Positive solutions of singular boundary value problems with indefinite weight

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Abstract

Using a generalized version of the method of lower and upper solutions, we prove existence of positive solutions for a class of boundary value problems for a nonlinear equation with singularities whose coefficients change sign.

1 Introduction

In this article we study the existence of positive solutions to a generalized Sturm-Liouville boundary value problem. We consider a second order scalar equation with coefficients that may be of indefinite sign and singular at the end points $t = 0$ and $t = 1$. In this way, we try to combine in a unique setting some features which were previously discussed in various different papers.

Positive solutions for second order nonlinear ordinary and partial differential equations have been widely investigated in the literature. We refer to [3], [11], [17] and the references quoted therein for some classical results in this area, dealing with the Dirichlet problem

$$\begin{array}{ll} -\Delta u = \lambda f(x,u), & x \in \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{array} \tag{1}$$

As is well known, the search of radially symmetric solutions to (1) when Ω is an annular domain or a ball in $\mathbb{R}^N$, yields, up to a suitable rescaling, to the study of a Dirichlet boundary value problem

$$\begin{gathered} -u'' = \lambda h_1(t,u), \\ u(0) = 0,\ u(1) = 0, \end{gathered} \tag{2}$$

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or of a mixed problem with a singularity at $t = 0$,

$$\begin{aligned} -(t^{N-1}u')' &= \lambda t^{N-1} h_2(t,u), \\ u'(0) = 0, \ & u(1) = 0. \end{aligned} \tag{3}$$

Motivated also by different examples, some authors, starting with [23], have considered the possibility of more general kind of singularities on the coefficients in order to cover problems of the form

$$\begin{aligned} -(p(t)u')' &= \lambda q(t) g(u), \\ au(0) - b(pu')(0) &= 0, \\ cu(1) + d(pu')(1) &= 0, \end{aligned} \tag{4}$$

with $p > 0$ on $]0,1[$ but possibly vanishing at $t = 0$ or $t = 1$ and $q > 0$, but possibly also unbounded near $t = 0$ or $t = 1$. Besides the classical Carathéodory conditions, it can be seen that an appropriate growth assumption in the t-variable for the right hand side of equation (2) may take the form (see [18]): *for each* $r > 0$, *there is a measurable function* $\rho_r > 0$ *such that* $\int_0^1 t(1-t)\rho_r(t)\,dt < +\infty$ *and* $|h_1(t,x)| \le \rho_r(t)$ *for a.e.* $t \in [0,1]$ *and for all* $|x| \le r$ (a condition which clearly generalizes the Carathéodory one). Note that the weak integrability assumption on ρ_r corresponds to

$$\int_0^1 G(t,t)\rho_r(t)\,dt < +\infty, \tag{5}$$

where $G(t,s)$ is the Green function associated to the problem

$$-u'' = w(t), \quad u(0) = u(1) = 0.$$

In the last ten years, a great deal of interest has been devoted to the investigation of equations with *indefinite weight*, for instance, studying problem (1) with $f(x,u) = a(x)g(u)$, and $a(\cdot)$ changing sign in Ω (see, e.g., [2], [4], [5], [6]). The study of positive radially symmetric solutions for nonlinearities indefinite in sign has been recently carried out in [1], [7]. Thus, a natural question that can be raised is that of discussing the problem of the existence of positive solutions for a generalized Sturm-Liouville boundary value problem of the form (4), in case the coefficients may be singular at $t = 0$ and $t = 1$ and $q(t)$ is also indefinite in sign.

In this article, we address our attention to such a question, by using the method of lower and upper solutions. More precisely, in Section 2, we study the problem

$$\begin{aligned} -(p(t)u')' &= f(t,u), \\ au(0) - b(pu')(0) &= 0, \\ cu(1) + d(pu')(1) &= 0, \end{aligned} \tag{6}$$

and give a theorem for the existence of a solution between a pair of well-ordered lower and upper solutions, under the main assumptions that $p(t)$ is positive on $]0,1[$ and f satisfies a growth assumption in t which is related to the Green function associated to the operator $u \mapsto -(p(t)u')'$ with the corresponding boundary conditions. Our assumption corresponds to (5) in the case when $p \equiv 1$ and $b = d = 0$, $ac \ne 0$. On the non-negative coefficients a, b, c, d we require that $a + b, a + c, c + d > 0$, so that we are able to deal with boundary conditions both like in (2) and in (3).

As a next step, we study equation (4), where we assume $g(u) > 0$ for all $u > 0$. Three different cases are then discussed. In Section 3, we study the situation when $g(0) > 0$ and prove the existence of small positive solutions when λ is small and positive and the weight $q(t)$ changes sign but is positive with respect to suitable averages. Our theorem generalizes recent results by Afrouzi and Brown [1] which in turn extended previous works in [7]. We also show that if there is a positive solution, then, necessarily, the corresponding averages for $q(t)$ must be nonnegative. From this point of view, our result can be considered as almost sharp. In Section 4 we study the case when $g(0) = 0$ with $g'(0) > 0$ and g bounded. Our result provides the existence of positive solutions for those values of $\lambda > 0$ for which the linear nonhomogeneous equation $(p(t)u')' + \lambda g'(0)q(t)u = 1$ has a non-negative solution satisfying our boundary conditions. This assumption is strictly related to a suitable form of the antimaximum principle (cf. [12], [13], [15] and the references therein) for the linear equation $(p(t)u')' + \mu q(t)u = w(t)$ with $q(t)$ possibly changing sign and the coefficients in the class of summability (weighted with the Green function) for which we have developed the result on lower and upper solutions of Section 2. It is immediate to see that under the regularity and growth assumptions on $p(t)$ and $q(t)$ usually considered in the literature, our result is satisfied for all those values of the parameter λ for which the antimaximum principle holds. Finally, in Section 5, we consider the case in which we can have $g(0^+) = +\infty$. This situation combines the effects of a singularity in the t-variable with a singularity in the u-variable with the possibility of a weight $q(t)$ which is not positive. For simplicity and also following [22], [25] (where the case of $q(t)$ positive and continuous on $[0, 1]$ was considered) and [9], [14], [19] (where $q(t)$ positive and satisfying (5) was assumed) we confine ourselves to the study of the Dirichlet problem. We show that in this case, under quite natural and mild assumptions, the existence of positive solutions may fail. On the other hand, we give some examples where the lower and upper solutions arguments developed in [9] and in [14], may be employed for the solvability of the singular two-point boundary value problem if $q(t)$ is not "too much" negative.

Throughout the article, we denote by $\mathbb{R}^+ = [0, +\infty[$, the set of non-negative real numbers. By a positive solution we usually mean a function $u \in \mathcal{C}([0, 1])$, with u and pu' absolutely continuous on $]0, 1[$, satisfying the differential equation almost everywhere in $]0, 1[$, as well as the boundary conditions (in the appropriate sense), and such that $u(t) > 0$ for all $t \in]0, 1[$.

2 The method of lower and upper solutions

In this section, we present results from the theory of lower and upper solutions that apply to the boundary value problem (6). We assume the following conditions are satisfied :

(H_1) $a,\ b,\ c,\ d \geq 0$ and $a + b,\ a + c,\ c + d > 0$;

(H_2) $p \in \mathcal{C}([0, 1])$ and $p > 0$ on $]0, 1[$. Further, $\frac{1}{p} \in L^1_{loc}(0, 1]$ if $a = 0$, $\frac{1}{p} \in L^1_{loc}[0, 1)$ if $c = 0$ and $\frac{1}{p} \in L^1(0, 1)$, if a and $c \neq 0$.

Define $w(t) = \displaystyle\int_{1/2}^{t} \frac{ds}{p(s)}$ and the Green function

$$
\begin{aligned}
G(t,s) &= \frac{[a(w(s)-w(0))+b][c(w(1)-w(t))+d]}{ac(w(1)-w(0))+ad+bc} && \text{if } s \le t,\\
&= \frac{[a(w(t)-w(0))+b][c(w(1)-w(s))+d]}{ac(w(1)-w(0))+ad+bc} && \text{if } s > t,
\end{aligned} \tag{7}
$$

associated with (6).

We shall consider nonlinearities f that satisfy the following assumptions.

(H_3) $f : [0,1] \times \mathbb{R} \to \mathbb{R}, (t,u) \mapsto f(t,u)$ is a Carathéodory function, i.e. measurable in t for all $u \in \mathbb{R}$ and continuous in u for a.e. $t \in [0,1]$.

(H_4) for any $r > 0$ there exists a measurable function $h(t)$ such that
(i) for a.e. $t \in [0,1]$ and all $u \in [-r,r]$, $|f(t,u)| \le h(t)$ and
(ii) $G(t,t)h(t) \in L^1(0,1)$, with $G(t,s)$ defined in (7).

The following lemma proves the problem (6) is equivalent to a fixed point problem.

Lemma 2.1. *Assume conditions (H_1), (H_2), (H_3) and (H_4) are satisfied. Then problem* (6) *is equivalent to the fixed point problem*

$$u = Tu,$$

where $T : \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ is defined by

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s))\, ds \tag{8}$$

and $G(t,s)$ is given in (7)*. Further, the operator T is completely continuous.*

Proof. The equivalence of (6) with the fixed point problem follows from straightforward computations. In particular, we observe that the fixed points of T are such that

$$
\begin{aligned}
-(pu')(t) &= c\int_0^t \tfrac{a(w(s)-w(0))+b}{ac(w(1)-w(0))+ad+bc} f(s,u(s))\, ds\\
&\qquad -a\int_t^1 \tfrac{c(w(1)-w(s))+d}{ac(w(1)-w(0))+ad+bc} f(s,u(s))\, ds.\\
&= \tfrac{c}{c(w(1)-w(t))+d}\int_0^t G(t,s) f(s,u(s))\, ds\\
&\qquad -\tfrac{a}{a(w(t)-w(0))+b}\int_t^1 G(t,s) f(s,u(s))\, ds.
\end{aligned}
$$

Hence $pu' \in \mathcal{C}(]0,1[)$. If $b > 0$ we have further $pu' \in \mathcal{C}([0,1[)$. Indeed, given u and using (H_4), we can write for $t \le 1/2$

$$\left| \tfrac{c}{c(w(1)-w(t))+d} \int_0^t G(t,s) f(s,u(s))\, ds \right| \le \tfrac{c}{cw(1)+d} \int_0^t G(s,s) h(s)\, ds,$$

which proves that this term goes to zero with t, so that

$$\lim_{t\to 0} (pu')(t) = \tfrac{a}{b} \int_0^1 G(0,s) f(s,u(s))\, ds = \tfrac{a}{b} u(0).$$

In a similar way, we prove that if $d > 0$,

$$pu' \in \mathcal{C}(]0,1]) \text{ and } \lim_{t\to 1}(pu')(t) = -\frac{c}{d}u(1).$$

Let us prove now the continuity of the operator T. We note that if $u_n \to u$ in $\mathcal{C}([0,1])$, then $|f(\cdot,u_n) - f(\cdot,u)| \to 0$ almost everywhere in $[0,1]$ and $|f(s,u_n(s)) - f(s,u(s))| \le h(s)$ for a suitable h with $G(s,s)h(s) \in L^1(0,1)$. Using

$$\|Tu_n - Tu\|_\infty \le \int_0^1 G(s,s)|f(s,u_n(s)) - f(s,u(s))|\, ds$$

and the Lebesgue dominated convergence Theorem, it follows that $Tu_n \to Tu$ in $\mathcal{C}([0,1])$.

Next, we prove the complete continuity of T. Let $\mathcal{B}$ be a bounded set in $\mathcal{C}([0,1])$. From (H_4), there exists a measurable function h such that $G(s,s)h(s) \in L^1(0,1)$ and $|f(t,u(t))| \le h(t)$ for a.e. $t \in [0,1]$ and all $u \in \mathcal{B}$. Given $\varepsilon > 0$, we can find $\delta_1 > 0$ such that $|(Tu)(t)| = |(Tu)(t) - (Tu)(1)| < \varepsilon/2$ for all $t \in [1-2\delta_1, 1]$ and all $u \in \mathcal{B}$. Similarly, we can find $\delta_0 > 0$ such that $|(Tu)(t) - (Tu)(0)| < \varepsilon/2$ for all $t \in [0, 2\delta_0]$ and all $u \in \mathcal{B}$. Then, we can fix a constant M such that $|(Tu)'(t)| \le M$, for all $t \in [\delta_0, 1-\delta_1]$ and all $u \in \mathcal{B}$. Hence, there is $\delta < \min\{\delta_0, \delta_1, \varepsilon/M\}$ such that, if $|t_1 - t_2| < \delta$, then $|(Tu)(t_1) - (Tu)(t_2)| < \varepsilon$, for all $u \in \mathcal{B}$. We can now apply Arzelà-Ascoli Theorem and conclude the proof. ■

Remark For the Dirichlet problem, $b = d = 0$, the condition $G(t,t)h(t) \in L^1(0,1)$ corresponds to $(w(t) - w(0))(w(1) - w(t))h(t) \in L^1(0,1)$. If $p = 1$ this reduces to $t(1-t)h(t) \in L^1(0,1)$.

For the mixed problem, $a = d = 0$, the condition $G(t,t)h(t) \in L^1(0,1)$ is equivalent to $(w(1) - w(t))h(t) \in L^1(0,1)$. If $p = 1$ this reduces to $(1-t)h(t) \in L^1(0,1)$.

Remark In case $a = 0$, the condition

$$\lim_{t\to 0} \frac{1}{p(t)} \int_0^t h(s)\, ds = 0$$

implies $u'(0) = 0$. Notice also that using Tonelli's and Fubini's Theorems, we can show that this condition also implies $G(t,t)h(t) \in L^1(0,1)$.

In the sequel, we shall use the following definitions of lower and upper solutions which allow these functions to have corners. A function $\alpha \in \mathcal{C}([0,1])$ is said to be a *lower solution* of (6) if:
(a) for any $t_0 \in]0,1[$, either $D_-\alpha(t_0) < D^+\alpha(t_0)$,
or there exist an open interval $I_0 \subset]0,1[$ such that $t_0 \in I_0$, $\alpha \in W^{1,1}(I_0)$, $p\alpha' \in W^{1,1}(I_0)$ and for a.e. $t \in I_0$

$$-(p(t)\alpha'(t))' \le f(t,\alpha(t)); \tag{9}$$

(b) if $a > 0$, we have $a\alpha(0) - b(pD^+\alpha)(0) \le 0$,
if $a = 0$, either $(pD^+\alpha)(0) > 0$, or $(pD^+\alpha)(0) = 0$ and there exists an $\varepsilon > 0$ such that $\alpha \in W^{1,1}_{loc}(0,\varepsilon)$, $p\alpha' \in W^{1,1}(0,\varepsilon)$ and (9) holds for a.e. $t \in [0,\varepsilon]$;
(c) if $c > 0$, we have $c\alpha(1) + d(pD^+\alpha)(1) \le 0$,
if $c = 0$, either $(pD_-\alpha)(1) < 0$, or $(pD_-\alpha)(1) = 0$ and there exists an $\varepsilon > 0$ such that $\alpha \in W^{1,1}_{loc}(1-\varepsilon,1)$, $p\alpha' \in W^{1,1}(1-\varepsilon,1)$ and (9) holds for a.e. $t \in [1-\varepsilon,1]$.

A function $\beta \in \mathcal{C}([0,1])$ is an *upper solution* of (6) if it satisfies the above definition with reversed inequalities and with the Dini derivatives $D^+\alpha$ and $D_-\alpha$ changed into $D_+\beta$ and $D^-\beta$.

A basic existence result is as follows.

Theorem 2.2. *Suppose assumptions* (H_1) *and* (H_2) *are satisfied. Let* α *and* $\beta \geq \alpha$ *be lower and upper solutions of* (6) *and define* $E = \{(t,u) \in [0,1] \times \mathbb{R} \mid \alpha(t) \leq u \leq \beta(t)\}$. *Assume*
(H_3') $f : E \to \mathbb{R}, (t,u) \mapsto f(t,u)$ *is a Carathéodory function, i.e. for all* $u \in \mathbb{R}$, *the function* $f(\cdot,u)$ *with domain* $\{t \in [0,1] \mid (t,u) \in E\}$ *is measurable and for a.e.* $t \in [0,1]$, *the function* $f(t,\cdot)$ *with domain* $\{u \in \mathbb{R} \mid (t,u) \in E\}$ *is continuous;*
(H_4') *there exists a measurable function* $h(t)$ *such that*
(i) for a.e. $t \in [0,1]$ *and all* u *with* $(t,u) \in E$, $|f(t,u)| \leq h(t)$;
(ii) $G(t,t)h(t) \in L^1(0,1)$, *with* $G(t,s)$ *defined in* (7).
Then the problem (6) *has at least one solution* u *such that* $\alpha \leq u \leq \beta$.

Remark The basic ideas of this result are essentially due to Scorza Dragoni [24] in 1931. The use of lower and upper solutions with corners goes back to M. Nagumo [20] in 1954. A more recent version which does not consider singularities can be found in De Coster and Habets [10, Theorem 1.4]. The singular problem at $t = 0$ has been considered in Njoku, Omari and Zanolin [21], see also [9].

Proof. Consider the modified problem

$$\begin{aligned} -(p(t)u')' &= f(t,\gamma(t,u)), \\ au(0) - b(pu')(0) &= 0, \\ cu(1) + d(pu')(1) &= 0, \end{aligned} \tag{10}$$

where $\gamma(t,u) = \max\{\alpha(t), \min\{u, \beta(t)\}\}$.

Claim 1 – The problem (10) *has at least one solution.* We can write (10) as the fixed point problem

$$u(t) = \bar{T}u,$$

where $\bar{T}u = T(\gamma(\cdot,u))$ and apply Schauder's Theorem.

Claim 2 – The solution u *of* (10) *is such that* $\alpha \leq u \leq \beta$. Notice first that $k = \alpha - u$ cannot be a positive constant. Indeed, in such a case and if $a > 0$, we deduce from the boundary conditions and the definition of lower solution

$$0 = a(\alpha(0) - k) - b(p\alpha')(0) \leq -ak < 0.$$

If $a = 0$, we have $c > 0$ and a similar contradiction follows from the boundary condition at $t = 1$.

Let us assume next that, for some $t_0 \in]0,1[$

$$\max_t(\alpha(t) - u(t)) = \alpha(t_0) - u(t_0) > 0$$

and for $t > t_0$, near enough t_0, $\alpha(t_0) - u(t_0) > \alpha(t) - u(t)$. It follows then that

$$D_-\alpha(t_0) - u'(t_0) \geq D^+\alpha(t_0) - u'(t_0)$$

and, by definition of a lower solution, there exists an open interval I_0, with $t_0 \in I_0, \alpha \in W^{2,1}(I_0)$ and for a.e. $t \in I_0$

$$-(p(t)\alpha'(t))' \leq f(t, \alpha(t)).$$

Further $\alpha'(t_0) - u'(t_0) = 0$ so that for $t \geq t_0$, near enough t_0,

$$p(t)(\alpha'(t) - u'(t)) \geq -\int_{t_0}^{t} [f(s, \alpha(s)) - f(s, \gamma(s, u(s)))]\, ds = 0.$$

This contradicts the definition of t_0.

A similar argument holds if $t_0 = 0$ and $a = 0$. If $t_0 = 0$ and $a \neq 0$, the contradiction follows from $0 < a[\alpha(0) - u(0)] - b[(pD^+\alpha)(0) - (pu')(0)] \leq 0$.

The same contradictions hold if $t_0 \in]0, 1]$ is such that for $t < t_0$, near enough t_0, $\alpha(t_0) - u(t_0) > \alpha(t) - u(t)$.

At last, we prove in a similar way that $u \leq \beta$.

Conclusion – As a consequence of Claim 2, the solution u of (10) solves (6) and is such that $\alpha \leq u \leq \beta$. ■

3 The case $g(0) > 0$

Consider the boundary value problem

$$\begin{aligned} (p(t)u')' + \lambda q(t)g(u) &= 0, \\ au(0) - b(pu')(0) &= 0, \\ cu(1) + d(pu')(1) &= 0. \end{aligned} \tag{11}$$

Theorem 3.1. *Assume* (H_1) *and* (H_2) *are satisfied. Suppose* $g : \mathbb{R} \to \mathbb{R}$ *is a continuous function such that* $g(0) > 0$ *and* $q : [0, 1] \to \mathbb{R}$ *is a measurable function so that* $G(t, t)|q(t)| \in L^1(0, 1)$, *where the Green function* $G(t, s)$ *is defined in* (7). *Assume further that for some* $\eta > 0$ *small enough and all* $t \in [0, 1]$

$$\int_0^1 G(t, s)q_+(s)\, ds \geq (1 + \eta) \int_0^1 G(t, s)q_-(s)\, ds, \tag{12}$$

where $q_+(t) = \max\{q(t), 0\}$ *and* $q_-(t) = \max\{-q(t), 0\}$.

Then for $\lambda > 0$ *small enough, problem* (11) *has at least one positive solution* u_λ *such that*

$$\lim_{\lambda \to 0} \|u_\lambda\|_\infty = 0.$$

Proof. Step 1 – Construction of a lower solution of (11) *for small values of* λ. Let

$$v_0(t) = \int_0^1 G(t, s)|q(s)|\, ds \quad \text{and} \quad u_0(t) = \int_0^1 G(t, s)q(s)\, ds. \tag{13}$$

Define $\varepsilon > 0$ from $1 + \eta = \frac{1+\varepsilon}{1-\varepsilon}$ and

$$\alpha(t) := \lambda g(0)(u_0(t) - \varepsilon v_0(t)).$$

Notice that

$$u_0(t) - \varepsilon v_0(t) = (1-\varepsilon)[\int_0^1 G(t,s)q_+(s)\,ds - (1+\eta)\int_0^1 G(t,s)q_-(s)\,ds] > 0.$$

Next, we choose $\lambda > 0$ small enough, i.e. $g(\alpha(t)) - g(0)$ small enough, so that for a.e. $t \in]0,1[$

$$(p(t)\alpha'(t))' + \lambda q(t)g(\alpha(t)) = \lambda[(g(\alpha(t)) - g(0))q(t) + \varepsilon g(0)|q(t)|\,] > 0. \qquad (14)$$

At last, we notice that $a\alpha(0) - b(p\alpha')(0) = 0$ and $c\alpha(1) + d(p\alpha')(1) = 0$. Further, if $a = 0$, we have $\alpha \in W^{1,1}_{loc}(0,1)$, $p\alpha \in W^{1,1}(0,\varepsilon)$ for some $\varepsilon > 0$, and (14) holds for a.e. $t \in [0,\varepsilon]$. A similar conclusion holds if $c = 0$. Hence, α is a lower solution of (11).

Step 2 – Construction of an upper solution of (11). The function $\beta(t) = \lambda^r v_0(t) > 0$, with $r \in]0,1[$, is an upper solution for small values of λ since in this case

$$(p(t)\beta'(t))' + \lambda q(t)g(\beta(t)) \le \lambda^r(-|q(t)| + \lambda^{1-r}q(t)g(\beta(t))) < 0.$$

Conclusion – Notice at last that, for small values of λ,

$$\alpha(t) \le \lambda g(0)u_0(t) \le \lambda g(0)v_0(t) \le \lambda^r v_0(t) = \beta(t).$$

The proof follows now from Theorem 2.2. ■

We can particularize this theorem to a non-singular Dirichlet problem where $p(t) \equiv 1$.

Corollary 3.2. *Assume $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(0) > 0$ and $q \in \mathcal{C}([0,1])$ satisfies*

(i) $\int_0^1 G(t,s)q(s)\,ds > 0$ *for* $t \in]0,1[$,
where $G(t,s) = s(1-t)$ *if* $s \in [0,t[$ *and* $G(t,s) = t(1-s)$ *if* $s \in [t,1[$;

(ii) $\int_0^1 sq(s)\,ds > 0$;

(iii) $\int_0^1 (1-s)q(s)\,ds > 0$.

Then, for $\lambda > 0$ small enough, the problem

$$\begin{gathered} u'' + \lambda q(t)g(u) = 0, \\ u(0) = 0, \ u(1) = 0, \end{gathered}$$

has at least one positive solution u_λ such that

$$\lim_{\lambda \to 0} \|u_\lambda\|_\infty = 0.$$

Proof. Define u_0 and v_0 from (13) and notice that (i), (ii) and (iii) imply

$$u_0(t) > 0 \text{ for } t \in]0,1[, \quad u_0'(0) > 0 \quad \text{and} \quad u_0'(1) < 0.$$

Further we compute

$$|v_0(t)| \leq \int_0^1 G(s,s)|q(s)|\,ds < +\infty,$$
$$|v_0'(0)| = \int_0^1 s|q(s)|\,ds < +\infty,$$
$$|v_0'(1)| = \int_0^1 (1-s)|q(s)|\,ds < +\infty.$$

It follows that for some $\varepsilon > 0$, we have $u_0 - \varepsilon v_0 \geq 0$, which implies (12) for $\eta = \frac{2\varepsilon}{1-\varepsilon}$. ∎

An other case of interest concerns radial solutions of the Laplacian. Here we have to consider a mixed problem, $a = d = 0$, together with $p(t) = t^n$ and $q(t) = t^n a(t)$.

Corollary 3.3. *Let $n \in \mathbb{N}$. Assume $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(0) > 0$ and assume $a \in \mathcal{C}([0,1])$ satisfies*

(i) $\int_t^1 \frac{1}{s^n}(\int_0^s r^n a(r)\,dr)\,ds > 0$ *for* $t \in [0,1[$;

(ii) $\int_0^1 s^n a(s)\,ds > 0$.

Then for $\lambda > 0$ small enough, the problem

$$(t^n u')' + \lambda t^n a(t) g(u) = 0,$$
$$u'(0) = 0, \; u(1) = 0,$$

has at least one positive solution u_λ such that

$$\lim_{\lambda \to 0} \|u_\lambda\|_\infty = 0.$$

Proof. We proceed as in the previous proof noticing that

$$u_0(t) = \int_t^1 \frac{1}{s^n}(\int_0^s r^n a(r)\,dr)\,ds > 0 \text{ on } [0,1[,$$
$$u_0'(1) = -\int_0^1 s^n a(s)\,ds < 0,$$
$$v_0(t) = \int_t^1 \frac{1}{s^n}(\int_0^s r^n |a(r)|\,dr)\,ds < +\infty,$$

and

$$|v_0'(1)| = \int_0^1 s^n |a(s)|\,ds < +\infty.$$

Hence, $u_0 - \varepsilon v_0 \geq 0$ for some small $\varepsilon > 0$. ∎

Remark This corollary generalizes a similar result in [1].

Remark The hypothesis (12) of Theorem 3.1 is almost optimal. If there exists a family of positive solutions $u_\lambda(t)$ of (11) that tends to zero with λ, we have

$$\frac{u_\lambda(t)}{\lambda} = \int_0^1 G(t,s) q(s) g(u_\lambda(s))\,ds \geq 0.$$

and going to the limit as λ goes to zero, we obtain

$$g(0) u_0(t) = \int_0^1 G(t,s)[q_+(s) - q_-(s)] g(0)\,ds \geq 0.$$

i.e. (12) is satisfied with $\eta = 0$.

4 The case $g(0) = 0$

Theorem 4.1. *Assume (H_1) and (H_2) are satisfied and let $q : [0,1] \to \mathbb{R}$ be a measurable function so that $G(t,t)|q(t)| \in L^1(0,1)$, where the Green function $G(t,s)$ is defined in* (7). *Suppose $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous bounded function such that $g(0) = 0$. Assume $\lambda > 0$ and there exists α_0, positive on $]0,1[$, that verifies*

$$\begin{aligned}(p(t)\alpha_0')' + \lambda q(t)g'(0)\alpha_0 = |q(t)|,\\ a\alpha_0(0) - b(p\alpha_0')(0) = 0,\\ c\alpha_0(1) + d(p\alpha_0')(1) = 0.\end{aligned} \tag{15}$$

Then there exists a positive solution of problem (11).

Proof. Step 1 – Construction of a lower solution of (11). Take $\alpha = A\alpha_0$, with $A > 0$, and compute

$$\begin{aligned}(p(t)\alpha')' + \lambda q(t)g(\alpha) = &\\ = A[(p(t)\alpha_0')' + \lambda q(t)(\tfrac{g(\alpha)}{\alpha} - g'(0))\alpha_0 + \lambda q(t)g'(0)\alpha_0]&\\ = A[|q(t)| + \lambda q(t)(\tfrac{g(\alpha)}{\alpha} - g'(0))\alpha_0] \geq 0&\end{aligned}$$

if α, i.e. A, is small enough.

Step 2 – Construction of an upper solution of (11). Define β_0 to be a solution of

$$\begin{aligned}(p(t)\beta_0')' + |q(t)|\,\|g\|_\infty = 0,\\ a\beta_0(0) - b(p\beta_0')(0) = 0,\\ c\beta_0(1) + d(p\beta_0')(1) = 0,\end{aligned}$$

and consider the function $\beta = \lambda\beta_0 + 1$. We compute then

$$(p(t)\beta')' + \lambda q(t)g(\beta) \leq \lambda[(p(t)\beta_0')' + |q(t)|\|g\|_\infty] = 0.$$

Conclusion. If A is small enough, we have $\alpha \leq \beta$ and the theorem follows from Theorem 2.2. ∎

Remark Notice that the existence of a positive solution of (15) can follow from a local antimaximum property (see [8] or [15]).

Remark In case $g : \mathbb{R}^+ \to \mathbb{R}$ has a positive root $u_0 > 0$, we can choose this root as an upper solution $\beta = u_0$ and delete the boundedness assumption on g. For example, one can consider this way the logistic nonlinearity $g(u) = u(1-u)$ as in [16].

5 Some remarks on the singular case

In this section, we shall restrict our attention to the boundary value problem

$$\begin{aligned}u'' + q(t)g(u) = 0,\\ u(0) = 0,\ u(1) = 0,\end{aligned} \tag{16}$$

where g can be unbounded as u goes to zero. For simplicity we assume here that $q(t)$ is continuous on $]0,1[$.

Our first result shows that positive solutions do not exist if $q(t)$ is negative in some neighbourhood of any of the end points of the interval $[0,1]$.

Proposition 5.1. *Let $g :]0,+\infty[\to]0,+\infty[$ and $q :]0,1[\to \mathbb{R}$ be continuous functions and define $G(x) = \int_1^x g(s)\,ds$. Assume*

$$\lim_{x\to 0^+} G(x) = -\infty \tag{17}$$

and suppose that there are $\varepsilon > 0$ and $\delta > 0$ such that $q(t) \le -\delta$ for all $t \in]0,\varepsilon]$ (or for all $t \in [1-\varepsilon, 1[$). Then, there is no positive solution of (16).

Proof. Assume that $-q(t) \ge \delta$ for all $t \in]0,\varepsilon]$ and consider a positive solution $u(t)$ of (16) for $t \in]0,\varepsilon]$. From $u''(t) = -q(t)g(u(t)) > 0$, we have that u is strictly convex on $]0,\varepsilon]$ and then, using the fact that $u(0) = 0$ and $u(t) > 0$ for $t \in]0,\varepsilon]$, we see that $u'(t) > 0$ on $]0,\varepsilon]$. Multiplying the equation in (16) by $u'(t)$ we have that $u''(t)u'(t) \ge \delta g(u(t))u'(t)$ and therefore, the map $t \mapsto \frac{1}{2}u'(t)^2 - \delta G(u(t))$ is nondecreasing on $]0,\varepsilon]$. Hence, we have

$$\frac{1}{2}u'(t)^2 - \delta G(u(t)) \le \frac{1}{2}u'(\varepsilon)^2 - \delta G(u(\varepsilon)) =: K_\varepsilon$$

for all $t \in]0,\varepsilon]$. Using (17), a contradiction follows, by letting $t \to 0^+$.

The case in which $-q(t) \ge \delta$ on $[1-\varepsilon, 1[$ is treated in the same way. ∎

This proposition is somewhat sharp since positive solutions can exist if $q(t) < 0$ on a set $]0,\varepsilon] \cup [1-\varepsilon, 1[$. Consider for example the problem

$$u'' + \frac{q(t)}{u} = 0, \quad u(0) = 0,\ u(1) = 0, \tag{18}$$

where $q(t) = -2(1 - 6t + 6t^2)t^2(1-t)^2$. This problem has the positive solution $u(t) = t^2(1-t)^2$.

Our second result shows that for positive solutions to exist the function q must have a positive mean value.

Proposition 5.2. *Let $g :]0,+\infty[\to]0,+\infty[$ be a continuously differentiable function with $g'(x) < 0$ and let $q \in \mathcal{C}(]0,1[) \cap L^1(0,1)$ be such that*

$$\int_0^1 q(s)\,ds \le 0. \tag{19}$$

Then, there is no positive solution of (16).

Proof. Let u be a solution of (16). Observe that there is a compact interval $J \subset]0,1[$ such that $\int_J u'(t)^2\,dt > 0$. We can choose two sequences $s_n \to 0$ and $t_n \to 1$ such that $u'(s_n) \ge 0$ and $u'(t_n) \le 0$. Without loss of generality, we can assume that $J \subset]s_n, t_n[$. We compute then

$$\begin{aligned}\int_{s_n}^{t_n} q(t)\,dt &= -\int_{s_n}^{t_n} \frac{u''(t)}{g(u(t))}\,dt \\ &= -\left.\frac{u'(t)}{g(u(t))}\right|_{s_n}^{t_n} - \int_{s_n}^{t_n} \frac{g'(u(t))}{g^2(u(t))}u'^2(t)\,dt \ge \eta > 0.\end{aligned}$$

Taking the limit as $n \to \infty$, we have that $\int_0^1 q(s)\,ds > 0$ which contradicts (19). ∎

The condition (19) is in a way best possible since the example (18) with $q(t) = -2\varepsilon^2(1 - 6t + 6t^2)t^2(1-t)^2$ has the positive solution $u(t) = \varepsilon t^2(1-t)^2$ and is such that $\displaystyle\int_0^1 q(s)\,ds$ is as small as we wish.

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