Resonant nonlinear boundary value problems with almost periodic nonlinearity*

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Abstract

In this paper we give a qualitative and quantitative description of the set of continuous functions h for which the resonant boundary value problem

$$-u''(x) - u(x) + g(u(x)) = h(x), \ x \in [0, \pi],$$
$$u(0) = u(\pi) = 0,$$

has solution. Here, g is a continuous and bounded function (not identically zero), with primitive G satisfying the following hypothesis: there exist sequences $\{x_n\} \to +\infty$, $\{y_n\} \to +\infty$ such that $G(x_n) \to \sup\ \{G(t) : t \geq 0\}$, $G(y_n) \to \inf\ \{G(t) : t \geq 0\}$. In particular, this is the case if g is continuous and bounded and G is an almost periodic function. A noteworthy example, from the point of view of the applications to some problems in Mechanics, is when the function g is of the form $g = \sum_{i=1}^{n} g_i$, where each function g_i is a continuous periodic function with period T_i and with zero mean value, i.e., $\int_0^{T_i} g_i(t)\ dt = 0$, $1 \leq i \leq n$. In the proofs we use the Liapunov-Schmidt reduction, the shooting method and a detailed study of the oscillatory properties of the integral expressions associated to the bifurcation equation.

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1 Introduction

Nonlinear boundary value problems of the form

$$\begin{gathered} -u''(x) - u(x) + g(u(x)) = h(x), \ x \in [0,\pi], \\ u(0) = u(\pi) = 0, \end{gathered} \tag{1.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function satisfying some additional oscillatory properties, and $h \in C[0,\pi]$, are very important in the applications ([13]). Let us remark that (1.1) is a resonant problem at the principal eigenvalue, with bounded nonlinearity. For example, if g is a periodic function with zero mean value, (1.1) models the motion of a clock pendulum ([12]).

From the pioneer work by Dancer ([8]), for the special function $g(u) = \sin u$, and Ward ([17]), for general periodic nonlinearities with zero mean value, different authors have contributed to the study of this significant case (i.e., the case where g is periodic and with zero mean value), providing answers to the question on the existence and multiplicity of solutions of (1.1) ([3], [5], [15], [16]). Nevertheless, it must be pointed out that some important questions remains open, above all those related to the possible extension of the results to partial differential equations.

A more complicated situation is when the nonlinear term g is a finite sum of periodic functions. It must be emphasized that this may be of great interest in the application. For instance, this may be the case of a mechanical system formed by two pendulum which are connected by a chain-pinion system ([14]. Here the nonlinearity g is given by an expression of the form $g(u) = \sin u + \sin(\lambda u)$, where $\lambda \in \mathbb{R}^+$. Obviously, if λ is not a rational number, this function is not periodic. In this paper we study this kind of problems, showing that, from the qualitative point of view, the conclusion about the solvability of (1.1) is like the case when g is a periodic function with zero mean value. In the proofs we use the Liapunov-Schmidt reduction, but after applying this method, the main difficulty is to prove that the bifurcation equation changes its sign. If g is periodic and with zero mean value, this difficulty may be overcome by using some ideas about connectivity ([1], [5]) and the fact that there exist sequences $\{x_n\} \to +\infty, \ \{y_n\} \to +\infty$ such that the function G, a primitive of g, satisfies $G(x_n) = \max\ \{G(t) : t \geq 0\}, \ G(y_n) = \min\ \{G(t) : t \geq 0\}$. However, if g is a finite sum of periodic functions, the previous property is not necessarily true (think, for instance in the function $g(u) = \sin u + \sin(\sqrt{2}\ u)$). The problem here is how to compare the different terms in the bifurcation equation. This way seems really difficult. Instead of it, we adopt in this paper a different point of view, doing a detailed analysis of the global oscillatory properties of g. In fact, we consider more general situations where g is a continuous and bounded function, and its primitive G fulfills the following condition: there exist sequences $\{x_n\} \to +\infty, \ \{y_n\} \to +\infty$ such that $G(x_n) \to \sup\ \{G(t) : t \geq 0\}, \ G(y_n) \to \inf\ \{G(t) : t \geq 0\}$. In particular, this last condition is true if G is an almost periodic function and includes the case where g is a finite sum of periodic functions with zero mean value ([9]). In the last section of the paper we use the shooting method for obtaining some quantitative estimations which may be helpful to decide, in concrete examples, if (1.1) has solution. It seems that this type of questions has not been previously considered in the literature for problems like (1.1), even in the case of periodic nonlinearities.

2 Qualitative description of the range

Let us consider the bvp (1.1) where, from now on, the function $g : \mathbb{R} \to \mathbb{R}$ is bounded and continuous. Every function $h \in C[0, \pi]$ has a (unique) decomposition $h(x) = a \sin x + \tilde{h}(x),\ x \in [0, \pi]$, such that $\int_0^\pi \tilde{h}(x) \sin(x) dx = 0$. Then, the bvp (1.1) may be written in the form:

$$\begin{gathered} -u''(x) - u(x) + g(u(x)) = a \sin x + \tilde{h}(x),\ x \in [0, \pi], \\ u(0) = u(\pi) = 0. \end{gathered} \tag{2.2}$$

Let V denote the Banach space $V = C([0, \pi], \mathbb{R})$, with the norm $\|v\|_0 = \max\limits_{x \in [0,\pi]} |v(x)|$, for any $v \in V$. By U we denote the Banach space $U = \{u \in V : u(0) = u(\pi) = 0\}$ with the same norm. If we define the operators

$$L : dom\ L \to V,\ dom\ L = U \cap C^2[0, \pi],\quad Lu = -u'' - u,\ \forall\ u \in\ dom\ L,$$

and

$$N : U \to V, (Nu)(x) = a \sin x + \tilde{h}(x) - g(u(x)), \forall\ u \in U,\ \forall\ x \in [0, \pi],$$

then, problem (2.2) is equivalent to the operator equation

$$Lu = Nu. \tag{2.3}$$

It is very well known that L is a linear Fredholm mapping of index zero, so that there exist continuous projections $P : U \to U$ and $Q : V \to V$, such that $Im\ P = ker\ L$, $Im\ L = ker\ Q$ and (2.3) is equivalent to the alternative system

$$\tilde{u} = K(I - Q)N(c \sin(.) + \tilde{u})\ \ \text{(auxiliary equation)}\ , \tag{2.4}$$

$$QN(c \sin(.) + \tilde{u}) = 0\ \ \text{(bifurcation equation)}\ , \tag{2.5}$$

where K is the inverse of the mapping $L : dom\ L \cap ker\ P \to Im\ L$ and any $u \in U$ is written in the form $u(x) = \overline{u}(x) + \tilde{u}(x) = c \sin x + \tilde{u}(x),\ c \in \mathbb{R}$,
$\displaystyle\int_0^\pi \tilde{u}(x) \sin x dx = 0.$
Applying the Schauder fixed point theorem, we get that for any fixed $c \in \mathbb{R}$, there exists at least one solution $\tilde{u} \in ker\ P$ of (2.4) ([10]).
Denote by Σ the "solution set"of equation (2.4), i.e.,

$$\Sigma = \{(c, \tilde{u}) \in \mathbb{R} \times ker\ P : \tilde{u} = K(I - Q)N(c \sin(.) + \tilde{u})\}.$$

Taking into account that

$$Qv(x) = \left(\frac{2}{\pi} \int_0^\pi v(x) \sin x\ dx\right) \sin x,\ \forall\ v \in V,$$

the bifurcation equation (2.5) becomes

$$a = \frac{2}{\pi} \left[\int_0^\pi g(c \sin x + \tilde{u}(x))\ \sin x\ dx\right]. \tag{2.6}$$

Hence, for a given $\tilde{h}$, bvp (2.2) has solution if and only if a belongs to the range of the function $\Gamma : \Sigma \to \mathbb{R}$, defined by $\Gamma(c,\tilde{u}) = \frac{2}{\pi}\int_0^\pi g(c\sin x + \tilde{u}(x)) \ \sin x \ dx$. It is very well known ([1], [4]) that $\Gamma(\Sigma)$ is a bounded interval, which may be denoted by $I_{\tilde{h}}$. Moreover, if we define

$$p_1 : \Sigma \to \mathbb{R}, \ p_2 : \Sigma \to ker \ P, \text{ by } p_1(c,\tilde{u}) = c, \quad p_2(c,\tilde{u}) = \tilde{u}, \ \forall \ (c,\tilde{u}) \in \Sigma,$$

then, since g is bounded, we deduce from (2.4) that there is a constant $M > 0$, independent of $c \in \mathbb{R}$, such that

$$\|\tilde{u}\|_0 \le M, \ \|(\tilde{u})'\|_0 \le M, \|(\tilde{u})''\|_0 \le M, \ \forall \ \tilde{u} \in p_2(\Sigma). \tag{2.7}$$

We may use similar ideas as in [3] (Lemma 3) or in [15] (formula (24)), to prove the following lemma, where an equivalent expression for $\Gamma(c,\tilde{u})$ is given. This expression will allow us to study its sign.

LEMMA 2.1. *If G is a primitive of g, then there exists $c_0 > 0$ (depending only on the constant M in (2.7)) such that if $(c,\tilde{u}) \in \Sigma$ with $c \ge c_0$, we have*

$$\Gamma(c,\tilde{u}) =$$

$$= \int_0^\pi [-G(c\sin x + \tilde{u}(x)) + G(\|c\sin(.) + \tilde{u}(.)\|_0)] \ \frac{c + \cos x \ \tilde{u}'(x) - \sin x \ \tilde{u}''(x)}{(c\cos x + \tilde{u}'(x))^2} \ dx. \tag{2.8}$$

Analogously, if $(c,\tilde{u}) \in \Sigma$ and $c \le -c_0$, then

$$\Gamma(c,\tilde{u}) =$$

$$= \int_0^\pi [-G(c\sin x + \tilde{u}(x)) + G(-\|c\sin(.) + \tilde{u}(.)\|_0)] \ \frac{c + \cos x \ \tilde{u}'(x) - \sin x \ \tilde{u}''(x)}{(c\cos x + \tilde{u}'(x))^2} \ dx. \tag{2.9}$$

Also, from the properties of the function sin(.) and (2.7), it is easily proved that if c_0 is sufficiently large, then

$$c(c\sin x + \tilde{u}(x)) > 0, \ \ \forall \ x \in (0,\pi), \ \ \forall \ (c,\tilde{u}) \in \ \Sigma : |c| \ge c_0. \tag{2.10}$$

Now, with the purpose of stating and proving the main result of this section, we introduce an additional hypothesis on G:

$$\begin{gathered} \exists \ \{x_n\} \to +\infty, \ \exists \ \{y_n\} \to +\infty, \text{ such that} \\ G(x_n) \to \sup \ \{G(t) : t \ge 0\}, \ G(y_n) \to \inf \ \{G(t) : t \ge 0\} \end{gathered} \tag{2.11}$$

where in the previous assumption, the quantities $\sup \ \{G(t) : t \ge 0\}$ and $\inf \ \{G(t) : t \ge 0\}$ are not necessarily finite numbers.

The main result of this section is given by the following theorem. Remember that for a given function $\tilde{h}$, $I_{\tilde{h}}$ is the set of values a for which (2.2) has solution.

THEOREM 2.2. *Let us consider the bvp (2.2), where g is continuous and bounded, not identically zero and G satisfies (2.11). Then, for any given $\tilde{h}$, the interval $I_{\tilde{h}}$ contains negative and positive values. Moreover, for each given natural number n, there is an $\epsilon_n > 0$ (depending on n, g and $\tilde{h}$), such that (2.2) has at least n solutions if $0 < |a| \leq \epsilon_n$. Finally, if $a = 0$, (2.2) has infinitely many solutions.*

Proof. We prove that $I_{\tilde{h}}$ contains positive values. In an analogous way it is possible to prove that it also contains negative values.
We distinguish three cases:
Case 1: $S \equiv \sup\{G(t) : t \geq 0\}$ is not achieved at $[0, +\infty)$.
Case 2: S is achieved at $[0, +\infty)$ and the set $A = \{x \geq 0 : G(x) = S\}$ is not a bounded set.
Case 3: S is achieved at $[0, +\infty)$ and the set $A = \{x \geq 0 : G(x) = S\}$ is a bounded set.
In both cases, 1 or 2, it is easily proved the following property:

$$\exists\, x \geq c_0 + M : G(x) = \max_{[0,x]} G \tag{2.12}$$

and moreover, G is not a constant function in $[0, x]$. In fact, (2.12) is trivial if we are in case 2. If we are in case 1, then $S_1 \equiv \max_{[0,c_0+M]} G < S$. From (2.11) we deduce the existence of $t_1 > c_0 + M$ such that $G(t_1) > S_1$. Let $x_1 \in [0, t_1]$ be such that $G(x_1) = \max_{[0,t_1]} G$. Then $x_1 > c_0 + M$, $G(x_1) = \max_{[0,x_1]} G$ and G is no constant in $[0, x_1]$.
Property (2.12) allows to prove that $\Gamma(\Sigma)$ contains positive values. This fact is established in the next lemma.

LEMMA 2.3. *If g is continuous and bounded and there exists some x satisfying (2.12) and G is not a constant function in $[0, x]$, then $\exists\, (c, \tilde{u}_c) \in \Sigma : \Gamma(c, \tilde{u}_c) > 0$ and $|c - x| \leq M$.*

Proof. Since (2.2) is a resonance problem at the principal eigenvalue and the nonlinearity g is bounded, it is possible to prove the existence of a connected subset Σ_1 of Σ such that $p_1(\Sigma_1) = [x-M, x+M]$ ([1]). Let us denote $r = x-M \geq c_0$, $s = x+M$ and choose $(r, \tilde{u}_r)$, $(s, \tilde{u}_s) \in \Sigma_1$. Then $\|r\sin(.)+\tilde{u}_r(.)\|_0 \leq r+M = x$, $\|s\sin(.)+\tilde{u}_s(.)\|_0 \geq s - M = x$. Therefore, $\exists\, (c, \tilde{u}_c) \in \Sigma_1 : \|c\sin(.) + \tilde{u}_c(.)\|_0 = x$. Consequently, from (2.8), we obtain $\Gamma(c, \tilde{u}_c) > 0$.

Let us suppose now that we are in case 3. Then, if $t_0 = \sup\ A$, we have $G(t_0) = S$ and $G(t) < S$, $\forall\, t > t_0$. Since the constant M given in (2.7) may be chosen arbitrary large, it is not restrictive to assume the following property

$$\exists\, t_0 < c_0 + M \ \text{ such that } G(t_0) = S \text{ and } G(t) < S,\ \forall\, t \geq c_0 + M \tag{2.13}$$

Now, property (2.13) allows to prove that $\Gamma(\Sigma)$ contains positive values. This fact is established in the next lemma.

LEMMA 2.4. *If g is continuous and bounded and (2.13) is satisfied, then $\exists\, (c, \tilde{u}_c) \in \Sigma : \Gamma(c, \tilde{u}_c) > 0$.*

Proof. Let $\bar{c} > 0$ be such that

$$\frac{\bar{c} - M}{c_0 + M} > 2 \tag{2.14}$$

and choose $\epsilon > 0$ such that $G(t) < S - 4\epsilon, \ \forall t \in [c_0 + M, \bar{c}]$. Let $\hat{c} > 0$ be satisfying the following properties:

$$\left.\begin{array}{ll} \text{(a)} & \dfrac{\bar{c} - M}{x} < 1/2 \ \ \forall x \geq \hat{c} - M \\[2ex] \text{(b)} & \dfrac{\dfrac{x - 2M}{(x+M)^2}}{\dfrac{x + 2M}{(\frac{\sqrt{3}}{2}x - M)^2}} > 1/2 \ \ \forall x \geq \hat{c} - M \\[2ex] \text{(c)} & G(\hat{c}) \geq S - \epsilon \end{array}\right\} \tag{2.15}$$

Think that (a) and (b) are fulfilled if $\hat{c}$ is sufficiently large. Moreover, property (c) may be established from (2.11).

Let $\delta > 0$ be such that $G(t) < S - \delta, \ \forall\, t \in [c_0 + M, \hat{c}]$ and let us define $\tilde{c} = \min\{x \geq c_0 + M : G(x) \geq S - \delta\} > \hat{c}$.

Clearly we have the following property

$$\tilde{c} \text{ satisfies } \ (2.15)(\text{ as } \hat{c}) \text{ and } G(\tilde{c}) \geq G(x), \ \forall x \in [c_0 + M, \tilde{c}] \tag{2.16}$$

As in cases 1 and 2, let $\Sigma_1 \subset \Sigma$ be a connected subset such that $p(\Sigma_1) = [\tilde{c} - M, \tilde{c} + M]$. Then, there exists $(c, \tilde{u}_c) \in \Sigma_1, \|c\sin(\cdot) + \tilde{u}_c(\cdot)\|_0 = \tilde{c}, \ |c - \tilde{c}| \leq M$. Next, we will prove that $\Gamma(c, \tilde{u}_c) > 0$.

First, it is known ([3]) that $\exists\, \alpha \in [\frac{\pi}{4}, \frac{3\pi}{4}]$ such that the function $c\sin(\cdot) + \tilde{u}_c(\cdot)$ attains its maximum only in the point α.Then,

$$\Gamma(c, \tilde{u}_c) = \int_0^\pi H(t)dt = \int_0^\alpha H(t)dt + \int_\alpha^\pi H(t)dt = I_1 + I_2$$

where

$$H(t) = [-G(c\sin(t) + \tilde{u}_c(t)) + G(\|c\sin(.) + \tilde{u}_c(.)\|_0)] \cdot \frac{c + \cos(t)\tilde{u}_c'(t) - \sin(t)\tilde{u}_c''(t)}{(c\cos(t) + \tilde{u}_c'(t))^2}$$

Let us prove that $I_1 > 0$. To see this, let us define $x_0 = \arcsin(\frac{c_0 + 2M}{c})$, $x_1 = \arcsin(\frac{\bar{c} - M}{c})$ ($x_0 < x_1 < \frac{\pi}{6}$ from (2.14) and (2.15), (a)). Then,

$$I_1 = \int_0^{x_0} H(t)dt + \int_{x_0}^{x_1} H(t)dt + \int_{x_1}^{\alpha} H(t)dt$$

Let us study these three integral terms .

1. From (2.15), (c) and the inequality $x_0 < \frac{\pi}{6}$, we obtain

$$\int_0^{x_0} H(t)\ dt \geq \int_0^{x_0} (-\epsilon\ ) \cdot \frac{c + \cos(t)\tilde{u}_c'(t) - \sin(t)\tilde{u}_c''(t)}{(c\cos(t) + \tilde{u}_c'(t))^2} dt$$

$$\geq -\epsilon\ \cdot \frac{(c + 2M)x_0}{(c\cos(x_0) - M)^2} \geq -\epsilon\ \cdot \frac{(c + 2M)x_0}{(c\sqrt{3}/2 - M)^2}$$

2. Since

$$x_0 < t < x_1 \Rightarrow \frac{c_0 + 2M}{c} < \sin(t) < \frac{\bar{c} - M}{c} \Rightarrow c_0 + 2M < c\sin(t) < \bar{c} - M \Rightarrow$$

$$c_0 + M < \tilde{u}_c(t) + c\sin(t) < \bar{c}$$

we deduce that if $t \in [x_0, x_1]$ then $G(\tilde{u}_c(t) + c\sin(t)) < S - 4\epsilon$; also, from (2.15), (c), $G(\tilde{c}) \geq S - \epsilon$. Therefore,

$$\int_{x_0}^{x_1} H(t)\ dt \geq \int_{x_0}^{x_1} 3\epsilon \cdot \frac{c - 2M}{(c+M)^2} dt = 3\epsilon \frac{c - 2M}{(c+M)^2}(x_1 - x_0)$$

3. Lastly, since $G(\tilde{c}) \geq G(x),\ \forall x \in [c_0 + M, \tilde{c}]$, we have $\int_{x_1}^{\alpha} H(t)\ dt \geq 0$

Just to conclude that $I_1 > 0$, it is sufficient to check that $x_1 - x_0 \geq x_0$. But this last relation is trivial, since

$$1/2 > \frac{\bar{c} - M}{c} > 2 \cdot \frac{c_0 + 2M}{c}$$

Now, taking into account that the function arcsin(.) is increasing in $[0, 1/2]$ and that $\arcsin(2t) \geq 2\arcsin(t)$ in this interval, we obtain $x_1 \geq 2x_0$.

To prove that $I_2 > 0$, we may define $y_0 = \pi - x_0,\ y_1 = \pi - x_1$ and then, to decompose I_2 in the form

$$I_2 = \int_{\alpha}^{y_1} H(t)dt + \int_{y_1}^{y_0} H(t)dt + \int_{y_0}^{\pi} H(t)dt$$

Then, an analogous reasoning to the previous one allows to demonstrate that $I_2 > 0$. By using the sequence $\{y_n\}$ of the hypothesis (2.11), instead of $\{x_n\}$, it is possible to prove, in an analogous way, that $\Gamma(\Sigma)$ contains negative values.

The results related to the multiplicity of solutions may be demonstrated by using the following ideas. Let us denote by the same letter c_1, the constant x of (2.12) and the constant $\hat{c}$ of (2.15). Then, the conclusion of the previous reasonings is that there exists a connected subset Σ_1 of Σ such that $p_1(\Sigma_1) = [c_1 - M, c_1 + M]$ and $(c, \tilde{u}_c) \in \Sigma_1$ with $|c_1 - c| \leq M$ such that $\Gamma(c, \tilde{u}_c) > 0$. Let us denote by d_1 the corresponding constant obtained from the hypothesis (2.11) by using the sequence $\{y_n\}$. Without loss of generality we may assume that $d_1 \geq c_1$. Then, if Π_1 is a connected subset of Σ such that $p_1(\Pi_1) = [c_1 - M, d_1 + M]$, we obtain that $\Gamma(\Pi_1) \supset [-\epsilon_1, \epsilon_1]$, with $\epsilon_1 > 0$. Repeating all the previous process but with the constant $d_1 + 2M$ instead of c_0, we obtain the existence of a connected subset Π_2 of Σ such that $\Pi_1 \cap \Pi_2 = \emptyset$ and $\Gamma(\Pi_2) \supset [-\epsilon_2, \epsilon_2]$. If $|a| \leq \min\{\epsilon_1, \epsilon_2\}$, (2.2) has, at least, two solutions. It is now clear how to obtain n solutions if $|a|$ is sufficiently small and infinitely many solutions if $a = 0$.

REMARK. It is clear that the method of the proof may be used in other situations. For instance, the same result may be obtained if the hypothesis (2.11) is replaced by

$$\begin{array}{c} \exists \ \{x_n\} \to -\infty, \ \exists \ \{y_n\} \to -\infty, \text{ such that} \\ G(x_n) \to \sup \ \{G(t) : t \geq 0\}, \ G(y_n) \to \inf \ \{G(t) : t \geq 0\} \end{array} \tag{2.17}$$

Next, we show that the conditions of Theorem 2.2 are fulfilled in the important case where g is continuous and bounded and G is an almost periodic function. Let us remember that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be almost periodic (a.p.) if for any $\epsilon > 0$ there is a $L = L(f, \epsilon) > 0$ such that in any interval of length L there is an element t such that $|f(x+t) - f(x)| < \epsilon, \ \forall \ x \in \mathbb{R}$ ([9], [11]).

COROLLARY 2.5. *Let g be a continuous and bounded function such that G is a nonconstant almost periodic function. Then the conclusion of Theorem 2.2 is true.*

Proof. If G is a.p., then $S \equiv \sup \ \{G(t) : t \geq 0\}$ is finite. Moreover, if $\epsilon > 0$ is given, there is $x = x(\epsilon)$ such that $|S - f(x)| < \epsilon$. Also, by using the definition of a.p. function, there exists $t(\epsilon) \in [\frac{1}{\epsilon}, \frac{1}{\epsilon} + L]$ verifying $|f(x) - f(x + t(\epsilon))| < \epsilon$. Therefore, $|f(x + t(\epsilon)) - S| < 2\epsilon$. Taking $\epsilon = 1/n, \ n \in \mathbb{N}$, we obtain the sequence $\{x_n\}$ of the hypothesis (2.11). In an analogous way may be obtained the sequence $\{y_n\}$.

Since continuous periodic functions are almost periodic functions and also f, g a.p. imply $f + g$ a.p. ([9], [11]), we have the following significant corollary.

COROLLARY 2.6. *Let us consider the bvp (2.2) where the function g is a not identically zero function of the form $g = \sum_{i=1}^{n} g_i$ and each function g_i, $1 \leq i \leq n$ is T_i-periodic and with zero mean value, i.e., $\int_0^{T_i} g_i(u) \ du = 0$. Then, the conclusion of the Theorem 2.2 is true. Moreover, in this case the interval $I_{\tilde{h}}$ is closed.*

Proof. In this case, $G = \sum_{i=1}^{n} G_i$ where each $G_i, 1 \leq i \leq n$, is periodic. Also, by using the Riemann-Lebesgue lemma, it is easily deduced that $I_{\tilde{h}}$ is closed.

REMARK. Under the conditions of the previous corollary, and if the constant a is equal to zero in (2.2), the functional $\Phi : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\Phi(u) = \int_0^{\pi} \left[\frac{1}{2}((u')^2 - u^2) + G(u) - hu \right] \ dt$$

is bounded from below ($H_0^1(\Omega)$ is the usual Sobolev space), but it is not coercive. Also, it does not satisfy, in general, the $(P - S)_m$ (Palais-Smale condition), where $m = \inf_{\ H_0^1(\Omega)} \Phi$ (see [2]). However, taking into account some ideas from [3], it can be proved (under more general hypotheses that those of the previous corollary) that Φ attains its infimum. This result, which is interesting from the point of view of the applications, does not seem trivial, since some fundamental properties used in [3] are not valid in the present situation. It will be shown elsewhere.

3 Quantitative estimations

Theorem 2.2 provides a qualitative description of the range of the operator $-u''(x)-u(x)+g(u(x))$ under the boundary conditions $u(0)=u(\pi)=0$. However, this result is not completely satisfactory from the point of view of the possible applications to concrete situations, since it does not give any quantitative estimation of the interval $I_{\tilde{h}}$. In this section we show some ideas about this problem. For clarity of the exposition, we restrict ourselves to the case of locally lipschitz and periodic nonlinearities, with zero mean value, but it is obvious that we can deal with more general situations. In the proofs, we combine the use of the shooting method with some formula of the previous section.

Let us consider the bvp (2.2) where $g:\mathbb{R}\to\mathbb{R}$ is a nontrivial locally lipschitzian function, T-periodic and with zero mean value; $a\in\mathbb{R}$ and $\tilde{h}\in C[0,\pi]$ satisfies $\int_0^\pi \tilde{h}(x)\sin\ x\ dx=0$. If (2.2) has solution u, then (2.6) is satisfied for the function u, so that

$$|a|\le\frac{4}{\pi}\ \max_{\mathbb{R}}\ |g|\equiv R \tag{3.18}$$

We assume the previous restriction from now on.

The initial value problem

$$\begin{gathered}-u''(x)-u(x)+g(u(x))=a\sin x+\tilde{h}(x),\ \ x\in[0,\pi],\\ u(0)=0,\ \ u'(0)=r,\end{gathered} \tag{3.19}$$

has, for any given real number r, a unique solution $u_a^r\in C^2(\mathbb{R})$ which depends continuously on a and r. Also, it is trivially checked ([7]) that u_a^r is given by the relation

$$u_a^r(x)=r\sin(x)+\tilde{u}_a^r(x) \tag{3.20}$$

where

$$\tilde{u}_a^r(x)=-\cos(x)\int_0^x\sin(t)[h(t)-g(u(t))]\ dt$$

$$+\ \sin(x)\int_0^x\cos(t)[h(t)-g(u(t))]\ dt$$

and $h(x)=a\sin x+\tilde{h}(x)$. Moreover, u_a^r satisfies (2.2) if and only if $u_a^r(\pi)=0$, or equivalently

$$a=\frac{2}{\pi}\int_0^\pi g(u_a^r(x))\sin x\ dx \tag{3.21}$$

Therefore, if for any fixed real number a satisfying (3.18), we define the mapping

$$\Gamma_a:\mathbb{R}\to\mathbb{R}$$

by

$$\Gamma_a(r)=\frac{2}{\pi}\int_0^\pi g(u_a^r(x))\sin(x)dx,$$

(2.2) has solution if and only if $a\in\ Im\ \Gamma_a$.

If for a given function $\tilde{h}$, we denote $p=\max_{\mathbb{R}}\ |g|+R+\max_{[0,\pi]}\ |\tilde{h}|$ $(p=p(g,\tilde{h}))$, then

$$|\tilde{u}_a^r(x)|\le 4p,\ \ |(\tilde{u}_a^r)'(x)|\le 4p,\ \ |(\tilde{u}_a^r)''(x)|\le 5p,\ \forall\ x\in[0,\pi], \tag{3.22}$$

Since the function G, primitive of g, is also a T-periodic function, there exists some value $\alpha > (5\sqrt{2}+4)p$ such that

$$G(\alpha) = \mathrm{Max}\{G(x) : x \in \mathbb{R}\} \tag{3.23}$$

We choose G such that it has also mean value zero. Let us fix some value $\alpha > (5\sqrt{2}+4)p,\ \alpha > T$, satisfying (3.23).
Now, we obtain an estimation (which depends on such $\alpha,\ g$ and $\tilde{h}$) of the positive range of the function Γ_a, for any fixed real number a satisfying (3.18). Similar ideas may be used to estimate the negative range.

LEMMA 3.7. *Let $\alpha > (5\sqrt{2}+4)p$ be such that (3.23) is satisfied and*

$$\varepsilon_\alpha = \frac{2}{\pi}\ Min\left\{\frac{r-5\sqrt{2}p}{(r+4p)^3} :\ r \in \{\alpha-4p, \alpha+4p\}\ \right\}\ (\max_{\mathbb{R}}\ G)(y-x)$$

where $[x,y] \subset [0,T]$ is any interval such that $G(t) \leq 0\ \forall\ t \in [x,y]$. Then, if $a \in [0, \varepsilon_\alpha]$, (2.2) has solution.

Proof. Let us choose $r \in \mathbb{R}^+$ such that $\|r\sin(\cdot) + \tilde{u}_a^r(\cdot)\|_0 = \alpha$. Note that this is always possible since $\|r\sin(\cdot) + \tilde{u}_a^r(\cdot)\|_0 \to +\infty$, in a continuous way if $r \to +\infty$. Also, $r > 5\sqrt{2}p$ and $\|\tilde{u}_a^r\|_2 \equiv \max\{\|\tilde{u}_a^r\|_0,\ \|(\tilde{u}_a^r)'\|_0,\ \|(\tilde{u}_a^r)''\|_0\} < 5p$. Taking into account the Lemma 1 and Lemma 3 in [3], for $\delta = \pi/4$, we have that since $r > 5\sqrt{2}p$, the formulas given in our Lemma (2.1) of section 2 are valid. Consequently, if $u = r\sin(.) + \tilde{u}_a^r(.)$, we obtain

$$\int_0^\pi g(u(t))\sin(t)dt = \int_0^\pi [-G(u(t)) + G(\alpha)] \cdot \frac{r + \cos(t)\tilde{u}'(t) - \sin(t)\tilde{u}''(t)}{(r\cos(t) + \tilde{u}'(t))^2}dt \geq$$

$$\geq \frac{r-5\sqrt{2}p}{(r+4p)^2}\int_0^\pi [G(\alpha) - G(u(t))]dt$$

(think that $\max_{[0,\pi]}\ |\cos t| + |\sin t| = \sqrt{2}$). Since G has zero mean value, there exists an interval $I = [x,y] \subset [0,T]$ such that $G(t) \leq 0\ \forall\ t \in [x,y]$. Also, we know that the function u takes all the values from 0 to α. Then, $\exists\ t_0, t_1 \in [0,\pi]$ verifying $u(t_1) = y,\ u(t_0) = x,\ u([t_0,t_1]) = [x,y]$. By using the Mean Value Theorem we have: $t_1 - t_0 = \dfrac{y-x}{u'(c)} \geq \dfrac{y-x}{r+4p}$. Consequently,

$$\int_0^\pi g(u(t))\sin(t)dt \geq \frac{r-5\sqrt{2}p}{(r+4p)^2}\int_{t_0}^{t_1}[G(\alpha) - G(u(t))] \geq \frac{r-5\sqrt{2}p}{(r+4p)^2}\ \max_{\mathbb{R}} G)\ \frac{y-x}{r+4p}$$

Finally, taking into account that $r \in [\alpha-4p, \alpha+4p]$ and that the function $\dfrac{r-5\sqrt{2}p}{(r+4p)^3}$ has not relative minimum in the interior of this interval, we have the conclusion of the lemma.

REMARKS.
1.- Lemma 1 and Lemma 3 in [3] are proved under the additional assumption $u(\pi) = 0$, which is not necessarily satisfied here. But is is trivially deduced from the proof given in [3] that this hypothesis is not necessary. The important fact, which is satisfied here, is that the functions $(\tilde{u}_a^r)', (\tilde{u}_a^r)''$ are uniformly bounded in $[0, \pi]$ ((3.22)).
2.- It is clearly deduced from the proof of the previous Lemma that if the function G is negative in more than one interval $[x, y]$, then we may improve the obtained estimations. This is what happens in the following example.
3.- It is also possible to use the expression (2.6) to estimate the range of values a for which (2.2) has solution. The ideas are practically the same, but the estimations obtained by this method are, in general, worse than those which may be obtained as previously, by using the shooting method (think that the bounds for $\tilde{u}$ which are derived from (2.4) are, in general, worse than those for $\tilde{u}_a^r$).

EXAMPLE.Let us consider the following problem:

$$\left.\begin{array}{c} -u'' - u + \frac{1}{2}\sin(u) = a\sin(x), \;\; x \in [0, \pi], \\ u(0) = u(\pi) = 0. \end{array}\right\} \tag{3.24}$$

It is easily checked that in this case one may take $R = 2/\pi$, $p = \dfrac{\pi + 4}{2\pi}$, $\alpha = 3\pi$ and $[x, y] = [0, \frac{\pi}{2}]$. An elementary calculation shows that $\varepsilon_\alpha = 0.0004$, so that if $a \in [0, \varepsilon_\alpha]$, (3.24) has solution. However, it must be pointed out that, taking into account the basic ideas of this section and the particular properties of the concrete problem that we are considering, it is possible, in general, to obtain better estimations than those which have been obtained for general problems. For instance, in the case of (3.24), we may restrict the values of the constant a such that $|a| < 10^{-1}$. Then, we get a smaller value of p and, consequently, a bigger value of ε_α. More precisely, if $|a| < 10^{-1}$, then we may choose $p = 0.6$, $\alpha = 3\pi$, $[x, y] = [0, \frac{\pi}{2}]$. In this case $\varepsilon_\alpha = 0.00131712$. Also, let us note that we can improve the value of ε_α by considering the union of intervals $I = [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, \frac{5\pi}{2}]$ instead of the interval $[0, \frac{\pi}{2}]$. It is clear that the proof of Lemma 3.7 is valid in this case.

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