On Generalized Quadrangles with Some Concurrent Axes of Symmetry

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Abstract

Let $\mathcal{S}$ be a finite Generalized Quadrangle (GQ) of order (s,t), $s \neq 1 \neq t$, and suppose L is a line of $\mathcal{S}$. A symmetry about L is an automorphism of $\mathcal{S}$ which fixes every line concurrent with L. A line L is an axis of symmetry if there is a full group of size s of symmetries about L. A point of a generalized quadrangle is a translation point if every line through it is an axis of symmetry. If there is a point p in a GQ $\mathcal{S} = (P, B, I)$ for which there is a group G of automorphisms of the GQ which fixes p linewise, and such that G acts regularly on the points of $P \setminus p^{\perp}$, then $\mathcal{S}$ is called an elation generalized quadrangle, and instead of $\mathcal{S}$, often the notations $(\mathcal{S}^{(p)}, G)$ or $\mathcal{S}^{(p)}$ are used. If G is abelian, then $(\mathcal{S}^{(p)}, G)$ is a translation generalized quadrangle (TGQ), and a GQ is a TGQ $\mathcal{S}^{(p)}$ if and only if p is a translation point, see [9].
We study the following two problems. (1) Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$. How many distinct axes of symmetry through the same point p are needed to conclude that every line through p is an axis of symmetry, and hence that $\mathcal{S}^{(p)}$ is a TGQ? (2) Given a TGQ $(\mathcal{S}^{(p)}, G)$, what is the minimum number of distinct lines through p such that G is generated by the symmetries about these lines?

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1 Introduction and Statement of the Main Results

In the monograph *Finite Generalized Quadrangles, S. E. Payne and J. A. Thas, Research Notes in Mathematics* **110**, *Pitman* it was proved that a GQ $\mathcal{S}$ is a TGQ $(\mathcal{S}^{(p)}, G)$ with translation point p if and only if every line through p is an axis of symmetry, that is, if p is a translation point, and then G is precisely the group generated by all symmetries about the lines incident with p. In [25] we noted that their proof was also valid for all lines through p minus one. This observation is one of the main motivations of the present paper: *what is — in general — the minimal number of distinct axes of symmetry through a point p of a GQ $\mathcal{S}$ forcing $\mathcal{S}^{(p)}$ to be a TGQ?*

In order to study the generalized quadrangles which have some distinct axes of symmetry through the point p, we will introduce Property (T) as follows. An ordered flag (L, p) satisfies Property (T) with respect to L_1, L_2, L_3, where L_1, L_2, L_3 are three distinct lines incident with p and distinct from L, if the following condition is satisfied: if (i, j, k) is a permutation of $(1, 2, 3)$, if $M \sim L$ and $M \not\mathrm{I} p$, and if $N \sim L_i$ and $N \not\mathrm{I} p$ with $M \not\sim N$, then the triads $\{M, N, L_j\}$ and $\{M, N, L_k\}$ are not both centric. TGQ's which satisfy Property (T) for some ordered flag always have order (s, s^2) for some s, and every TGQ of order (s, t) which has a subGQ of order s through the translation point satisfies Property (T) for some ordered flag(s) containing the translation point. Suppose that the GQ $\mathcal{S}$ satisfies Property (T) for the ordered flag (L, p) w.r.t. the distinct lines L_1, L_2, L_3, all incident with p. Moreover, suppose that L, L_1, L_2, L_3 are axes of symmetry. Then we will show that $\mathcal{S}^{(p)}$ is a TGQ and that the translation group G is generated by the symmetries about L, L_1, L_2, L_3.

We will also study the following related problem: *given a general TGQ $\mathcal{S}^{(p)}$, what is the minimal number of lines through p such that the translation group is generated by the symmetries about these lines?* We will show that there is a connection between the minimal number of lines through a translation point of a TGQ such that the translation group is generated by the symmetries about these lines, and the kernel of the TGQ; in particular, if $(\mathcal{S}^{(p)}, G)$ is a TGQ of order (s, t), $1 \neq s \neq t \neq 1$, with $(s, t) = (q^{na}, q^{n(a+1)})$ where a is odd and where $\mathbf{GF}(q)$ is the kernel of the TGQ, and if $k + 3$ is the minimum number of distinct lines through p such that G is generated by the symmetries about these lines, then we will show that $k \leq n$.

Another main result reads as follows: let $\mathcal{S}$ be a thick GQ of order (s, t), $t - s \geq 1$, and let p be a point of $\mathcal{S}$ incident with more than $t - s + 2$ axes of symmetry. Then $\mathcal{S}^{(p)}$ is a translation generalized quadrangle.

We will also introduce Property (T') as follows. An ordered flag (L, p) satisfies Property (T') with respect to L_1, L_2, L_3, where L_1, L_2, L_3 are distinct lines incident with p and distinct from L, if the following condition holds: if $M \sim L$ and $M \not\mathrm{I} p$, and if q and q' are distinct arbitrary points on M which are not incident with L, then there is a permutation (i, j, k) of $(1, 2, 3)$ such that there are lines M_i, M_j, M_k, with $M_r \sim L_r$ and $r \in \{i, j, k\}$, for which $M \in \{M_i, M_k, L\}^\perp$ and $M_j \in \{M_i, M_k, L_j\}^\perp$, and such that $q \mathrm{I} M_i$ and $q' \mathrm{I} M_k$.

It is a main goal of the present paper to state elementary combinatorial and group theoretical conditions for a GQ $\mathcal{S}$ such that $\mathcal{S}$ arises from a flock, see J. A. Thas [16], [18], [22], [20], [23] and also K. Thas [29]. We will show that a combination of Property (T') and Property (T) leads to Property (G) for TGQ's $\mathcal{S}$, and hence that

it is possible to prove that such a TGQ $\mathcal{S}$ is related to a flock by the main theorem of [22], see further. A classification result is eventually obtained.
Many other results will be proved, including a new divisibility condition for GQ's which have a point incident with at least three axes of symmetry, see Section 4.
Finally, in an appendix we will provide a new short proof of the following theorem of Payne and Thas [9]: 'If p is a point of the thick GQ $\mathcal{S}$ of order s which is incident with three distinct axes of symmetry L_1, L_2, L_3, then $\mathcal{S}^{(p)}$ is a translation generalized quadrangle for which the translation group is generated by all symmetries about L_1, L_2, L_3'.

2 Introducing Generalities and Notation

2.1 Some basics

A (finite) *generalized quadrangle (GQ) of order* (s,t) is an incidence structure $\mathcal{S} = (P, B, I)$ in which P and B are disjoint (nonempty) sets of objects called *points* and *lines* respectively, and for which I is a symmetric point-line incidence relation satisfying the following axioms.

(GQ1) Each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line.

(GQ2) Each line is incident with $s+1$ points ($s \geq 1$) and two distinct lines are incident with at most one point.

(GQ3) If p is a point and L is a line not incident with p, then there is a unique point-line pair (q, M) such that $pIMIqIL$.

Generalized quadrangles were introduced by J.Tits [30] in his celebrated work on triality, in order to have a better understanding of the Chevalley groups of rank 2. The main results, up to 1983, on finite generalized quadrangles are contained in the monograph *Finite Generalized Quadrangles* by S. E. Payne and J. A. Thas (FGQ, [9]). A survey of some 'new' developments on this subject in the period 1984-1992, can be found in the article *Recent developments in the theory of finite generalized quadrangles* [15]. We should also mention [21].
Let $\mathcal{S} = (P, B, I)$ be a (finite) generalized quadrangle of order (s,t), $s \neq 1 \neq t$. Then $|P| = (s+1)(st+1)$ and $|B| = (t+1)(st+1)$. Also, $s \leq t^2$ and, dually, $t \leq s^2$, and $s+t$ divides $st(s+1)(t+1)$.
There is a point-line duality for GQ's of order (s,t) for which in any definition or theorem the words "point" and "line" are interchanged and also the parameters. Normally, we assume without further notice that the dual of a given theorem or definition has also been given. Also, sometimes a line will be identified with the set of points incident with it without further notice.
A GQ is called *thick* if every point is incident with more than two lines and if every line is incident with more than two points. A *flag* of a GQ is an incident point-line pair. Let p and q be (not necessarily distinct) points of the GQ $\mathcal{S}$; we write $p \sim q$

and say that p and q are *collinear*, provided that there is some line L such that $pILIq$ (so $p \not\sim q$ means that p and q are not collinear). Dually, for $L, M \in B$, we write $L \sim M$ or $L \not\sim M$ according as L and M are *concurrent* or nonconcurrent. If $p \neq q \sim p$, the line incident with both is denoted by pq, and if $L \sim M \neq L$, the point which is incident with both is sometimes denoted by $L \cap M$.
For $p \in P$, put $p^\perp = \{q \in P \parallel q \sim p\}$ (note that $p \in p^\perp$). For a pair of distinct points $\{p,q\}$, the *trace* of $\{p,q\}$ is defined as $p^\perp \cap q^\perp$, and we denote this set by $\{p,q\}^\perp$. Then $|\{p,q\}^\perp| = s+1$ or $t+1$, according as $p \sim q$ or $p \not\sim q$. More generally, if $A \subset P$, $A^\perp$ is defined by $A^\perp = \bigcap\{p^\perp \parallel p \in A\}$. For $p \neq q$, the *span* of the pair $\{p,q\}$ is $sp(p,q) = \{p,q\}^{\perp\perp} = \{r \in P \parallel r \in s^\perp$ for all $s \in \{p,q\}^\perp\}$. When $p \not\sim q$, then $\{p,q\}^{\perp\perp}$ is also called *the hyperbolic line* defined by p and q, and $|\{p,q\}^{\perp\perp}| = s+1$ or $|\{p,q\}^{\perp\perp}| \leq t+1$ according as $p \sim q$ or $p \not\sim q$.
A *triad* of points (respectively lines) is a triple of pairwise noncollinear points (respectively pairwise disjoint lines). Given a triad T, a *center* of T is just an element of $T^\perp$. If $p \sim q$, $p \neq q$, or if $p \not\sim q$ and $|\{p,q\}^{\perp\perp}| = t+1$, we say that the pair $\{p,q\}$ is *regular*. The point p is *regular* provided $\{p,q\}$ is regular for every $q \in P \setminus \{p\}$. Regularity for lines is defined dually. One easily proves that either $s = 1$ or $t \leq s$ if $\mathcal{S}$ has a regular pair of noncollinear points.
If (p, L) is a nonincident point-line pair of a GQ, then by *[p,L]* we denote the unique line of the GQ which is incident with p and concurrent with L. Finally, if $\mathcal{S}$ is a GQ of order (s,t) with $s = t$, then $\mathcal{S}$ is also said to be *of order* s.

2.2 Elation generalized quadrangles and translation generalized quadrangles

A *whorl about the point* p of $\mathcal{S}$ is a collineation of $\mathcal{S}$ which fixes each line through p. An *elation about the point* p is a whorl about p that fixes no point of $P \setminus p^\perp$. By definition, the identical permutation is an elation (about every point). If p is a point of the GQ $\mathcal{S}$, for which there exists a group of elations G about p which acts regularly on the points of $P \setminus p^\perp$, then $\mathcal{S}$ is said to be an *elation generalized quadrangle (EGQ)* with *elation point* p and *elation group* (or *base group*) G, and we sometimes write $(\mathcal{S}^{(p)}, G)$ for $\mathcal{S}$. A *symmetry* about a point p of the GQ $\mathcal{S}$ (of order (s,t) with $s, t \neq 1$) is an elation about p which fixes any point of $p^\perp$ (in fact, a nontrivial whorl about p which fixes every element of $p^\perp$ automatically is a symmetry about p). A group of symmetries about a point can have at most t elements, and if this bound is reached, the point is called a *center of symmetry*. Dually, we define the notion *axis of symmetry*. Any center (respectively axis) of symmetry is regular; see FGQ [9]. If a GQ $(\mathcal{S}^{(p)}, G)$ is an EGQ with elation point p, and if any line incident with p is an axis of symmetry, then we say that $\mathcal{S}$ is a *translation generalized quadrangle (TGQ)* with *translation point* p and *translation group* (or *base group*) G. In such a case, G is uniquely defined; G is generated by all symmetries about every line incident with p, and G is the set of all elations about p, see FGQ [9].
TGQ's were introduced by J. A. Thas in [13] for the case $s = t$ and, in the general case, by S. E. Payne and J. A. Thas in FGQ.

Theorem 2.1 (FGQ, 8.1.3). *Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, and suppose that L and M are distinct concurrent lines. If α is a symmetry about L and if β is a symmetry about M, then $\alpha\beta = \beta\alpha$.*

Theorem 2.2 (FGQ, 8.3.1). *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s,t), $s,t \geq 2$. Suppose each line through some point p is an axis of symmetry, and let G be the group generated by the symmetries about the lines through p. Then G is elementary abelian and $(\mathcal{S}^{(p)}, G)$ is a TGQ.*

For the case $s = t$, we have the following result of [9], see also [25] and the appendix for several shorter proofs.

Theorem 2.3 (FGQ, 11.3.5). *Let $\mathcal{S} = (P, B, I)$ be a GQ of order s, with $s \neq 1$. Suppose that there are at least three axes of symmetry through a point p, and let G be the group generated by the symmetries about these lines. Then G is elementary abelian and $(\mathcal{S}^{(p)}, G)$ is a TGQ.*

Theorem 2.4 (FGQ, 8.2.3 and 8.5.2). *Suppose $(\mathcal{S}^{(x)}, G)$ is an EGQ of order (s,t), $s \neq 1 \neq t$. Then $(\mathcal{S}^{(x)}, G)$ is a TGQ if and only if G is an (elementary) abelian group. Also in such a case there is a prime p and there are natural numbers n and k, where k is odd, such that either $s = t = p^n$ or $s = p^{nk}$ and $t = p^{n(k+1)}$. It follows that G is a p-group.*

Remark 2.5. We have classified the GQ's with two distinct translation points in [29].

2.3 4-Gonal families and some useful theorems

Suppose $(\mathcal{S}^{(p)}, G)$ is an EGQ of order (s,t), $s,t \neq 1$, with elation point p and elation group G, and let q be a point of $P \setminus p^\perp$. Let $L_0, L_1, \ldots, L_t$ be the lines incident with p, and define r_i and M_i by $L_i I r_i I M_i I q$, $0 \leq i \leq t$. Put $A_i = \{\theta \in G \parallel M_i^\theta = M_i\}$, $A_i^* = \{\theta \in G \parallel r_i^\theta = r_i\}$, and $\mathcal{J} = \{A_i \parallel 0 \leq i \leq t\}$. Then $|G| = s^2t$ and $\mathcal{J}$ is a set of $t+1$ subgroups of G, each of order s. Also, for each i, A_i^* is a subgroup of G of order st containing A_i as a subgroup. Moreover, the following two conditions are satisfied:

- **(K1)** $A_iA_j \cap A_k = \mathbf{1}$ for distinct i, j and k;
- **(K2)** $A_i^* \cap A_j = \mathbf{1}$ for distinct i and j.

Conversely, if G is a group of order s^2t, where $s \neq 1 \neq t$, and $\mathcal{J}$ (respectively $\mathcal{J}^*$) is a set of $t+1$ (respectively $t+1$) subgroups A_i (respectively A_i^*) of G of order s (respectively of order st), and if the conditions (K1) and (K2) are satisfied, then the A_i^* are uniquely defined by the A_i, and $(\mathcal{J}, \mathcal{J}^*)$ is said to be a *4-gonal family of type (s,t) in G.*
Let $(\mathcal{J}, \mathcal{J}^*)$ be a 4-gonal family of type (s,t) in the group G of order s^2t. For any $h \in G$ let us define θ_h by

$$g^{\theta_h} = gh, \qquad (A_ig)^{\theta_h} = A_igh,$$

$$(A_i^* g)^{\theta_h} = A_i^* gh,\ [A_i]^{\theta_h} = [A_i],\ (\infty)^{\theta_h} = (\infty),$$

with $g \in G$, $A_i \in \mathcal{J}$, $A_i^* \in \mathcal{J}^*$. Then θ_h is an automorphism of $\mathcal{S}(G, \mathcal{J})$ which fixes the point (∞) and all lines of type (b). If $G' = \{\theta_h \parallel h \in G\}$, then clearly $G' \cong G$ and G' acts regularly on the points of type (1). Hence, a group of order $s^2 t$ admitting a 4-gonal family can be represented as a general elation group of a suitable elation generalized quadrangle. This was first noted by Kantor [7].

The following two results (and their generalizations, see Theorem 2.8 and Theorem 2.9) will appear to be very useful for the sequel.

Theorem 2.6 (X. Chen and D. Frohardt [3]). *Let G be a group of order $s^2 t$ admitting a 4-gonal family $(\mathcal{J}, \mathcal{J}^*)$ of type (s,t), $s \neq 1 \neq t$. If there exist two distinct members in $\mathcal{J}$ which are normal subgroups of G, then s and t are powers of the same prime number p and G is an elementary abelian p-group.*

Theorem 2.7 (D. Hachenberger [4]). *Let G be a group of order $s^2 t$ admitting a 4-gonal family $(\mathcal{J}, \mathcal{J}^*)$ of type (s,t), $s \neq 1 \neq t$. If G is a group of even order, and if there exists a member of $\mathcal{J}$ which is a normal subgroup of G, then s and t are powers of 2 and G is an elementary abelian 2-group.*

In geometrical terms, Theorem 2.6 reads as follows: *Let $(\mathcal{S}^{(x)}, G)$ be an EGQ of order (s,t), $s, t \neq 1$, and suppose that there are at least two axes of symmetry L and M through the elation point x, such that the full groups of symmetries about L and M are completely contained in G. Then s and t are powers of the same prime number p and G is an elementary abelian p-group.* Hence, by Theorem 2.4, $\mathcal{S}^{(x)}$ is a translation generalized quadrangle. In geometrical terms, we have the following for Theorem 2.7: *Let $(\mathcal{S}^{(x)}, G)$ be an EGQ of order (s,t), with $s, t \neq 1$ and s or t even, and suppose that there is at least one axis of symmetry L through the elation point x, such that the full group of symmetries about L is completely contained in G. Then s and t are powers of 2 and G is an elementary abelian 2-group.* Thus, $\mathcal{S}^{(x)}$ is a translation generalized quadrangle.

In Theorem 2.6 and Theorem 2.7, the group theoretical descriptions have one major restriction; they demand that the groups of symmetries about the lines must be **completely contained** in the elation group. In [26], we improved these theorems as follows.

Theorem 2.8 (K. Thas [26]). *Let $(\mathcal{S}^{(x)}, G)$ be an EGQ of order (s,t), $s, t \neq 1$. If there are two distinct regular lines through the point x, then s and t are powers of the same prime number p, G is an elementary abelian p-group and hence $\mathcal{S}^{(x)}$ is a TGQ with translation group G.*

Theorem 2.9 (K. Thas [26]). *Let $(\mathcal{S}^{(x)}, G)$ be an EGQ of order (s,t), $s, t \neq 1$. If there is a regular line through the point x, and G is a group of even order, then s and t are powers of 2, G is an elementary abelian 2-group and hence $\mathcal{S}^{(x)}$ is a TGQ with translation group G.*

3 Generalized Quadrangles with Concurrent Axes of Symmetry

In Payne and Thas [9] it was proved that a GQ $\mathcal{S}$ is a TGQ $(\mathcal{S}^{(p)}, G)$ with translation point p if and only if every line through p is an axis of symmetry, that is, if p is a translation point, and G is precisely the group generated by all symmetries about the lines incident with p. In [25] we noted that their proof was also valid for all lines through p minus one. Here, we will study the following natural problem: what is — in general — the minimal number of distinct axes of symmetry through a point p of a GQ $\mathcal{S}$ such that $\mathcal{S}^{(p)}$ is a TGQ? There are only such results known for (thick) GQ's of order s, and there, three axes of symmetries appears to be sufficient. In the appendix, we will give a short new geometrical proof of this theorem without using the coordinatization method for GQ's — the only known proof of this theorem is contained in Chapter 11 of [9], is rather long and complicated and uses this method. For thick GQ's of order (s,t) with $s \neq t$, the problem is a lot harder; we will show that $t-s+3$ axes of symmetry are sufficient in the general case, see Section 7. In the next paragraph we will introduce a purely combinatorial property, namely *Property (T)*, involving four concurrent axes of symmetry. Property (T) is closely related to *Property (G)*, see Section 6, and seems to be more general in the case of the translation generalized quadrangles. Moreover, every known translation generalized quadrangle of order (s,t), $1 \neq s \neq t \neq 1$, or its translation dual has Property (T). TGQ's which satisfy Property (T) for some ordered flag always have order (s, s^2) for some s, see MAIN THEOREM 1, and every TGQ of order (s,t) which has a subGQ of order s through the translation point satisfies Property (T) for some ordered flags containing the translation point, see Theorem 3.3. In § 3.2 we will also introduce *Property (T')*, which, in combination with Property (T), leads to Property (G) for TGQ's, and hence to *flock generalized quadrangles* in the odd case by [22, 7.3.4], see § 6.3. In the even case, a slightly different statement will be proved.
We start with defining Property (T).

3.1 Property (T)

Suppose $\mathcal{S}$ is a GQ of order (s,t), $s,t \neq 1$, and let p be a point of the GQ. We introduce Property (T) as follows.

Property (T). An ordered flag (L,p) satisfies *Property (T) with respect to* L_1, L_2, L_3, where L_1, L_2, L_3 are three distinct lines incident with p and distinct from L, if the following condition holds: if (i,j,k) is a permutation of $(1,2,3)$, if $M \sim L$ and $M \not\mathrm{I}\, p$, and if $N \sim L_i$ and $N \not\mathrm{I}\, p$ with $M \not\sim N$, then the triads $\{M, N, L_j\}$ and $\{M, N, L_k\}$ are not both centric.
If the ordered flag (L,p) satisfies Property (T) with respect to L_1, L_2, L_3, then we also say that $\mathcal{S}$ *satisfies Property (T) for the ordered flag* (L,p) *w.r.t.* L_1, L_2, L_3.

The following theorem will appear to be very useful for the sequel.

Theorem 3.1. *Suppose $\mathcal{S}$ is a thick GQ of order (s,t), and let p be a point of $\mathcal{S}$ incident with three distinct axes of symmetry. If G is the group generated by the* *symmetries about these axes, then G is a group of elations with center p of order s^3.*

Proof. Suppose that L_1, L_2, L_3 are three distinct axes of symmetry incident with p, and let G_i be the full group of symmetries about L_i. With α_1, α_2 and β contained in respectively G_1, G_2 and G_3 (and none of these collineations trivial), suppose that the following holds:

$$\alpha_1\alpha_2 = \beta.$$

If q is not collinear with p, then $(q, q^\beta, q^{\alpha_1})$ are the points of a triangle, contradiction. This observation shows us that $|G| = s^3$, and that each element of G is an elation (because no two elements of G have the same action on a point of $P \setminus p^\perp$), and so G is a group of elations about p. ∎

MAIN THEOREM 1. *Suppose $\mathcal{S} = (P, B, I)$ is a thick GQ of order (s, t), and let $p \in P$ be a point incident with four distinct axes of symmetry L_1, L_2, L_3 and L_4. Moreover, suppose that Property (T) holds for the ordered flag (L_4, p) w.r.t. the lines L_1, L_2, L_3. Then $t = s^2$, every line through p is an axis of symmetry, and if G is the group generated by all symmetries about L_1, L_2, L_3, L_4, then $(\mathcal{S}^{(p)}, G)$ is a TGQ.*

Proof. Suppose that G_i is the full group of symmetries about the line L_i, $i \in \{1, 2, 3, 4\}$, and consider the group G generated by all symmetries about the lines L_1, L_2, L_3 and L_4. We define G' as the group generated by the symmetries about the axes L_j with $j = 1, 2, 3$. A general element of G' always can be written in the form $\phi_k\phi_j\phi_i$, with $(i, j, k) = (1, 2, 3)$ and ϕ_i a symmetry about the line L_i, see Theorem 2.1. Also, because of Theorem 3.1 G' is an elation group with center p and of size s^3.

Suppose, for a non-trivial symmetry $g \in G_4$ and an elation $\alpha = \alpha_1\alpha_2\alpha_3$ of G', with $\alpha_i \in G_i$, that g and α have the same action on a point q of $P \setminus p^\perp$. Then $q^g = q^\alpha =: q'$, and thus we have that $q' \in [q, L_4]$[1]. It is clear that none of the symmetries α_i are identical ($i = 1, 2, 3$), otherwise we would have a composition of at most three symmetries about distinct axes of symmetry with a common intersection point which acts trivially on a point of $P \setminus p^\perp$, a contradiction. If we now consider the triads of lines $\{[q, L_4], [q^{\alpha_1}, L_2], L_3\}$ and $\{[q, L_4], [q^{\alpha_1}, L_2], L_1\}$, then the assumption we just made implies that they are both centric, in contradiction with Property (T). Thus we have proved that an element of G' and an element of G_4 can never have the same action on a point of $P \setminus p^\perp$. Therefore, we have that $|G| = s^4$ and that every element of G is an elation with center p. Since $|P \setminus p^\perp| = s^2t$ and since $t \leq s^2$ by Theorem 1.2.3 of [9], there follows that $t = s^2$. Also, there follows that G acts regularly on the points of $P \setminus p^\perp$, and hence that $(\mathcal{S}^{(p)}, G)$ is an EGQ with elation point p. Since the lines L_i are regular, the proof is complete by Theorem 2.8. ∎

The following property is well-known.

Observation 3.2. *Let $\mathcal{S}$ be a thick GQ of order (s, s^2), and suppose $\mathcal{S}'$ is a subquadrangle of $\mathcal{S}$ of order s. Then every line of $\mathcal{S}$ is either contained in $\mathcal{S}'$, or intersects $\mathcal{S}'$ in exactly one point.*

Proof. Easy counting. ∎

[1]Remark that $q' \neq q$.

Theorem 3.3. *1. Suppose $\mathcal{S}$ is a thick GQ of order (s,t), and let p be a point incident with four distinct axes of symmetry, three of which are contained in a proper subquadrangle $\mathcal{S}'$ of $\mathcal{S}$ of order (s,t'), $t' \neq 1$, but not the fourth. Suppose G is the group generated by all symmetries about these four axes. Then $t = s^2$ and $(\mathcal{S}^{(p)}, G)$ is a TGQ. Also, $\mathcal{S}'$ is a TGQ.*

2. Suppose $\mathcal{S}$ is a thick GQ of order (s,t), and let p be a point incident with four distinct axes of symmetry L_1, L_2, L_3, L three of which are contained in a proper subquadrangle $\mathcal{S}'$ of $\mathcal{S}$ of order (s,t'), $t' \neq 1$, but not the fourth, say L. Then (L,p) satisfies Property (T) w.r.t. L_1, L_2, L_3.

Proof. It is clear that an axis of symmetry of a GQ of order (s,t) also is an axis of symmetry of a proper subGQ of order (s,t') which contains this line. An axis of symmetry of a GQ is a regular line, and since $t' \neq 1$, there follows by [9, 2.2.2] that $s \leq t'$ and by the same theorem that $t' = s$. Hence $t = s^2$. By Theorem 2.3 there holds that $\mathcal{S}'$ is a TGQ.
Let L_1, L_2, L_3 be three distinct axes of symmetry through the point p which are contained in $\mathcal{S}'$, and let L_4 be an axis of symmetry through p which is not contained in $\mathcal{S}'$. Suppose $L \sim L_4 \neq L$ is arbitrary, $L \not\mathrm{I}\, p$, and let L' be an arbitrary line of $L_i^{\perp} \setminus \{L_i\}$ not through p for a fixed i in $\{1,2,3\}$. Suppose that the triads of lines $\{L, L', L_j\}$ and $\{L, L', L_k\}$ are both centric, with $\{j,k\} = \{1,2,3\} \setminus \{i\}$. First suppose that $L' \in \mathcal{S}'$. If M is a center of $\{L, L', L_j\}$, then $|M \cap \mathcal{S}'| \geq 2$, and hence M is a line of $\mathcal{S}'$. There follows that $|L \cap \mathcal{S}'| \geq 2$, and hence L is also a line of $\mathcal{S}'$. This immediately leads to the fact that L_4 is a line of $\mathcal{S}'$, contradiction. The same holds for $\{L, L', L_k\}$.
Next, suppose that $L' \notin \mathcal{S}'$, that $M \in \{L, L', L_j\}^{\perp}$ and $N \in \{L, L', L_k\}^{\perp}$. The line L intersects $\mathcal{S}'$ in one point q. Consider a symmetry θ about L_4 which maps the point $q' = L \cap M$ onto q. Then the line M^{θ} is contained in $\mathcal{S}'$, and hence also L'^{θ}. By the first part of this proof, we get a contradiction. Hence, Property (T) is satisfied for the ordered flag (L_4, p) w.r.t. the lines L_1, L_2, L_3, and the theorem follows from MAIN THEOREM 1. ■

3.2 Property (T')

Suppose $\mathcal{S}$ is a GQ of order (s,t), $s,t \neq 1$, and let p be a point of the GQ.

Property (T'). An ordered flag (L,p) satisfies *Property (T') with respect to the lines* L_1, L_2, L_3, where L_1, L_2, L_3 are different lines incident with p and distinct from L, with the following condition: if $M \sim L$ and $M \not\mathrm{I}\, p$, and if q and q' are distinct arbitrary points on M which are not incident with L, then there is a permutation (i,j,k) of $(1,2,3)$, such that there are lines M_i, M_j, M_k, with $M_r \sim L_r$ and $r \in \{i,j,k\}$, for which $M \in \{M_i, M_k, L\}^{\perp}$, $M_j \in \{M_i, M_k, L_j\}^{\perp}$ and such that $q \mathrm{I} M_i$ and $q' \mathrm{I} M_k$.
If the ordered flag (L,p) satisfies Property (T') w.r.t. the lines L_1, L_2, L_3, then we also say that $\mathcal{S}$ *satisfies Property (T') for the ordered flag* (L,p) *w.r.t.* L_1, L_2, L_3.

Theorem 3.4. *Suppose that $\mathcal{S} = (P, B, I)$ is a thick GQ of order (s, t), and let p be a point of P which is incident with four distinct axes of symmetry L, L_1, L_2, L_3 such that Property (T') is satisfied w.r.t. the lines L_1, L_2, L_3. If G_L, G_1, G_2, G_3 are the full groups of symmetries about these lines, then $G_L \subseteq \langle G_1, G_2, G_3 \rangle$.*

Proof. Put $G = \langle G_1, G_2, G_3 \rangle$. Suppose $M \sim L$ is arbitrary with $M \not{I} p$, and suppose q and q' are distinct arbitrary points on M which are not incident with L. Since Property (T') is satisfied for the ordered flag (L, p) w.r.t. the lines L_1, L_2, L_3, there is a permutation (i, j, k) of $(1, 2, 3)$, such that there are lines M_i, M_j, M_k, with qIM_i and $q'IM_k$, for which $M \in \{M_i, M_k, L\}^\perp$ and $M_j \in \{M_i, M_k, L_j\}^\perp$, and such that $M_r \sim L_r$ with $r \in \{i, j, k\}$. For convenience, put $(i, j, k) = (1, 2, 3)$.
By the considerations above, the following collineations exist:

1. θ_1 is the symmetry about L_1 which sends $q = M \cap M_1$ to $M_1 \cap M_2$;
2. θ_2 is the symmetry about L_2 which sends $M_1 \cap M_2$ to $M_2 \cap M_3$;
3. θ_3 is the symmetry about L_3 which maps $M_2 \cap M_3$ to $q' = M \cap M_3$.

Define the following collineation of $\mathcal{S}$: $\theta := \theta_1\theta_2\theta_3$. Then θ is an automorphism of $\mathcal{S}$ which is contained in G, and hence θ is an elation about p by Theorem 3.1. Also, θ fixes M and maps q onto q'. Now by K. Thas [26], there follows that θ is a symmetry about L. It follows now easily that $G_L \subseteq G$ since q and q' were arbitrary. ■

Remark 3.5. By K. Thas [26] it is sufficient to ask that L is a regular line to conclude it is an axis of symmetry.

Note. We emphasize the fact that Property (T) and Property (T') are purely combinatorial properties which are defined without the use of collineations.

4 A Divisibility Condition for GQ's with Three Distinct Concurrent Axes of Symmetry

Theorem 4.1 (FGQ, 8.1.2). *If a thick GQ $\mathcal{S}$ has a nonidentity symmetry θ about some line, then $st(1 + s) \equiv 0 \mod s + t$.*

Theorem 4.2. *Suppose $\mathcal{S}$ is a GQ of order (s, t), $s \neq 1 \neq t$, and let L, M and N be three different axes of symmetry incident with the same point p. Then $s|t$ and $\frac{t}{s} + 1|(s + 1)t$.*

Proof. Define G' as the group generated by the symmetries about the lines L, M and N. By Theorem 3.1 we have that G' is a group of elations with center p, and the size of G' is s^3. If we consider the permutation group $(P \setminus p^\perp, G')$, then there follows that $|G'|$ divides $|P \setminus p^\perp|$, or that $s^3|s^2t$. So t is divisible by s. By Theorem 4.1, the theorem now follows. ■

5 Property (SUB) and Property (T)

Property (SUB). Suppose $\mathcal{S}$ is a thick GQ of order (s,t), and suppose M_1, M_2, M_3, M_4 are four distinct lines of $\mathcal{S}$, incident with the same point. Then these four lines satisfy *Property (SUB)* if $\mathcal{S}$ does not have a subGQ of order s containing these four lines.

Theorem 5.1. *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, and let M_1, M_2, M_3 and M_4 be four distinct axes of symmetry through a point p such that there is a line $M \in \{M_1, \ldots, M_4\} = \mathcal{M}$ for which Property (T) is satisfied for the ordered flag (M,p) w.r.t. the lines of $\mathcal{M} \setminus M$. Then these four lines also satisfy Property (SUB).*

Proof. Suppose that the lines M_1, M_2, M_3 and M_4 are contained in a proper subGQ $\mathcal{S}'$ of order s and suppose that $M = M_4$ is such that (M,p) satisfies Property (T) w.r.t. the lines M_1, M_2, M_3. Consider lines L with $L \mathrm{I} M_4$ and L' with $L' \mathrm{I} M_1$, both contained in $\mathcal{S}'$. The lines M_i are axes of symmetry in the quadrangle $\mathcal{S}'$, so they are regular. There follows now that each triad of lines in $\mathcal{S}'$ which contains one of those axes, is centric. This leads to a contradiction. ∎

Theorem 5.2. *Suppose that $(\mathcal{S}^{(p)}, G)$ is a TGQ of order (s, s^2), $s > 1$, and suppose that $\mathcal{S}$ has a subquadrangle $\mathcal{S}'$ of order s which contains the point p. Then there exist four lines incident with p, such that G is the group generated by the symmetries about these lines.*

Proof. The quadrangle $\mathcal{S}'$ is a TGQ of order s with translation point p, and so there exist three lines in $\mathcal{S}'$ incident with p, such that the translation group G' of $\mathcal{S}'$ is generated by all symmetries about these three lines. Consider an axis of symmetry L of $\mathcal{S}$ incident with p and not contained in $\mathcal{S}'$. This line intersects $\mathcal{S}'$ only in p. Take an arbitrary line M of $\mathcal{S}$ intersecting L and not incident with p. This line is not contained in $\mathcal{S}'$, and thus it intersects $\mathcal{S}'$ in just one point. Since L is an axis of symmetry, the group G_L of symmetries about L acts transitively on the points of $M \setminus \{L \cap M\}$; it follows now that every G'-orbit intersects M in exactly one point. Also, there follows that $G'' = \langle G', G_L \rangle$ has size s^4. Since G'' is a group of elations with center p, the theorem is proved. ∎

Remark 5.3. It is clear that four is the minimum number of lines such that the translation group of a TGQ of order (s,t), $1 < s \neq t$, is generated by the symmetries about these lines, see e.g. the proof of MAIN THEOREM 1.

Suppose that $\mathcal{S}$ is a thick TGQ of order (s,t) which satisfies property (T) for some ordered flag (L,p) w.r.t. the lines L_1, L_2, L_3. Then by MAIN THEOREM 1 we have that $t = s^2$. Now suppose that C is the class of all thick TGQ's $\mathcal{S}^{(p)}$, with p fixed, for which the following condition holds:

(C) If L_1, L_2, L_3, L_4 are different lines through p for which Property (SUB) is satisfied, then there is a line $L \in \{L_1, L_2, L_3, L_4\} = \mathcal{L}$ such that $\mathcal{S}^{(p)}$ satisfies property (T) for the ordered flag (L,p) w.r.t. the lines of $\mathcal{L} \setminus \{L\}$.

Note that by Theorem 5.1, C is exactly the class of TGQ's for which Property (T) and Property (SUB) are 'equivalent' (with respect to certain lines incident with p). If $\mathcal{S}^{(p)} \in \mathsf{C}$ is a TGQ of order (s,t) **not** satisfying Property (T) for every ordered flag (L,p) (w.r.t. every three distinct lines through p and different from L), then Property (SUB) does not hold, and so there is a subGQ of order s. It follows that $t = s$ or $t = s^2$ by [9, 2.2.2]. Hence we have the following.

Observation 5.4. *Every element of* C *has classical order, that is, is either of order* s *or of order* (s,s^2).

Let $\mathcal{S}^{(p)} \in \mathsf{C}$. Suppose that there are four distinct axes of symmetry incident with the translation point p such that for one of these lines M Property (T) is satisfied for the ordered flag (M,p) w.r.t. the other three lines. Then the translation group is generated by all symmetries about these four lines by MAIN THEOREM 1. If no ordered flag satisfies Property (T) w.r.t. every three distinct lines through p, then neither Property (SUB) will be satisfied for every four lines, hence there is a subquadrangle $\mathcal{S}'$ of $\mathcal{S}$ of order s. If $\mathcal{S}$ is of order s, then every three axes through the translation point will determine the translation group. If $t > s$, then $t = s^2$, and by Theorem 5.2, there exist four axes such that the translation group is generated by the symmetries about the four axes. We have the following observation.

Observation 5.5. *If* $\mathcal{S} \in \mathsf{C}$*, then there always exist four axes which 'determine' the translation group, that is, the translation group is generated by the symmetries about the four axes.*

6 Property (T), Property (T') and Property (G)

6.1 3-Regularity

Suppose $\mathcal{S}$ is a GQ of order (s,s^2), $s \neq 1$. Then for any triad of points $\{p,q,r\}$, $|\{p,q,r\}^{\perp}| = s+1$, see 1.2.4 of [9]. Evidently $|\{p,q,r\}^{\perp\perp}| \leq s+1$. We say that $\{p,q,r\}$ is 3-*regular* provided that $|\{p,q,r\}^{\perp\perp}| = s+1$. A point p is 3-*regular* if each triad of points containing p is 3-regular. Let $\mathcal{S}$ be a generalized quadrangle of order (s,s^2), $s \neq 1$. Let x_1, y_1 be distinct collinear points. We say that the pair $\{x_1,y_1\}$ has *Property (G)*, or that $\mathcal{S}$ has *Property (G) at* $\{x_1,y_1\}$, if every triad $\{x_1,x_2,x_3\}$ of points for which $y_1 \in \{x_1,x_2,x_3\}^{\perp}$ is 3-regular.

6.2 T(n,m,q)'s

Suppose $H = \mathbf{PG}(2n+m-1,q)$ is the finite projective $(2n+m-1)$-space over $\mathbf{GF}(q)$, and let H be embedded in a $\mathbf{PG}(2n+m,q)$, say H'. Now define a set $\mathcal{O} = \mathcal{O}(n,m,q)$ of subspaces as follows: $\mathcal{O}$ is a set of q^m+1 $(n-1)$-dimensional subspaces of H, say $\mathbf{PG}(n-1,q)^{(i)}$, every three of which generate a $\mathbf{PG}(3n-1,q)$, and such that the following condition is satisfied: for every i there is a subspace $\mathbf{PG}(n+m-1,q)^{(i)}$ of H of dimension $n+m-1$, which contains $\mathbf{PG}(n-1,q)^{(i)}$ and which is disjoint from any other $\mathbf{PG}(n-1,q)^{(j)}$ if $j \neq i$. If $\mathcal{O}$ satisfies all these conditions, then it is called a *generalized ovoid*, or an *egg*. The spaces $\mathbf{PG}(n+m-1,q)^{(i)}$

are the *tangent spaces* of the egg, or just the *tangents*.
Generalized ovoids were introduced by J. A. Thas in [12].
Now let $\mathcal{O}(n,m,q)$ be an egg of $H = \mathbf{PG}(2n+m-1,q)$, and define a point-line incidence structure $T(n,m,q)$ as follows.

- The **POINTS** are of three types.
  1. A symbol (∞).
  2. The subspaces $\mathbf{PG}(n+m,q)$ of H' which intersect H in a $\mathbf{PG}(n+m-1,q)^{(i)}$.
  3. The points of $H' \setminus H$.
- The **LINES** are of two types.
  1. The elements of the egg $\mathcal{O}(n,m,q)$.
  2. The subspaces $\mathbf{PG}(n,q)$ of $\mathbf{PG}(2n+m,q)$ which intersect H in an element of the egg.
- **INCIDENCE** is defined as follows: the point (∞) is incident with all the lines of type (1) and with no other lines; the points of type (2) are incident with the unique line of type (1) contained in it and with all the lines of type (2) which they contain (as subspaces), and finally, a point of type (3) is incident with the lines of type (2) that contain it.

Then S. E. Payne and J. A. Thas, and J. A. Thas, prove in [13, 9] that $T(n,m,q)$ is a TGQ of order (q^n, q^m), and that, conversely, any TGQ can be seen in this way. Hence, *the study of translation generalized quadrangles is equivalent to the study of the generalized ovoids.*

If $n \neq m$, then by 8.7.2 of [9] the q^m+1 tangent spaces of $\mathcal{O}(n,m,q)$ form an $\mathcal{O}^*(n,m,q)$ in the dual space of $\mathbf{PG}(2n+m-1,q)$. So in addition to $T(n,m,q)$ there arises a TGQ $T(\mathcal{O}^*)$, also denoted $T^*(n,m,q)$, or $T^*(\mathcal{O})$. The TGQ $T^*(\mathcal{O})$ is called the *translation dual* of the TGQ $T(\mathcal{O})$.
Each TGQ $\mathcal{S}$ of order $(s, s^{\frac{a+1}{a}})$, with translation point (∞), where a is odd and $s \neq 1$, has a *kernel* $\mathbb{K}$, which is a field with a multiplicative group isomorphic to the group of all collineations of $\mathcal{S}$ fixing the point (∞), and any given point not collinear with (∞), linewise. We have $|\mathbb{K}| \leq s$, see [9]. The field $\mathbf{GF}(q)$ is a subfield of $\mathbb{K}$ if and only if $\mathcal{S}$ is of type $T(n,m,q)$, see [9]. The TGQ $\mathcal{S}$ is isomorphic to a $T_3(\mathcal{O})$ of Tits with $\mathcal{O}$ an ovoid of $\mathbf{PG}(3,s)$ if and only if $|\mathbb{K}| = s$. It is well-known that the TGQ $T(\mathcal{O})$ and its translation dual $T(\mathcal{O}^*)$ have isomorphic kernels. An egg $\mathcal{O} = \mathcal{O}(n,m,q), n \neq m$, is called *good* at an element $\pi \in \mathcal{O}$ if for every two distinct elements π^* and π'' of $\mathcal{O} \setminus \{\pi\}$ the $(3n-1)$-space $\pi\pi^*\pi''$ contains exactly q^n+1 elements of $\mathcal{O}$ (and is disjoint from the other elements). If the egg $\mathcal{O}$ contains a good element, then the egg is subconsequently called *good*, and for a good egg $\mathcal{O}(n,m,q)$ there holds that $m = 2n$. For convenience, we will sometimes say that

the TGQ $T(\mathcal{O})$ is *good at its element* π if $\mathcal{O}$ is good at its element π. If a TGQ $\mathcal{S}^{(\infty)}$ contains a good element π, then its translation dual satisfies Property (G) for the corresponding flag $((\infty)', \pi')$, with π' the tangent space of $\mathcal{O}$ at π.

Theorem 6.1 (J. A. Thas [16]). *If the TGQ $\mathcal{S}^{(\infty)}$ contains a good element π, then its translation dual satisfies Property (G) for the corresponding flag $((\infty)', \pi')$.*

Theorem 6.2. *Let $\mathcal{S} = T(n, m, q)$ be a TGQ for which there exist four distinct elements π_i, $i = 1, 2, 3, 4$, of $\mathcal{O} = \mathcal{O}(n, m, q)$ which generate a $(4n - 1)$-space. Then $\mathcal{S}$ is a GQ of order (q^n, q^{2n}) and $\mathcal{S}$ satisfies Property (T) for every ordered flag $(\pi_r, (\infty))$, $r \in \{1, 2, 3, 4\}$, w.r.t. π_i, π_j, π_k, where $\{i, j, k\} = \{1, 2, 3, 4\} \setminus \{r\}$.*

Proof. Since $4n - 1 \leq 2n + m - 1$, there follows immediately that $\mathcal{S}$ is a GQ of order (q^n, q^{2n}). Now, let $L_k \sim \pi_k$ be lines of $\mathcal{S}$ such that $\mathsf{V}_3 = \{L_1, L_2, \pi_3\}$ and $\mathsf{V}_4 = \{L_1, L_2, \pi_4\}$ are centric triads with $L_3 \in \mathsf{V}_3^{\perp}$ and $L_4 \in \mathsf{V}_4^{\perp}$. Note that we assume that $L_1 \not\sim L_2$. It follows that the n-spaces L_3 and L_4 intersect the $(2n+1)$-dimensional space generated by the n-spaces L_1 and L_2 in a space of minimum dimension 1, thus, if π is the space generated by the n-spaces L_k, $1 \leq k \leq 4$, then π has dimension at most $4n - 1$. If we intersect π with $\mathbf{PG}(4n - 1, q)$, then we obtain a space with maximal dimension $4n - 2$ which contains the π_k's, a contradiction. Hence, for every ordered flag $(\pi_r, (\infty))$, Property (T) is satisfied w.r.t. π_i, π_j, π_k, where $\{i, j, k\} = \{1, 2, 3, 4\} \setminus \{r\}$. ■

The following theorem is a converse of Theorem 6.2.

Theorem 6.3. *Suppose $\mathcal{S}^{(p)}$ is the thick TGQ of order (q^n, q^{2n}) which corresponds to the generalized ovoid $\mathcal{O}$ of $\mathbf{PG}(4n-1, q)$. If $L\mathrm{I}p$ is a line such that Property (T) is satisfied for the ordered flag (L, p) with respect to the lines L_1, L_2, L_3 through p, and if π, respectively π_i, is the element of $\mathcal{O}$ which corresponds to L, respectively L_i, $i = 1, 2, 3$, then $\langle \pi, \pi_1, \pi_2, \pi_3 \rangle = \mathbf{PG}(4n - 1, q)$.*

Proof. Immediate from the proof of MAIN THEOREM 1 and the interpretation in the projective model $T(n, 2n, q)$. ■

Theorem 6.4. *A TGQ $\mathcal{S}^{(\infty)} = T(\mathcal{O})$ which is good at an element $\pi \in \mathcal{O}$ satisfies Property (T) for the ordered flag $(\pi, (\infty))$.*

Proof. If $\mathcal{O}$ is good at some element π, then there are always four elements — including π — of the corresponding egg, which generate the projective $(4n-1)$-space $\mathbf{PG}(4n - 1, q)$. ■

Thus, if $\mathcal{S} = T(\mathcal{O})$ is a TGQ for which $\mathcal{O}$ has a good element, then there are always four lines such that the translation group is generated by the symmetries about these lines.

Corollary 6.5. *The $T_3(\mathcal{O})$ of Tits (see Chapter 3 of FGQ) satisfies Property (T).*

Proof. $\mathcal{O}$ is good at every point by [9, 8.7.4]. ■

Lemma 6.6. *Suppose $(\mathcal{S}^{(p)}, G)$ is a thick TGQ of order (s,t), $t \geq 3$. Then $\mathcal{S}$ is of order s if and only if there are lines $L_1 Ip, L_2 Ip, L_3 Ip$ such that for every other line LIp, with G_i the full group of symmetries about L_i and G_L the full group of symmetries about L, the group $\langle G_1, G_2, G_3, G_L \rangle$ has size s^3.*

Proof. It is clear that a TGQ $(\mathcal{S}^{(p)}, G)$ of order s, $s > 1$, has the desired property, since $|G| = s^3$ and since by Theorem 2.3 G is generated by the symmetries about three arbitrary distinct lines through p.
Let $(\mathcal{S}^{(p)}, G)$ be a TGQ, and suppose that the required conditions are satisfied. Suppose that G_i is the full group of symmetries about the line L_i, with $i \in \{1, 2, \ldots, t+1\}$ and $L_i Ip$. If $t = 3$, then for any thick TGQ of order (s,t) we have $s = 3$ and so $|G| = s^3$. Hence, let $t \geq 4$. Define the group H_j as $H_j = \langle G_1, G_2, G_3, \ldots, G_j \rangle$ with $j \in \{4, 5, \ldots, t+1\}$. Then $|H_4| = s^3$. Considering that a group generated by the symmetries about three concurrent axes of symmetry is a group of elations of order s^3 about their intersection point by Theorem 3.1, we have that $H_4 = \langle G_1, G_2, G_3 \rangle$. But $H_5 = \langle H_4, G_5 \rangle$, so $H_5 = H_4$, and thus also $H_4 = H_5 = \ldots = H_{t+1}$. Since $H_{t+1} = G$ there follows that $|G| = s^3$, and hence $\mathcal{S}$ is of order s. ∎

The following theorem implies that Property (T') is a characteristic property for TGQ's of order s.

Theorem 6.7. *Suppose $(\mathcal{S}^{(p)}, G)$ is a thick TGQ of order (s,t), $t \geq 3$. Then $\mathcal{S}$ is of order s if and only if there is a line LIp such that $\mathcal{S}$ satisfies Property (T') for the ordered flag (L,p) w.r.t. every three distinct lines L_1, L_2, L_3 through p which are different from L.*

Proof. Immediately by Lemma 6.6 and Theorem 3.4. ∎

Theorem 6.8. *Suppose that $\mathcal{S}^{(p)} = T(\mathcal{O})$ is a thick TGQ of order (s,t) with $s \neq 1 \neq t$, such that there is a line LIp so that for every three distinct lines L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T') is satisfied for the ordered flag (L,p) w.r.t. L_1, L_2, L_3, and suppose that there is at least one 3-tuple (M,N,U) such that (L,p) satisfies Property (T) w.r.t. (M,N,U).*
Then $T(\mathcal{O})$ is good at its element L.

Proof. Since there is at least one 3-tuple (M,N,U) such that (L,p) satisfies Property (T) w.r.t. (M,N,U), there follows by MAIN THEOREM 1 that $t = s^2$. By Theorem 6.3, the definitions of Property (T) and Property (T'), Lemma 6.6 and Theorem 6.7, there follows that the generalized ovoid $\mathcal{O}$ satisfies the condition that for every three distinct lines L_1, L_2, L_3 through p and different from L, the projective space generated by the four corresponding elements of $\mathcal{O}$ is either a $\mathbf{PG}(4n-1,q)$ or a $\mathbf{PG}(3n-1,q)$. Thus every $(3n-1)$-space which is generated by the element π of $\mathcal{O}$ which corresponds to L and two other elements of $\mathcal{O}$, has the property that it either is disjoint with any other element of $\mathcal{O}$ or completely contains it. Suppose that we denote the elements of $\mathcal{O}$ by $\pi, \pi^1, \ldots, \pi^{q^{2n}}$, where π corresponds to L, and fix for instance π^i, i arbitrary. Then all the $(3n-1)$-spaces of $\mathbf{PG}(4n-1,q)$ which contain π, π^i and an element of $\mathcal{O} \setminus \{\pi, \pi^i\}$ intersect two by two in $\pi\pi^i$ and cover $\mathbf{PG}(4n-1,q)$. By the preceding remarks, every element of $\mathcal{O} \setminus \{\pi, \pi^i\}$ is completely contained in one of these $(3n-1)$-spaces, and is disjoint with any other of the

$(3n-1)$-spaces. Since the number of these spaces is q^n+1, there follows that every of these $(3n-1)$-spaces contains exactly q^n+1 elements of $\mathcal{O}$. The theorem now follows since i was arbitrary. ∎

Note. It is also possible to easily prove Theorem 6.8 with the use of 8.7.2 of FGQ [9].

Theorem 6.9. *Suppose that $\mathcal{S}^{(p)} = T(\mathcal{O})$ is a thick TGQ of order (s,t) with $s \neq t$, such that there is a line LIp so that for every three distinct lines L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T') is satisfied for the ordered flag (L,p) w.r.t. L_1, L_2, L_3. Then $t = s^2$ and the translation dual $\mathcal{S}^{*(p')} = T(\mathcal{O}^*)$ satisfies Property (G) for the flag (p', L'), where (p', L') corresponds to (p, L).*

Proof. Immediate by Theorem 6.8 and Theorem 6.1. ∎

6.3 Flocks, Property (T) and Property (T')

Let $\mathcal{F}$ be a *flock* of a quadratic cone $\mathcal{K}$ with vertex v of $\mathbf{PG}(3,q)$, that is, a partition of $\mathcal{K} \setminus \{v\}$ into q disjoint irreducible conics. Then we have the following theorem.

Theorem 6.10 (J. A. Thas [14]). *To any flock of the quadratic cone of $\mathbf{PG}(3,q)$ corresponds a GQ of order (q^2, q).*

The following important theorem on flock GQ's is due to Payne (for notations not explicitly given here, see e.g. [22]).

Theorem 6.11 (S. E. Payne [10]). *Any flock GQ satisfies Property (G) at its point (∞).*

Now we come to the main theorem of the masterful sequence of papers [16], [18], [22], [20]; it is a converse of the previous theorem and the solution of a longstanding conjecture.

Theorem 6.12 (J. A. Thas [22]). *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (q^2, q), $q > 1$, and assume that $\mathcal{S}$ satisfies Property (G) at the flag (x, L). If q is odd then $\mathcal{S}$ is the dual of a flock GQ. If q is even and all ovoids $\mathcal{O}_z$ (see Section 5 of [22]) are elliptic quadrics, then we have the same conclusion.*

Finally, we recall the following.

Theorem 6.13 (J. A. Thas [16]). *Suppose $\mathcal{S}^{(p)} = T(\mathcal{O})$ is a TGQ of order (s, s^2), $s > 1$, for which $\mathcal{O}$ is good at its element π. Then $\mathcal{S}$ contains at least $s^3 + s^2$ subGQ's of order s (all containing the point p). For s odd these subGQ's are isomorphic to the classical GQ $\mathcal{Q}(4,s)$ (which arises from a nonsingular quadric in $\mathbf{PG}(4,s)$).*

Theorem 6.14 (J. A. Thas [24]). *Let $\mathcal{S}^{(p)} = T(\mathcal{O})$ be a TGQ of order (s, s^2), s even, such that $\mathcal{O}$ is good at its element πIp. If $\mathcal{S}$ contains at least one subGQ of order s which is isomorphic to the GQ $\mathcal{Q}(4,s)$, then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$ (which arises from a nonsingular elliptic quadric in $\mathbf{PG}(5,s)$).*

There is a very nice corollary of Theorem 6.12 and Theorem 6.8.

Corollary 6.15. *Suppose that $\mathcal{S}^{(p)} = T(\mathcal{O})$ is a thick TGQ of order (s,t) with $s \neq t$ and s odd, such that there is a line LIp so that for every three distinct lines L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T') for the ordered flag (L,p) is satisfied w.r.t. L_1, L_2, L_3. Then $t = s^2$ and the translation dual $\mathcal{S}^{*(p')} = T(\mathcal{O}^*)$ of $\mathcal{S}$ is the point-line dual of a flock GQ.*

Proof. By Theorem 6.9 we have that $t = s^2$, and that the translation dual $\mathcal{S}^{*(p')}$ satisfies Property (G) at its flag (p', L') which corresponds with (p, L). By Theorem 6.12, the proof is complete. ∎

If the flock GQ $\mathcal{S}(\mathcal{F})$ is the classical GQ $H(3,q^2)$, then it is a TGQ with base line L, L any line of $\mathcal{S}(\mathcal{F})$. The dual $\mathcal{S}(\mathcal{F})^D$ of $\mathcal{S}(\mathcal{F})$ is isomorphic to $T_3(\mathcal{O})$, $\mathcal{O}$ an elliptic quadric of $\mathbf{PG}(3,q)$. Hence the kernel $\mathbb{K}$ is the field $\mathbf{GF}(q)$. Also, $\mathcal{S}(\mathcal{F})^D$ is isomorphic to its translation dual $(\mathcal{S}(\mathcal{F})^D)^*$. Let $\mathcal{K}$ be the quadratic cone with equation $X_0X_1 = X_2^2$ of $\mathbf{PG}(3,q)$, q odd. Then the q planes π_t with equation $tX_0 - mt^\sigma X_1 + X_3 = 0$, $t \in \mathbf{GF}(q)$, m a given non-square in $\mathbf{GF}(q)$ and σ a given automorphism of $\mathbf{GF}(q)$, define a flock $\mathcal{F}$ of $\mathcal{K}$; see [14]. All the planes π_t contain the exterior point $(0,0,1,0)$ of $\mathcal{K}$. This flock is *linear*, that is, all the planes π_t contain a common line, if and only if $\sigma = \mathbf{1}$. Conversely, every nonlinear flock $\mathcal{F}$ of $\mathcal{K}$ for which the planes of the q conics share a common point, is of the type just described, see [14].
The corresponding GQ $\mathcal{S}(\mathcal{F})$ was first discovered by W. M. Kantor, and is therefore called the *Kantor (flock) generalized quadrangle*. This quadrangle is a TGQ for some baseline, and the following was shown by Payne in [10][2].

Theorem 6.16 (S. E. Payne [10]). *Suppose a TGQ $\mathcal{S} = T(\mathcal{O})$ is the point-line dual of a GQ $\mathcal{S}(\mathcal{F})$ which arises from a Kantor flock $\mathcal{F}$. Then $T(\mathcal{O})$ is isomorphic to its translation dual $T^*(\mathcal{O})$.*

The kernel $\mathbb{K}$ is the fixed field of σ, see [11].
The following recent and very interesting theorem classifies TGQ's arising from flocks in the odd case.

Theorem 6.17 (Blokhuis, Lavrauw and Ball [2]). *Let $T(\mathcal{O})$ be a TGQ of order (q^n, q^{2n}), where $\mathbf{GF}(q)$ is the kernel, and suppose $T(\mathcal{O})$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$, with the additional condition that $q \geq 4n^2 - 8n + 2$ and q is odd. Then $T(\mathcal{O})$ is isomorphic to the point-line dual of a Kantor flock GQ.*

We are now able to classify the thick TGQ's for which there is a line LIp so that for every three distinct lines L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T') is satisfied for the ordered flag (L,p) w.r.t. L_1, L_2, L_3.

[2]Recently, Bader, Lunardon and Pinneri [1] proved that a TGQ which arises from a flock is isomorphic to its translation dual if and only if it is the point-line dual of a Kantor flock GQ. It should be noted however that their proof relies heavily on results of J. A. Thas and H. Van Maldeghem [19] and J. A. Thas [18].

Corollary 6.18. *Suppose that $\mathcal{S}^{(p)}$ is a thick TGQ of order (s,t), $s \neq 1 \neq t$, such that there is a line $L\mathrm{I}p$ so that for every three distinct lines L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T') is satisfied for the ordered flag (L,p) w.r.t. L_1, L_2, L_3. Then we have the following classification.*

1. *$s = t$ and $\mathcal{S}$ is a TGQ with no further restrictions.*
2. *$t = s^2$, s is an even prime power, and $\mathcal{O}$ is good at its element π which corresponds to L, where $\mathcal{S} = T(\mathcal{O})$. Also, $\mathcal{S}$ has precisely $s^3 + s^2$ subGQ's of order s which contain the line L, and if one of these subquadrangles is classical, i.e. isomorphic to the GQ $\mathcal{Q}(4,s)$, then $\mathcal{S}$ is classical, that is, isomorphic to the GQ $\mathcal{Q}(5,s)$.*
3. *$t = s^2$ and $s = q^n$, q odd, where $\mathbf{GF}(q)$ is the kernel of the TGQ $\mathcal{S}^{(p)}$, with $q \geq 4n^2 - 8n + 2$, and $\mathcal{S}$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ where $\mathcal{F}$ is a Kantor flock.*
4. *$t = s^2$ and $s = q^n$, q odd, where $\mathbf{GF}(q)$ is the kernel of the TGQ $\mathcal{S}^{(p)}$, with $q < 4n^2 - 8n + 2$, and $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ for some flock $\mathcal{F}$.*

Proof. If for every three distinct lines L_1, L_2, L_3 through p and different from L, Property (T') is always satisfied for the ordered flag (L,p) w.r.t. L_1, L_2, L_3, then by Lemma 6.7 there follows that $\mathcal{S}$ is a TGQ of order s with no further restrictions. Suppose this is not the case. Then there is a 3-tuple (M, N, U) of distinct lines through p such that Property (T) is satisfied for the ordered flag (L,p) w.r.t. M, N, U. Hence by MAIN THEOREM 1 there follows that $t = s^2$, and if $\mathcal{S} = T(\mathcal{O})$, then $\mathcal{O}$ is good at its element π which corresponds to L by Theorem 6.8. Suppose that s is even. Then (2) follows from Theorem 6.13 and Theorem 6.14. Next suppose that s is odd. By Theorem 6.12 $T^*(\mathcal{O})$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$. Then (3) and (4) follow from Theorem 6.17. ∎

7 The General Problem

We start this section with the following result.

Theorem 7.1 (FGQ, 9.4.2). *Suppose that $L_0, L_1, \ldots, L_r$, $r \geq 1$, are $r+1$ lines incident with a certain point p in the GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$. Suppose that $\mathcal{O}$ is the set of points different from p, which are on the lines of $L_0, L_1, \ldots, L_r$, and denote $P \setminus p^\perp$ by Ω. Suppose that G is a group of elations with center p, and suppose G has the property that, if M is an arbitrary line which intersects $\mathcal{O}$ in one point m, then G acts transitively on the points of Ω lying on M. If $r > t/s$, then G acts transitively — and so also regularly — on the points of Ω.*

■

Corollary 7.2. *Suppose that $(\mathcal{S}^{(p)}, G)$ is a TGQ of order (s,t), $s, t \neq 2$, and suppose $k > t/s$, $k \in \mathbb{N}$. Then the translation group G is generated by the symmetries about $k+1$ arbitrary lines through the translation point p.*

■

Corollary 7.3. *Let $\mathcal{S}$ be a thick GQ of order (s,t) with the property that there is a point p incident with at least $s+2$ distinct axes of symmetry, and suppose that the group G generated by all symmetries about some $s+2$ axes of symmetry through p is a group of elations. Then $(\mathcal{S}^{(p)}, G)$ is a TGQ.*

Proof. From the inequality of Higman [9] there follows that $t/s \leq s$, hence $s+1 > t/s$. So, the conditions of Theorem 7.1 for the group G and the $s+2$ lines $L_i I p$ are satisfied. Hence, the group G acts regularly on the points of $P \setminus p^\perp$, and $(\mathcal{S}^{(p)}, G)$ is an EGQ. There are at least two regular lines through the elation point p, and by Theorem 2.8 the proof is complete. ■

Theorem 7.4 (FGQ, 8.2.4). *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s,t) with $s \leq t$ and $s > 1$, and let p be a point for which $\{p, x\}^{\perp\perp} = \{p, x\}$ for all $x \in P \setminus p^\perp$. Let G be a group of whorls about p.*

1. *If $y \sim p$, $y \neq p$, and if θ is a nonidentity whorl about p and y, then all points fixed by θ lie on py and all lines fixed by θ meet py.*
2. *If θ is a nonidentity whorl about p, then θ fixes at most one point of $P \setminus p^\perp$.*
3. *If G is generated by elations about p, then G is a group of elations, i.e. the set of elations about p is a group.*
4. *If G acts transitively on $P \setminus p^\perp$ and $|G| > s^2 t$, then G is a Frobenius group on $P \setminus p^\perp$, so that the set of all elations about p is a normal subgroup of G of order $s^2 t$ acting regularly on $P \setminus p^\perp$, i.e. $\mathcal{S}^{(p)}$ is an EGQ with some normal subgroup of G as elation group.*
5. *If G is transitive on $P \setminus p^\perp$ and G is generated by elations about p, then $(\mathcal{S}^{(p)}, G)$ is an EGQ.*

Theorem 7.5. *Suppose $\mathcal{S}$ is a GQ of order (s,t), $t > s^2/2$ and $s \neq 1$, with x a point incident with $r+1$ axes of symmetry, $r \geq s+1$. If G is the group generated by all symmetries about these $r+1$ lines, then $(\mathcal{S}^{(x)}, G)$ is a TGQ.*

Proof. By [9, 1.4.2] and Theorem 7.4 the conditions of Theorem 7.1 are satisfied, which implies that $(\mathcal{S}^{(x)}, G)$ is an EGQ. Theorem 2.8 finishes the proof. ∎

Theorem 7.6. *Suppose $\mathcal{S} = (\mathcal{S}^{(p)}, G)$ is a thick TGQ of order (s,t), and let $\mathbf{GF}(q)$ be the kernel of the TGQ. Next, suppose that $L_1, L_2, \ldots, L_{t+1}$ are the lines incident with p, and let G_i be the group of all symmetries about the line L_i, $i \in \{1, 2, \ldots, t+1\}$. Define k as the minimum number such that $G = \langle G_{i_1}, G_{i_2}, \ldots, G_{i_{(3+k)}} \rangle$, with $\{i_1, i_2, \ldots, i_{(3+k)}\} \subseteq \{1, 2, \ldots, t+1\}$. Then we have the following inequality:*

$$k \leq \log_q \frac{t}{s}.$$

Proof. Denote the groups $\langle G_{i_1}, G_{i_2}, G_{i_3} \rangle$ and $\langle G_{i_1}, G_{i_2}, G_{i_3}, \ldots, G_{i_{(j+3)}} \rangle$ respectively by G'_0 and G'_j, with $j \in \{1, 2, \ldots, k\}$. By Theorem 3.1 we have that $|G'_0| = s^3$. Since k is defined as a minimum, we have the following strict chain of groups:

$$G'_0 < G'_1 < \ldots < G'_k = G.$$

Now fix a point $y \in P \setminus p^\perp$, where $\mathcal{S} = (P, B, I)$. The groups G'_i are all groups of elations about p, and hence for the G'_i-orbits G'^*_i which contain y, there holds that $|G'^*_i| = |G'_i|$, $i = 0, 1, \ldots, k$, and that

$$G'^*_0 \subset G'^*_1 \subset \ldots \subset G'^*_k = P \setminus p^\perp.$$

The TGQ $\mathcal{S}$ is a $T(\mathcal{O})$ for some egg $\mathcal{O}$ in $\mathbf{PG}(2n+m-1, q) \subset \mathbf{PG}(2n+m, q)$, where $\mathbf{GF}(q)$ is the kernel of the TGQ. If we interpret the strict chain of orbits in $\mathbf{PG}(2n+m-1, q)$, then we obtain a strict chain of affine spaces over $\mathbf{GF}(q)$:

$$\mathbf{AG}'_0 \subset \mathbf{AG}'_1 \subset \ldots \subset \mathbf{AG}'_k = \mathbf{AG}(2n+m, q),$$

and we have that $|\mathbf{AG}'_j| \geq q|\mathbf{AG}'_{j-1}|$ for every $j \in \{1, 2, \ldots, k\}$, which implies that $|G| \geq q^k s^3$. Since $|G| = s^2 t$, the theorem follows. ∎

This leads to one of our main theorems, which is a considerable improvement of the best known (general) result.

MAIN THEOREM 2. *Let $(\mathcal{S}^{(p)}, G)$ be a TGQ of order (s,t), $1 \neq s \neq t \neq 1$, with $(s,t) = (q^{na}, q^{n(a+1)})$, where $\mathbf{GF}(q)$ is the kernel of the TGQ and where a is odd. If $k+3$ is the minimum number of distinct lines through p such that G is generated by the symmetries about these lines, then*

$$k \leq n.$$

∎

Remark 7.7 (An alternative approach for $T_3(\mathcal{O})$). Suppose $(\mathcal{S}^{(p)}, G)$ is a TGQ of order (s,t), $s \neq 1 \neq t$. Then the kernel $\mathbb{K}$ of the TGQ is isomorphic to $\mathbf{GF}(s)$ if and only if $\mathcal{S}$ is a $T_3(\mathcal{O})$ with $\mathcal{O}$ some ovoid of $\mathbf{PG}(3,s)$, and $t = s^2$, see [9, 8.7.4]. There immediately follows from MAIN THEOREM 2 that there are four distinct lines incident with p such that G is generated by all symmetries about these four lines. Hence, the knowledge of the size of the kernel is already sufficient to completely solve problem (2) for the $T_3(\mathcal{O})$ of Tits.

The following is an analogue of MAIN THEOREM 2 in a more general context.

Theorem 7.8. *Suppose $\mathcal{S}$ is a thick GQ of order (s,t), and let x be a point of $\mathcal{S}$ incident with $r+1$ axes of symmetry $L_0, \ldots, L_r$. Suppose G is the group generated by all symmetries about the lines L_i. We denote the full group of symmetries about L_i by G_i, $0 \leq i \leq r$. Define k as the smallest natural number such that $|G| \leq s^3 k$, and suppose $r \geq 2$. Then there are at least $m = r - 2 - \log_p k$ groups of $G_0, G_1, \ldots, G_r$ which are abelian, and G is generated by the symmetries about at most $3 + \log_p k$ elements of $\{L_0, L_1, \ldots, L_r\}$. Here p is the smallest prime number dividing s.*

Proof. By Theorem 3.1 we have that s^3 divides $|G|$, and hence $|G| = s^3 k$. Suppose $r + 1 - m$ is the minimal number of groups of $\{G_0, G_1, \ldots, G_r\}$ which generate G. Then we have by the proof of Theorem 4.2 that

$$s^3 p^{r-2-m} \leq |G| = s^3 k,$$

and hence

$$p^{r-2-m} \leq k \implies m \geq r - 2 - \log_p k.$$

Next, suppose that $L_0, L_1, \ldots, L_r$ are axes of symmetry through p, indexed in such a way that $G = \langle G_j \parallel m \leq j \leq r \rangle$ (recall that G is generated by all symmetries about $r + 1 - m$ axes of symmetry incident with x), and define H_i as $H_i = \langle G_j \parallel j \neq i,\ 0 \leq j \leq r \rangle$, $i \in \{0, 1, \ldots, r\}$. Then we have that $G = H_i G_i = G_i H_i$ for every i, $i \in \{0, 1, \ldots, r\}$, since symmetries about different concurrent axes commute, and if $j \in \{0, 1, \ldots, m-1\}$, it follows that $G = H_j G_j = H_j$ $(G_j \leq H_j = G)$. Also, since symmetries about different concurrent axes commute, there follows that every group G_j, with $j \in \{0, 1, \ldots, m-1\}$, is abelian, see the proof of [9, 8.3.1]. This proves the result. ■

The next (related) theorem generalizes the fact that if $\mathcal{S}$ is a (thick) TGQ of order (s,t), then s and t are powers of the same prime.

Theorem 7.9. *Suppose $\mathcal{S}$ is a GQ of order (s,t), $t > s^2/2$ and $s \neq 1$, and let x be a point which is incident with $r + 1$ axes of symmetry $L_0, L_1, \ldots, L_r$, $r \geq 2$, such that the following condition is satisfied.*

- *If G_i is the full group of symmetries about L_i, $i = 0, 1, \ldots, r$, and $H_i = \langle G_j \parallel j \neq i, 0 \leq j \leq r \rangle$, then $G_i \not\leq H_i$.*

If $r + 1 \geq 3 + \log_p \frac{t}{s}$, then $r + 1 = 3 + \log_p \frac{t}{s}$ and $(\mathcal{S}^{(x)}, G)$ is a TGQ. Here p is the smallest prime number dividing s, and $G = \langle G_0, G_1, \ldots, G_r \rangle$. Also, $\frac{t}{s}$ is a power of p, and s and t are powers of p if $t = s^2$.

Proof. From Theorem 7.8 and Theorem 4.2, there follows that $|G| \geq p^{r-2}s^3$. Since $r+1 \geq 3 + \log_p \frac{t}{s}$, there follows that $|G| \geq s^2t$. Since $t > s^2/2$, it follows by Theorem 7.4 that G is a group of elations with center x, hence $|G| = s^2t$ and $r = 2 + \log_p \frac{t}{s}$. The equality implies that $\frac{t}{s}$ is a power of p, and in particular, if $t = s^2$, there holds that s is a power of p, and then also t . Since $|G| = s^2t$, there also follows that $(\mathcal{S}^{(x)}, G)$ is an EGQ, and by Theorem 2.8 and the fact that $r \geq 2$, we also know that $\mathcal{S}^{(x)}$ is a TGQ. ■

Theorem 7.10 (FGQ, 8.1.1). *Let θ be a nonidentity whorl about the point p of the GQ $\mathcal{S} = (P, B, I)$ of order (s,t), $s \neq 1 \neq t$. Then one of the following must hold for the fixed elements structure $\mathcal{S}_\theta = (P_\theta, B_\theta, I_\theta)$:*

1. *$y^\theta \neq y$ for each $y \in P \setminus p^\perp$.*
2. *There is a point y, $y \not\sim p$, for which $y^\theta = y$. Put $V = \{p, y\}^\perp$ and $U = V^\perp$. Then $V \cup \{p, y\} \subseteq P_\theta \subseteq V \cup U$, and $L \in B_\theta$ if and only if L joins a point of V with a point of $U \cap P_\theta$.*
3. *$\mathcal{S}_\theta$ is a subGQ of order (s', t), where $2 \leq s' \leq s/t \leq t$, and hence $t < s$.*

Theorem 7.11. *Suppose that $\mathcal{S} = (P, B, I)$ is a thick GQ, with x a point which is incident with at least $r+1$ axes of symmetry, $L_0, \ldots, L_r$, $r \geq 3$. Let G_i be the full group of symmetries about L_i. Define $G = \langle G_i \parallel i = 0, 1, \ldots, r\rangle$, and put $H_i = \langle G_j \parallel j \neq i, i, j \in \{0, 1, \ldots, r\}\rangle$. Suppose that G_* is an arbitrary G-orbit in $P \setminus x^\perp$. If $\forall i = 0, 1, \ldots, r$ there holds that H_i acts transitively on the points of G_*, then G acts regularly on the points of G_*, G is abelian and G is a group of elations about x. The same properties hold immediately for every G-orbit in $P \setminus x^\perp$.*

Proof. Suppose $|G_*| = m$; then $m \geq s^3$ by Theorem 3.1, and it follows that $|G| = mk$, with $k = |G_y|$ for an arbitrary point $y \in G_*$. In the following y will be fixed, as well as $i \in \{0, 1, \ldots, r\}$. There holds that $|H_i| = mk'$ with $k' = |(H_i)_y|$, and clearly that $k'|k$, say $k = k'n$.

So we have that:

$$|G| = mk = |G_iH_i| = \frac{|G_i| \times |H_i|}{|G_i \cap H_i|} = \frac{smk'}{|G_i \cap H_i|}. \tag{1}$$

There follows that $s = n|G_i \cap H_i|$, and thus that $n|s$. Suppose now that p is a prime which divides n; then there exists a $\theta \in G_y$ of order p. Suppose M is a line through y meeting L_i in x_i. The orbits of $\langle\theta\rangle$ on M (seen as a point set) are cycles of length p, and since p is a divisor of n and of s, we have that there are at least $p+1$ points incident with M which are fixed by θ. By Theorem 7.4 θ has to be the identity, since $s \leq t$ and since every axis of symmetry is regular. It follows that $n = 1$, and that $H_i = G$. Every two symmetries about distinct concurrent lines commute, and since we proved that $G_jH_j = H_j = G$, with $i = 0, 1, \ldots, r$, G_j is commutative for every j. Hence G is abelian, and since G acts transitively on G_*, G also acts regularly on G_*, see e.g. [5]. Finally suppose that H_* is an arbitrary G-orbit in $P \setminus x^\perp$. Since G is abelian and since G acts transitively on this orbit, we can again conclude that G acts regularly on H_*. This proves the assertion. ■

Theorem 7.12. *Suppose $\mathcal{S} = (P, B, I)$ is a generalized quadrangle with parameters (s, t), $s \neq 1 \neq t$, and that p is a point which is incident with at least three axes of symmetry. Also, suppose that G is the group generated by the symmetries about* **every** *axis of symmetry through p. Suppose G_* is an arbitrary G-orbit of the permutation group $(P \setminus p^\perp, G)$. Now define the incidence structure $\mathcal{S}'(G_*) = \mathcal{S}' = (P', B', I')$ as follows. The elements of P' are of three types: (1) the point p; (2) the points of G_*; (3) any point which is incident with an axis of symmetry through p. We have two types of lines: (a) the axes of symmetry through p; (b) the lines of $\mathcal{S}$ which intersect a line of the first type and contain at least one point of G_*. The incidence relation $I' \subseteq I$ is the restriction of I to $(P' \times B') \cup (B' \times P')$.*
Then we have the following properties.

1. *There are constants l and k such that any point of the first two types is incident with $l + 1$ lines of $\mathcal{S}'$, and every point of the last type is incident with $k + 1$ lines;*

2. *A line of $\mathcal{S}'$ contains $s + 1$ points of $\mathcal{S}'$;*

3. *$|G_*| = s^2 k$;*

4. *k is divisible by s, and in particular we have that $s \leq k$. Also, $l \leq k$;*

5. *The number of points of $\mathcal{S}'$ is $ks^2 + (l + 1)s + 1$, and the number of lines of $\mathcal{S}'$ is $(l + 1)(sk + 1)$.*

Proof.

(i) Let L be an arbitrary line through p, and consider an arbitrary point qIL, $q \neq p$. Suppose that q is incident with $k+1$ lines of $\mathcal{S}'$. Since G acts transitively on the points of $L \setminus \{p\}$ (p is incident with at least one axis of symmetry), and since G_* is fixed by G, we can conclude that every point of $L \setminus \{p\}$ is incident with $k + 1$ lines of $\mathcal{S}'$. Next consider an arbitrary line L' such that $L' \neq L$, and an arbitrary point $q'IL'$, $q' \neq p$ (so q' is a point of $\mathcal{S}'$). If $k' + 1$ is the number of lines of $\mathcal{S}'$ incident with q', then we can easily see that $k' \geq k$ (the $k + 1$ lines through q' of $\mathcal{S}$, meeting the $k + 1$ lines through q of $\mathcal{S}'$ are also lines of $\mathcal{S}'$), and conversely we have that $k \geq k'$. It follows that there exists a $k \in \mathbb{N}$ such that each point of $\mathcal{S}'$ of type (3) is incident with $k + 1$ lines of $\mathcal{S}'$. Suppose that p is incident with $l + 1$ lines of $\mathcal{S}'$, and consider a point p' of G_*. From the definition of $\mathcal{S}'$, we can immediately see that p' is also incident with $l + 1$ lines of $\mathcal{S}'$. This proves part (1) of the theorem.

(ii) Immediately by the definition of $\mathcal{S}'$.

(iii) Consider an arbitrary line $L \in B'$ of type (a); then each point of G_* is incident with a unique line of $\mathcal{S}'$ (of type (b)) which is concurrent with L. Then the statement follows easily by part (1).

(iv) If G' is a group generated by all symmetries about three distinct axes of symmetry through a point p in a GQ $\mathcal{S}$, then $|G'| = s^3$ by Theorem 3.1, and G' is a group of elations with center p. Thus s^3 is a divisor of $|G_*| = s^2 k$. Now suppose that M is a fixed line of $\mathcal{S}'$ of type (b). Counting the number of pairs (q, M), with qIM a point of G_* lying on a line of $\mathcal{S}'$ not through p, we obtain that

$$s + sl(s-1) + (k-1)s \leq |G_*| = s^2 k,$$

or that

$$sl - l \leq sk - k.$$

Hence part (4) of the theorem follows.

(v) The numbers of points and lines of $\mathcal{S}'$ follow immediately by (1) and (2). ■

Remark 7.13. Theorem 7.12 is taken from K. Thas [25], and was one of the main motivations for introducing *semi quadrangles*, see [28].

We are now ready to state the following (main) result.

MAIN THEOREM 3. *Suppose $\mathcal{S}$ is a GQ of order (s,t), $t \geq s \neq 1$, and let p be a point of $\mathcal{S}$ which is incident with more than $t-s+2$ axes of symmetry. Then $\mathcal{S}^{(p)}$ is a translation quadrangle. If $s \neq t$, and G is the group generated by $t-s+2$ arbitrary axes of symmetry through p, then G is the translation group.*

Proof. If $s = t$, then the theorem follows from Theorem 2.3, so suppose that $s \neq t$. Consider $r+1$ axes of symmetry through p, with $r = t-s+1$, and suppose that G_* is an arbitrary G-orbit in $P \setminus p^{\perp}$, with G as above. Since p is incident with at least three axes of symmetry, we have by Theorem 4.2 that t is divisible by s, and since $r \geq 2$, we can use Theorem 7.12 (in the following we use the same notations as in Theorem 7.12). If $\mathcal{S}'$ is the incidence structure associated to G_* as defined in Theorem 7.12, then there follows that $k \geq t-s+1$ since $k \geq l \geq r$. However, by Theorem 7.12 it follows that $s|k$, and we already remarked that $s|t$. Since $k \leq t$, there holds that $k = t$, thus $|G_*| = s^2 k = s^2 t$, so G acts transitively on $P \setminus p^{\perp}$. Since the $t-s+2$ axes were arbitrary chosen, the conditions of Theorem 7.11 hold. Hence we can conclude that the group G acts regularly on $P \setminus p^{\perp}$, and the theorem follows by Theorem 2.8. ■

Remark 7.14. Regarding GQ's which have nonconcurrent axes of symmetry (the so-called *span-symmetric generalized quadrangles*), we refer to [27] and [29].

Appendix: a New Short Proof of a Theorem of Payne and Thas

Theorem 7.15. *Suppose $\mathcal{S} = (P, B, I)$ is a thick GQ of order s, and let p be a point of P incident with three distinct axes of symmetry L_1, L_2, L_3. Then every line through p is an axis of symmetry, and so $\mathcal{S}$ is a TGQ.*

The proof of this theorem, mentioned in Chapter 11 of FGQ [9] is not easy, and uses a coordinatization method for GQ's of order s and the theory of planar ternary rings. We will give a new geometrical proof without the use of coordinatization.

In the following we define G as the group generated by all symmetries about L_1, L_2 and L_3. Furthermore, G_i will be the full group of symmetries about the axis L_i.

Proof of the theorem. By Theorem 3.1 there follows that G is a group of elations with center p of order s^3, thus, $(\mathcal{S}^{(p)}, G)$ is an EGQ since $\mathcal{S}$ is of order s. Since an axis of symmetry is a regular line, the theorem now follows from Theorem 2.8. ■

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