexive Banach space;*

(ii) $\phi : X \longmapsto \mathbb{R}$ *is a locally Lipschitz function;*

(iii) $X = Y \oplus V$, *where* $\dim Y < +\infty$*;*

(iv) *there exists* $R > 0$ *such that*

$$\max\{\phi(y) : y \in Y, \|y\| = R\} \leq \inf\{\phi(v) : v \in V\};$$

(v) ϕ *satisfies the nonsmooth* PS_{c_0}*-condition, where*

$$c_0 \stackrel{df}{=} \inf_{\gamma\in\Gamma} \max_{y\in D} \phi(\gamma(y)),$$

> *where* $D \stackrel{df}{=} \{y \in Y : \|y\| \leq R\}$ *and* $\Gamma \stackrel{df}{=} \{\gamma \in C(D;X) : \gamma(y) = y$ *whenever* $\|y\| = R\}$,

then $c_0 \geq \inf_{v\in V} \phi(v)$ *and* c_0 *is a critical value of* ϕ. *Moreover, if* $c_0 = \inf_{v\in V} \phi(v)$, *then there exists a critical point* $x \in V$ *of* ϕ *such that* $c_0 = \phi(x)$.

Remark 2.2. *Usually hypothesis* (iv) *is stated with a strict inequality. However, the result is true with more general condition (relaxed boundary condition, see Ghoussoub [8] and Kourogenis-Papageorgiou [12]).*

In our hypotheses, we will use the first nonzero eigenvalue λ_1 of the negative p-Laplacian $-\Delta_p x \stackrel{df}{=} -\left(\|x'\|^{p-2}x'\right)'$ with periodic boundary condition on the interval $[0,b]$. So we consider the following quasilinear eigenvalue problem:

$$(EP) \qquad \begin{cases} -\left(\|x'(t)\|^{p-2}x'(t)\right)' = \lambda\|x(t)\|^{p-2}x(t) \\ \qquad\qquad\qquad \text{almost everywhere on (0,b)} \\ x(0) = x(b), \quad x'(0) = x'(b). \end{cases}$$

It is well-known that $\lambda_0 = 0$ is an eigenvalue of (EP) and is simple and isolated. So, if $\lambda_1 \stackrel{df}{=} \inf\{\lambda > 0 : \ \lambda \text{ is an eigenvalue of } (EP)\}$, then $\lambda_1 > 0$ and

$$\|v'\|_p^p \geq \lambda_1\|v\|_p^p \quad \forall v \in V \tag{1}$$

where $V \stackrel{df}{=} \{v \in W^{1,p}([0,b];\mathbb{R}^N) : \ \int_0^b v(t)\,dt = 0\}$.

Finally let us recall the Ekeland variational principle (compare De Figueiredo [4], Hu-Papageorgiou [10], p. 519 or Clarke [2], Chapter 7.5).

Theorem 2.3. If (Y,d) *is a complete metric space and* $\phi : Y \longmapsto \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ *is lower semicontinuous and bounded from below,*
then *for any* $\varepsilon > 0$ *there exists* $y_\varepsilon \in Y$ *such that*

$$\begin{cases} \phi(y_\varepsilon) & \leq & \inf\limits_{y\in Y}\phi(y) + \varepsilon, \\ \phi(y_\varepsilon) & < & \phi(y) + \varepsilon d(y,y_\varepsilon) \qquad \forall y \in Y,\ y \neq y_\varepsilon. \end{cases}$$

3 Auxiliary results

Let $2 \leq p < +\infty$. We consider the following problem

$$(HVI) \qquad \begin{cases} \left(|x'(t)|^{p-2}x'(t)\right)' \in \partial j(t,x(t)) \quad \text{almost everywhere on } [0,b] \\ x(0) = x(b) \quad x'(0) = x'(b). \end{cases}$$

Here $j : [0,b] \times \mathbb{R} \longmapsto \mathbb{R}$ is a potential function measurable in the first variable and locally Lipschitz in the second one. So $\partial j(t,\zeta)$ represents the Clarke subdifferential of $j(t,\cdot)$. Our precise hypotheses on j are the following.

<u>$H(j)$</u> $j : [0,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a functional, such that

- (i) for every $\zeta \in \mathbb{R}$, the functional $\mathbb{R} \ni t \longmapsto j(t,\zeta) \in \mathbb{R}$ is measurable and $j(\cdot,0) \in L^{p'}([0,b])$ (where $\frac{1}{p} + \frac{1}{p'} = 1$);
- (ii) for almost all $t \in [0,b]$, the functional $\mathbb{R} \ni \zeta \longmapsto j(t,\zeta) \in \mathbb{R}$ is locally Lipschitz with L^1-locally Lipschitz constant;
- (iii) for almost all $t \in [0,b]$, all $\zeta \in \mathbb{R}$ and all $u^* \in \partial j(t,\zeta)$, we have $|u^*| \leq a(t) + c_1|\zeta|^{r-1}$ with $a \in L^{p'}([0,b])$, $c_1 > 0$ and $1 \leq r < p$;
- (iv) there exist two functions $j_\pm \in L^1([0,b])$, such that $\lim\limits_{\zeta\to\pm\infty} j(t,\zeta) = j_\pm(t)$ uniformly for almost all $t \in [0,b]$;
- (v) for almost all $t \in [0,b]$ and all $\zeta \in \mathbb{R}$, we have $pj(t,\zeta) \geq -\lambda_1|\zeta|^p$;

(vi) there exist two constants $c_\pm > 0$, such that $\int_0^b j(t, c_\pm)\, dt < \int_0^b j_\pm(t)\, dt < 0$.

Let $W^{1,p}_{\mathrm{per}}([0,b]) \stackrel{df}{=} \{x \in W^{1,p}([0,b]) :\ x(0) = x(b)\}$. Since $W^{1,p}([0,b]) \subseteq C([0,b])$, we see that the pointwise evaluation at $t = 0$ and $t = b$ make sense. Let ϕ : $W^{1,p}_{\mathrm{per}}([0,b]) \longmapsto \mathbb{R}$ be defined by

$$\phi(x) \stackrel{df}{=} \frac{1}{p}\|x'\|_p^p + \int_0^b j(t, x(t))\, dt.$$

By virtue of Lemma III.6.24, p. 313 of Hu-Papageorgiou [10], ϕ is locally Lipschitz.

Proposition 3.1. If *hypotheses* $H(j)$ *hold,*
then ϕ *satisfies the nonsmooth* PS_c*-condition for any* $c \neq \int_0^b j_\pm(t)\, dt$.

Proof : Let $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}([0,b])$ be a sequence, such that $\phi(x_n) \longrightarrow c$ (with some $c \neq \int_0^b j_\pm(t)\ dt$) and $m(x_n) \longrightarrow 0$ as $n \to +\infty$. We will show that $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}([0,b])$ is bounded. Suppose that this is not the case. By passing to a subsequence if necessary, we may assume that $\|x_n\| \longrightarrow +\infty$ as $n \to +\infty$. Let us set $y_n \stackrel{df}{=} \frac{x_n}{\|x_n\|}$ for $n \geq 1$. Again, by passing to a further subsequence if necessary, we may assume that $y_n \longrightarrow y$ weakly in $W^{1,p}_{\mathrm{per}}([0,b])$ and $y_n \longrightarrow y$ in $C([0,b])$ as $n \to +\infty$ (recall that the embedding $W^{1,p}_{\mathrm{per}}([0,b]) \subseteq C([0,b])$ is compact). From the choice of the sequence $\{x_n\}_{n\geq 1}$, we know that there exists $M_1 > 0$, such that $|\phi(x_n)| \leq M_1$ for all $n \geq 1$ and so

$$\left|\frac{1}{p}\|x_n'\|_p^p + \int_0^b j(t, x_n(t))\, dt\right| \leq M_1.$$

We divide last inequality by $\|x_n\|^p$ and obtain

$$\left|\frac{1}{p}\|y_n'\|_p^p + \int_0^b \frac{j(t, x_n(t))}{\|x_n\|^p}\, dt\right| \leq \frac{M_1}{\|x_n\|^p}. \tag{2}$$

Invoking Lebourg mean value theorem (see Lebourg [13] or Clarke [2], p. 41) and using hypothesis $H(j)(iii)$, we see that for almost all $t \in [0,b]$ and all $\zeta \in \mathbb{R}$, we have

$$|j(t,\zeta)| \leq |j(t,0)| + a(t)\|\zeta\| + c_1\|\zeta\|^r.$$

So it follows that

$$\begin{aligned} &\left|\int_0^b \frac{j(t, x_n(t))}{\|x_n\|^p}\, dt\right| \\ \leq\ & \int_0^b \frac{|j(t,0)|}{\|x_n\|^p} + \int_0^b \frac{|a(t)|}{\|x_n\|^{p-1}}|y_n(t)|\, dt + \int_0^b \frac{c_1}{\|x_n\|^{p-r}}|y_n(t)|^r\, dt, \end{aligned}$$

and thus

$$\int_0^b \frac{j(t, x_n(t))}{\|x_n\|^p}\, dt \longrightarrow 0 \qquad \text{as } n \to +\infty.$$

Also from the weak lower semicontinuity of the norm functional in a Banach space, we have $\|y'\|_p^p \leq \liminf_{n\to+\infty} \|y_n'\|_p^p$. Thus, by passing to the limit in (2) as $n \to +\infty$, we obtain that $\|y'\|_p = 0$, hence $y(t) = \overline{c}$ for all $t \in \mathbb{R}$, with some $\overline{c} \in \mathbb{R}$ (i.e. y is constant). Recall that $W^{1,p}_{\mathrm{per}}([0,b]) = \mathbb{R} \oplus V$, where $V \stackrel{df}{=} \left\{v \in W^{1,p}_{\mathrm{per}}([0,b]) :\ \int_0^b v(t)\, dt = 0\right\}$

(see Hu-Papageorgiou [10], p. 502). Set $x_n = \overline{x}_n + \widehat{x}_n$ with $\overline{x}_n \in \mathbb{R}$ and $\widehat{x}_n \in V$ for $n \geq 1$. Then $\overline{y}_n = \frac{\overline{x}_n}{\|x_n\|}$ and $\widehat{y}_n = \frac{\widehat{x}_n}{\|x_n\|}$ for $n \geq 1$. Since $y_n \longrightarrow \overline{c}$ weakly in $W^{1,p}_{\mathrm{per}}([0,b])$, we must have $\overline{y}_n \longrightarrow \overline{c}$ in $\mathbb{R}$ and $\widehat{y}_n \longrightarrow 0$ weakly in $W^{1,p}_{\mathrm{per}}([0,b])$ as $n \to +\infty$.

We will show that $\overline{c} \neq 0$. Suppose that this is not true, i.e. $\overline{c} = 0$. We have

$$1 = \|y_n\| = \frac{\|\overline{x}_n + \widehat{x}_n\|}{\|x_n\|} \leq \frac{\|\overline{x}_n\|}{\|x_n\|} + \frac{\|\widehat{x}_n\|}{\|x_n\|}$$

so, we obtain

$$1 \leq \liminf_{n\to+\infty} \frac{\|\widehat{x}_n\|}{\|x_n\|} \leq \limsup_{n\to+\infty} \frac{\|x_n - \overline{x}_n\|}{\|x_n\|} \leq \limsup_{n\to+\infty} \left(1 + \frac{\|\overline{x}_n\|}{\|x_n\|}\right) = 1,$$

thus

$$\frac{\|\widehat{x}_n\|}{\|x_n\|} \longrightarrow 1 \quad \text{as } n \to +\infty.$$

Hence, we can find $n_0 \geq 1$, such that for all $n \geq n_0$, we have

$$\frac{1}{2}\|x_n\| \leq \|\widehat{x}_n\| \leq 2\|x_n\|.$$

Recall that for all $n \geq 1$, we have

$$\begin{aligned} \|x_n'\|_p^p &\leq pM_1 + \int_0^b p\,|j(t,x_n(t))|\ dt \\ &\leq pM_1 + \int_0^b \left(p|j(t,0)| + p|a(t)||x_n(t)| + pc_1|x_n(t)|^r\right)\ dt \\ &\leq \beta_1 + \beta_2\|\widehat{x}_n\|^r, \end{aligned}$$

with some $\beta_1, \beta_2 > 0$. Using the Poincaré-Wirtinger inequality (see Hu-Papageorgiou [10], p. 866), we obtain

$$\|\widehat{x}_n'\|_p^p \leq \beta_3\left(1 + \|\widehat{x}_n'\|_p^r\right),$$

for some $\beta_3 > 0$. So we see that the sequence $\{\widehat{x}_n'\}_{n\geq 1} \subseteq L^p([0,b])$ is bounded (recall that $r < p$). Using once more the Poincaré-Wirtinger inequality, we obtain that $\{\widehat{x}_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}([0,b])$ is bounded.

From this, it follows that $\frac{\widehat{x}_n}{\|x_n\|} \longrightarrow 0$ in $W^{1,p}_{\mathrm{per}}([0,b])$, hence $y_n = \frac{x_n}{\|x_n\|} \longrightarrow 0$ in $W^{1,p}_{\mathrm{per}}([0,b])$. So we obtain a contradiction, since $\|y_n\| = 1$ for all $n \geq 1$. So $\overline{c} \neq 0$. This means in particular that for all $t \in [0,b]$, we have $|x_n(t)| \longrightarrow +\infty$ as $n \to +\infty$. Assume that $x_n(t) \longrightarrow +\infty$ (the proof is similar when $x_n(t) \longrightarrow -\infty$).

Let $\mathrm{J} : L^p([0,b];\mathbb{R}) \longmapsto \mathbb{R}$ be defined by $\mathrm{J}(x) \overset{df}{=} \int_0^b j(t,x(t))\ dt$. From Proposition III.6.28, p. 315 of Hu-Papageorgiou [10] (see also Chang [1], Theorem 2.2), we have that $\partial\left(\mathrm{J}|_{W^{1,p}_{\mathrm{per}}([0,b])}\right)(x) \subseteq L^{p'}([0,b])$, while from Theorem 2.7.5, p. 83 of Clarke [2], we have that if $u \in \partial\left(\mathrm{J}|_{W^{1,p}_{\mathrm{per}}([0,b])}\right)(x)$, then $u(t) \in \partial j(t,x(t))$ for almost all $t \in [0,b]$. Let $x_n^* \in \partial\phi(x_n)$ be such that $\|x_n^*\|_* = m(x_n)$ for $n \geq 1$. Its existence follows from the weak compactness of $\partial\phi(x_n)$ and the weak lower semicontinuity of the norm functional. We have $x_n^* = \mathrm{A}\,x_n + u_n^*$ for $n \geq 1$, where $u_n^* \in \partial\left(\mathrm{J}|_{W^{1,p}_{\mathrm{per}}([0,b])}\right)(x_n)$, hence $u_n^*(t) \in \partial j(t,x_n(t))$ for almost all $t \in [0,b]$ and

$\mathrm{A}: W^{1,p}_{\mathrm{per}}([0,b]) \longmapsto \left(W^{1,p}_{\mathrm{per}}([0,b])\right)^*$ is the nonlinear operator defined by $\langle \mathrm{A}\, x, y\rangle \stackrel{df}{=} \int_0^b |x'(t)|^{p-2}x'(t)y'(t)\ dt$ for all $x, y \in W^{1,p}_{\mathrm{per}}([0,b])$ (by $\langle \cdot,\cdot\rangle$ we denote the duality brackets for the pair $\left(W^{1,p}_{\mathrm{per}}([0,b]), \left(W^{1,p}_{\mathrm{per}}([0,b])\right)^*\right)$). From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}([0,b])$, passing to a subsequence if necessary, we have

$$\langle \mathrm{A}\, x_n, \widehat{x}_n\rangle + (u_n^*, \widehat{x}_n)_{pp'} \leq \frac{1}{n}\|\widehat{x}_n\|,$$

(by $(\cdot,\cdot)_{pp'}$ we denote the duality brackets for the pair $\left(L^p([0,b]), L^{p'}([0,b])\right)$). Recall that $\langle\cdot,\cdot\rangle|_{W^{1,p}_{\mathrm{per}}([0,b])\times L^{p'}([0,b])} = (\cdot,\cdot)_{pp'}$. So, we obtain

$$\|\widehat{x}_n'\|_p^p + \int_0^b u_n^*(t)\widehat{x}_n(t)\ dt \leq \frac{1}{n}\|\widehat{x}_n\|. \tag{3}$$

Using also hypothesis $H(j)(iii)$ and Poincaré-Wirtinger inequality, we obtain

$$\|\widehat{x}_n'\|_p^p \leq \beta_4\left(1 + \|\widehat{x}_n'\|_p^r\right),$$

with some $\beta_4 > 0$ and so the sequence $\{\widehat{x}_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}([0,b])$ is bounded.

Since $x_n(t) \longrightarrow +\infty$, we must have $\overline{x}_n \longrightarrow +\infty$ as $n \to +\infty$. By definition, we have

$$u_n^*(t)\widehat{x}_n(t) \leq j^0(t, x_n(t); \widehat{x}_n(t)) = \limsup_{\substack{z_n \to x_n(t)\\ \varepsilon \searrow 0}} \frac{j(t, z_n + \varepsilon\widehat{x}_n(t)) - j(t, z_n)}{\varepsilon} \tag{4}$$

Because $x_n(t) \longrightarrow +\infty$ as $n \to +\infty$, we must have that $z_n \longrightarrow +\infty$ as $n \to +\infty$. Let $N_1 \subseteq [0,b]$ be a Lebesgue-nul set outside which hypothesis $H(j)(vi)$ holds. Now for all $t \in [0,b] \setminus N_1$ and for any $\varepsilon > 0$, we can find $n_0(\varepsilon) \geq 1$, such that for all $n \geq n_0$, we have

$$j_+(t) - \varepsilon^2 \leq j\left(t, z_n + \varepsilon\widehat{x}_n(t)\right) \leq j_+(t) + \varepsilon^2$$

and

$$j_+(t) - \varepsilon^2 \leq j\left(t, z_n\right) \leq j_+(t) + \varepsilon^2.$$

Using these estimates in (4), we see that for all $n \geq n_0$, we have

$$|u_n^*(t)\widehat{x}_n(t)| \leq \frac{2\varepsilon^2}{\varepsilon} = 2\varepsilon.$$

Thus, we have that $u_n^*(t)\widehat{x}_n(t) \longrightarrow 0$ uniformly for almost all $t \in [0,b]$ and also $\int_0^b u_n^*(t)\widehat{x}_n(t)\ dt \longrightarrow 0$ as $n \to +\infty$. Then, from (3), it follows that $\|\widehat{x}_n'\|_p \longrightarrow 0$ as $n \to +\infty$ and so invoking once more the Poincaré-Wirtinger inequality, we have that $\widehat{x}_n \longrightarrow 0$ in $W^{1,p}_{\mathrm{per}}([0,b])$ as $n \to +\infty$.

Let

$$\begin{aligned}\Gamma_n(t) \stackrel{df}{=} \{&(v,\lambda) \in \mathbb{R}\times(0,1) : \\ &v \in \partial j\left(t, \overline{x}_n + \lambda\widehat{x}_n(t)\right),\ j\left(t, \overline{x}_n + \widehat{x}_n(t)\right) - j\left(t, \overline{x}_n\right) = v\widehat{x}_n(t)\}.\end{aligned}$$

From Lebourg mean value theorem, we have that $\Gamma_n(t) \neq \emptyset$ almost everywhere on $[0,b]$. By redefining Γ_n on the exceptional Lebesgue null set, we may assume that $\Gamma_n(t) \neq \emptyset$ for all $t \in [0,b]$. We claim that for every direction $h \in \mathbb{R}$, the function $(t,\lambda) \longmapsto j^0\left(t, \overline{x}_n + \lambda\widehat{x}_n(t); h\right)$ is measurable. Indeed from the definition of the directional derivative, we have

$$\begin{aligned} & j^0\left(t, \overline{x}_n + \lambda\widehat{x}_n(t); h\right) \\ &= \inf_{m\geq 1} \sup_{r,s\in\mathbb{Q}\cap\left(-\frac{1}{m},\frac{1}{m}\right)} \frac{j\left(t, \overline{x}_n + \lambda\widehat{x}_n(t) + r + sh\right) - j\left(t, \overline{x}_n + \lambda\widehat{x}_n(t) + r\right)}{s}. \end{aligned}$$

Since j is jointly measurable (see Hu-Papageorgiou [10], p. 142), it follows that the function $(t,\lambda) \longmapsto j^0\left(t, \overline{x}_n + \lambda\widehat{x}_n(t); h\right)$ is measurable. Let $S_n(t,\lambda) \stackrel{df}{=} \partial j\left(t, \overline{x}_n + \lambda\widehat{x}_n(t)\right)$ and $\{h_m\}_{m\geq 1} \subseteq \mathbb{R}$ be a countable dense set. Because $j^0(t, \overline{x}_n + \lambda\widehat{x}_n(t); \cdot)$ is continuous, we have

$$\begin{aligned} \mathrm{Gr} S_n &= \{(t,\lambda,u) \in [0,b]\times(0,1)\times\mathbb{R} :\ u \in S_n(t,\lambda)\} \\ &= \bigcap_{m\geq 1} \Big\{(t,\lambda,u) \in [0,b]\times(0,1)\times\mathbb{R} : \\ &\qquad\qquad (u,h_m)_{\mathbb{R}^N} \leq j^0\left(t, \overline{x}_n + \lambda\widehat{x}_n(t); h_m\right)\Big\} \end{aligned}$$

and so $\mathrm{Gr} S_n \in \mathcal{L}([0,b]) \times \mathcal{B}(0,1) \times \mathcal{B}(\mathbb{R})$, with $\mathcal{L}([0,b])$ being the Lebesgue σ-field of $[0,b]$ and $\mathcal{B}(0,1)$, $\mathcal{B}(\mathbb{R})$ being the Borel σ-field of $(0,1)$ and $\mathbb{R}$ respectively. Hence $\mathrm{Gr}\Gamma_n = \{(t,v,\lambda) \in [0,b]\times\mathbb{R}\times(0,1) :\ (v,\lambda) \in \Gamma_n(t)\} \in \mathcal{L}([0,b]) \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(0,1)$. So we can apply the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [10], Theorem II.2.14, p. 158), to obtain measurable functions $v_n : [0,b] \longmapsto \mathbb{R}$ and $\lambda_n : [0,b] \longmapsto (0,1)$ such that $(v_n(t), \lambda_n(t)) \in \Gamma_n(t)$ for all $t \in [0,b]$. We have $j\left(t, \overline{x}_n + \widehat{x}_n(t)\right) - j\left(t, \overline{x}_n\right) = v_n(t)\widehat{x}_n(t)$ and $v_n(t) \in \partial j\left(t, \overline{x}_n + \lambda_n(t)\widehat{x}_n(t)\right)$ almost everywhere on $[0,b]$. Thus we can write that

$$\phi(x_n) = \frac{1}{p}\|x_n'\|_p^p + \int_0^b v_n(t)\widehat{x}_n(t)\ dt + \int_0^b j\left(t, \overline{x}_n\right)\ dt. \tag{5}$$

As before, we can check that $\int_0^b v_n(t)\widehat{x}_n(t)\ dt \longrightarrow 0$ as $n \to +\infty$. Also we know that $\|x_n'\| \longrightarrow 0$. So by passing to the limit in (5) as $n \to +\infty$, we obtain, that $c = \int_0^b j_+(t)\ dt$, what is a contradiction. This proves that the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}([0,b])$ is bounded and so we may assume that $x_n \longrightarrow x$ weakly in $W^{1,p}_{\mathrm{per}}([0,b])$ and $x_n \longrightarrow x$ in $C([0,b])$ as $n \to +\infty$. From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}([0,b])$, at least for a subsequence, we have

$$\langle Ax_n, x_n - x\rangle + (u_n^*, x_n - x)_{pp'} \leq \frac{1}{n}\|x_n - x\|$$

and thus

$$\limsup_{n\to+\infty} \langle Ax_n, x_n - x\rangle \leq 0.$$

But it is easy to see that A is demicontinuous, monotone, thus maximal monotone and so generalized monotone too (see Hu-Papageorgiou [10], p. 365). Hence $\langle Ax_n, x_n\rangle \longrightarrow \langle Ax, x\rangle$ and so $\|x_n'\|_p \longrightarrow \|x'\|_p$ as $n \to +\infty$. Since $x_n' \longrightarrow x'$ weakly in $L^p([0,b])$ as $n \to +\infty$ and the space $L^p([0,b])$, being uniformly convex, has the Kadec-Klee property (see Hu-Papageorgiou [10], p. 28), we infer that $x_n' \longrightarrow x'$ in $L^p([0,b])$ and so $x_n \longrightarrow x$ in $W^{1,p}_{\text{per}}([0,b])$ as $n \to +\infty$.

Q.E.D.

Proposition 3.2. If *hypotheses* $H(j)$ *hold,*
then ϕ *is bounded below and* $\phi|_V \geq 0$.

Proof : By virtue of hypothesis $H(j)(iv)$, we can find Lebesgue-null set $N_2 \subseteq [0,b]$ and $M_2 > 0$, such that for all $t \in [0,b] \setminus N_2$, we have

$$\begin{cases} |j(t,\zeta) - j_+(t)| \leq 1 & \text{for } \ \zeta \geq M_2 \\ |j(t,\zeta) - j_-(t)| \leq 1 & \text{for } \ \zeta \leq -M_2. \end{cases}$$

Also from hypothesis $H(j)(iii)$ and Lebourg mean value theorem, we see that for almost all $t \in [0,b]$ and all $|\zeta| < M_2$, we have that $|j(t,\zeta)| \leq k(t)$ with some $k \in L^{p'}([0,b])$. Then for all $x \in W^{1,p}_{\text{per}}([0,b])$, we have

$$\begin{aligned} \phi(x) &= \frac{1}{p}\|x'\|_p^p + \int_0^b j(t,x(x))\,dt \\ &= \int_{\{x \geq M_2\}} j(t,x(t))\,dt + \int_{\{x < -M_2\}} j(t,x(t))\,dt + \int_{\{|x| \leq M_2\}} j(t,x(t))\,dt \\ &\geq -\|j_+\|_1 - \|j_-\|_1 - 2 - \|k\|_1. \end{aligned}$$

Hence ϕ is bounded below.

Also using hypothesis $H(j)(v)$ and (1), we have for all $v \in V$, that

$$\phi(v) = \frac{1}{p}\|v'\|_p^p + \int_0^b j(t,v(t))\,dt \geq \frac{1}{p}\|v'\|_p^p - \frac{\lambda_1}{p}\|v\|_p^p \geq 0.$$

Q.E.D.

4 Multiplicity theorem

Using the auxiliary results of the previous section, we can prove the following multiplicity result for (HVI).

Theorem 4.1. <u>If</u> *hypotheses* $H(j)$ *hold,*
<u>then</u> *problem* (HVI) *has at least three distinct solutions.*

Proof : Consider the open sets $U^{\pm} \stackrel{df}{=} \{x = \pm\zeta + v : \ \zeta > 0, v \in V\}$. Let $m_{\pm} \stackrel{df}{=} \inf\{\phi(x) : \ x \in U^{\pm}\}$. From Proposition 3.2 we know that $m_{\pm} > -\infty$. We set

$$\overline{\phi}_+(x) \stackrel{df}{=} \begin{cases} \phi(x) & \text{if } x \in \overline{U^+} \\ +\infty & \text{otherwise.} \end{cases}$$

Evidently $\overline{\phi}_+$ is lower semicontinuous and bounded below (see Proposition 3.2). Thus we can apply the Ekeland variational principle (see Theorem 2.3) and produce a sequence $\{x_n\}_{n \geq 1} \subseteq U^+$, such that $\overline{\phi}_+(x_n) = \phi(x_n) \searrow m_+$ and

$$\overline{\phi}_+(x_n) \leq \overline{\phi}_+(y) + \varepsilon_n\|x_n - y\| \quad \text{for all } y \in W^{1,p}_{\text{per}}([0,b]), \text{ with } \varepsilon_n \searrow 0$$

and so also

$$\phi(x_n) \le \phi(y) + \varepsilon_n \|x_n - y\| \quad \text{for all } y \in \overline{U^+}.$$

So $x_n \in U^+$ minimizes the functional $y \longmapsto \phi(y) + \varepsilon_n\|x_n - y\|$ on $\overline{U^+}$. Because $x_n \in U^+$ and U^+ is an open set, we have that

$$0 \in \partial\left(\phi + \varepsilon_n\|\cdot\|\right)(x_n) \subseteq \partial\phi(x_n) + \varepsilon_n\overline{B_1^*},$$

where $\overline{B_1^*} \stackrel{df}{=} \left\{u^* \in \left(W^{1,p}_{\rm per}([0,b])\right)^* : \|u^*\|_* \le 1\right\}$ (see Clarke [2], p. 38). Thus we can find $x_n^* \in \partial\phi(x_n)$, such that $\|x_n^*\|_* \le \varepsilon_n \searrow 0$. Since $m(x_n) \le \|x_n^*\|_*$ for $n \ge 1$, we have that $m(x_n) \longrightarrow 0$ as $n \to +\infty$. Also from hypothesis $H(j)(vi)$, we see that $m_+ \le \phi(c_+) = \int_0^b j(t, c_+)\, dt < \int_0^b j_\pm(t)\, dt$ and so we can apply Proposition 3.1 and deduce that there exists a subsequence of $\{x_n\}_{n\ge1}$ (still denoted with the same index), such that $x_n \longrightarrow u_1$ in $W^{1,p}_{\rm per}([0,b])$ as $n \to +\infty$, with some $u_1 \in W^{1,p}_{\rm per}([0,b])$. Then $\phi(u_1) = m_+$. If $u_1 \in \partial U^+$, then $u_1 \in V$ and so by Proposition 3.2, we have that $\phi(u_1) \ge 0 > m_+$ (see hypothesis $H(j)(vi)$), what is a contradiction. Hence $u_1 \in U^+$ and so, it follows that $u_1 \ne 0$ and $0 \in \partial\phi(u_1)$.

In a similar fashion, working with the open set U^- and the corresponding function $\overline{\phi}_-$, we obtain $u_2 \in U^-$, such that $\phi(u_2) = m_-$ and $0 \in \partial\phi(u_2)$. Evidently $u_1 \ne u_2$.

From Proposition 3.2, we know that $\phi|_V \ge 0 > \phi(c_\pm)$, with $c_- < 0 < c_+$. Also, if c_0 is the minimax quantity of Theorem 2.1, then

$$c_0 \ge \inf_V \phi = 0 > \int_0^b j_\pm(t)\, dt.$$

So ϕ satisfies the nonsmooth PS_{c_0}-condition (see Proposition 3.1). Applying Theorem 2.1, we obtain $u_0 \in W^{1,p}_{\rm per}([0,b])$, such that $\phi(u_0) = c_0$ and $0 \in \partial\phi(u_0)$. Since $m_\pm < 0 \le c_0$, it follows that $u_0 \ne u_1$ and $u_0 \ne u_2$. Also, if $c_0 = 0$, then $u_0 \in V$.

Therefore, for every $i \in \{0,1,2\}$, we have that $0 \in \partial\phi(u_i)$. For every $\vartheta \in C_0^\infty(0,b)$, we have

$$\langle Au_i, \vartheta\rangle + \int_0^b v(t)\vartheta(t)\, dt = 0,$$

with $v \in L^{p'}([0,b])$ and $v(t) \in \partial j(t, u_i(t))$ almost everywhere on $[0,b]$. So, we also have

$$\int_0^b |u_i'(t)|^{p-2}u_i'(t)\vartheta'(t)\, dt + \int_0^b v(t)\vartheta(t)\, dt = 0.$$

Since $|u_i'|^{p-2}u_i' \in L^{p'}([0,b])$, we have that $(|u_i'|^{p-2}u_i')' \in W^{-1,p'}([0,b]) = \left(W_0^{1,p}([0,b])\right)^*$ (see e.g. Hu-Papageorgiou [10], p. 866). Using integration by parts, we obtain

$$\langle\left(|u_i'|^{p-2}u_i'\right)', \vartheta\rangle = \langle v, \vartheta\rangle.$$

Because $C_0^\infty([0,b])$ is dense in $W_0^{1,p}([0,b])$, so we obtain

$$\begin{cases} (|u_i'(t)|^{p-2}u_i'(t))' = v(t) \in \partial j(t, u_i(t)) \\ \qquad\qquad\qquad\qquad \text{almost everywhere on } [0,b] \\ u_i(0) = u_i(b). \end{cases}$$

On the other hand, if $\vartheta \in C^1_{\rm per}([0,b])$, by Green's identity, we have

$$|u_i'(b)|^{p-2}u_i'(b)\vartheta'(b) - |u_i'(0)|^{p-2}u_i'(0)\vartheta'(0) + \langle\left(|u_i'|^{p-2}u_i'\right)', \vartheta\rangle = \langle v, \vartheta\rangle$$

and so

$$|u_i'(b)|^{p-2}u_i'(b)\vartheta'(b) = |u_i'(0)|^{p-2}u_i'(0)\vartheta'(0) \quad \text{for all } \vartheta \in C^1_{\mathrm{per}}([0,b])$$

and

$$|u_i'(b)|^{p-2}u_i'(b) = |u_i'(0)|^{p-2}u_i'(0).$$

As the map $\xi \longmapsto |\xi|^{p-2}\xi$ is homeomorphism, so we have that $u_i'(0) = u_i'(b)$. Therefore, we conclude that u_0, u_1 and u_2 are three distinct solutions of (HVI).

Q.E.D.

Remark 4.2. *Note that u_1 and u_2 are also nontrivial. In general, we can not guarantee the nontriviality of u_0. However, if we know that for all $t \in T_0 \subseteq [0,b]$, with $|T_0| > 0$, we have $0 \notin \partial j(t,0)$, then we can conclude that u_0 is also nontrivial. It will be very interesting to extend Theorem 4.1 to systems. It seems that our approach encounters serious technical difficulties, when we try to extend it to vector problems, since we cannot say anymore that $\partial U^+ = V$. We do not know if there are reasonable hypotheses on j, which will allow us to overcome this difficulty.*

We will end this section with a simple example illustrating the applicability of our result. The example is in the spirit of those of Panagiotopoulos [16], analyzed there in the context of mechanical systems.

First, let us consider function $f : \mathbb{R} \longmapsto \mathbb{R}$, defined by

$$f(\zeta) \stackrel{df}{=} \begin{cases} -\frac{c}{2\zeta^2} & \text{if} \qquad \zeta < -c \\ \frac{2\overline{r}|\zeta|^{\overline{r}-1}}{c^{\overline{r}}-1} & \text{if} \quad -c \leq \zeta < -1 \\ -\widehat{r}|\zeta|^{\widehat{r}-1} & \text{if} \quad -1 \leq \zeta < 0 \\ \widehat{r}\zeta^{\widehat{r}-1} & \text{if} \quad 0 \leq \zeta < 1 \\ -\frac{2\overline{r}\zeta^{\overline{r}-1}}{c^{\overline{r}}-1} & \text{if} \quad 1 \leq \zeta < c \\ \frac{c}{2\zeta^2} & \text{if} \quad c \leq \zeta, \end{cases} \tag{6}$$

where $1 < \overline{r} < p$, $1 < \widehat{r} < p$ and $c > 1$ is sufficiently large (so as to guarantee that hypothesis $H(j)(v)$ holds, e.g. $c \geq \left(2\left(\frac{p}{\lambda_1}\right)^{\frac{1}{p}} - 1\right)^{\frac{1}{\overline{r}}}$ is a sufficient condition; see Fig. 1).

Let $j : \mathbb{R} \longmapsto \mathbb{R}$ be defined by $j(\zeta) \stackrel{df}{=} \int_0^\zeta f(\xi)\, d\xi$ (see Fig. 2). Then

$$j(\zeta) = \begin{cases} -\frac{1}{2}\left(1 + \frac{c}{|\zeta|}\right) & \text{if} \quad c < |\zeta| \\ \frac{-2|\zeta|^{\overline{r}} + c^{\overline{r}} + 1}{c^{\overline{r}}-1} & \text{if} \quad 1 < |\zeta| \leq c \\ |\zeta|^{\widehat{r}} & \text{if} \qquad |\zeta| \leq 1. \end{cases}$$

Note that j is not differentiable at $-c$, -1, 1 and c. At these points f exhibits jump discontinuities. Let us define multifunction $\widehat{f} : \mathbb{R} \longmapsto 2^{\mathbb{R}}$, by "filling in the gaps" at the discontinuity points of f (see Fig. 3). From Clarke [2] (p. 34), we know that j is locally Lipschitz and $\partial j(\zeta) = \widehat{f}(\zeta)$ for all $\zeta \in \mathbb{R}$.

Now we can consider the following problem

$$\begin{cases} \left(|x'(t)|^{p-2}x'(t)\right)' = f(x(t)) & \text{almost everywhere on } (0,b) \\ x(0) = x(b), \quad x'(0) = x'(b), \end{cases} \tag{7}$$

Fig. 1

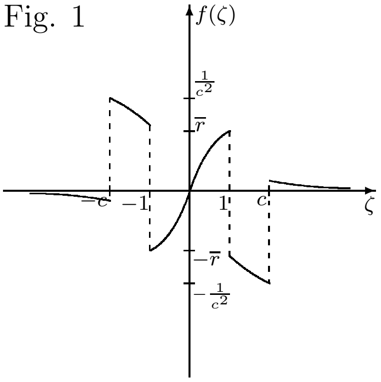


Fig. 2

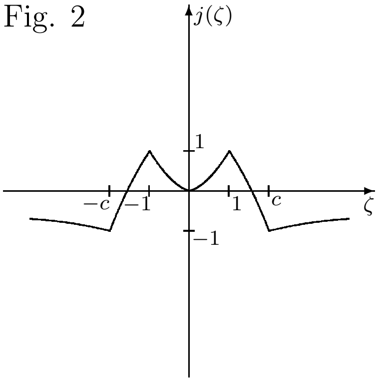
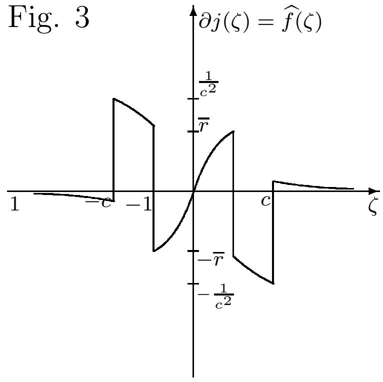


Fig. 3



which we can easily transform into a multivalued problem (elliptic inclusion), namely

$$\begin{cases} (|x'(t)|^{p-2}x'(t))' \in \partial j(x(t)) & \text{almost everywhere on } (0,b) \\ x(0) = x(b), \quad x'(0) = x'(b). \end{cases} \tag{8}$$

It is easy to verify that j satisfies hypotheses $H(j)$ and so by Theorem 4.1, problem (8) admits at least three distinct solutions.

We can also have another example, by replacing function f in the right hand side of (7), by function $g : [0,b] \times \mathbb{R} \longmapsto \mathbb{R}$, defined by $g(t,\zeta) \overset{df}{=} h(t)f(\zeta)$, where $h \in L^1([0,b])$, with $0 < h(t) \leq 1$ for almost all $t \in [0,b]$ and f is defined by (6). Then we can also produce function $j(t,\zeta)$ in an analogous way and check that hypotheses $H(j)$ are satisfied. Again from Theorem 4.1, we obtain the existence of three distinct solutions of (8).

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