Newton Polyhedra and the Poles of Igusa's Local Zeta Function

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Abstract

We give a very explicit formula for Igusa's local zeta function $Z_f(s,\chi)$ associated to a polynomial f in several variables over the p-adic numbers and to a character χ of the units of the p-adic integers (with conductor 1). This formula holds when f is sufficiently non-degenerated with respect to its Newton polyhedron $\Gamma(f)$. Using this formula, we give a set of possible poles of $Z_f(s,\chi)$, together with upper bounds for their orders. Moreover this formula implies that $Z_f(s) = Z_f(s,\chi_{\text{triv}})$ has always at least one real pole.

1 Introduction

For p prime, denote the field of p-adic numbers by $\mathbb{Q}_p$, the ring of p-adic integers by $\mathbb{Z}_p$, and the finite field of p elements by $\mathbb{F}_p$. If R is a commutative ring with identity, we will denote the set of its units by $R^\times$.

Definition 1.1. *Let $f(x) = f(x_1,\dots,x_n) \in \mathbb{Z}_p[x_1,\dots,x_n]$ with p prime. For $z \in \mathbb{Q}_p$, $\operatorname{ord} z \in \mathbb{Z} \cup \{\infty\}$ denotes the valuation, $|z| = p^{-\operatorname{ord} z}$ and $\operatorname{ac}(z) = p^{-\operatorname{ord} z} z$ denotes the angular component. Let $\chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ be a character of $\mathbb{Z}_p^\times$, i.e., a group homomorphism with finite image. We formally put $\chi(0) = 0$. To the above data we associate the following two* Igusa local zeta functions *(the global and the local one):*

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$$Z_f(s,\chi) = \int_{\mathbb{Z}_p^n} \chi(\mathrm{ac}\, f(x))|f(x)|^s|dx|,$$
$$\text{and } Z_{f,0}(s,\chi) = \int_{(p\mathbb{Z}_p)^n} \chi(\mathrm{ac}\, f(x))|f(x)|^s|dx|,$$

for $s \in \mathbb{C}$, $\mathrm{Re}\,(s) > 0$, where $|dx|$ denotes the Haar measure on $\mathbb{Q}_p^n$ so normalized that $\mathbb{Z}_p^n$ has measure 1. If χ is the trivial character χ_{triv}, then we write $Z_f(s)$ (resp. $Z_{f,0}(s)$) instead of $Z_f(s,\chi_{\mathrm{triv}})$ (resp. $Z_{f,0}(s,\chi_{\mathrm{triv}})$).

Igusa's local zeta function $Z_f(s)$ is directly related to the numbers of solutions of the congruences $f(x) \equiv 0 \bmod p^m$, $m = 1,2,3,\ldots$; see, e.g., [Igu78, pp. 97–98], [Den91, Section 1.2]. Using resolutions of singularities, Igusa [Igu74] proved that $Z_f(s,\chi)$ is a rational function of p^{-s} (see also [Igu78]). An entirely different proof was obtained ten years later by Denef [Den84] using p-adic cell decomposition instead of resolutions of singularities. We denote the meromorphic continuation of $Z_f(s,\chi)$ again by $Z_f(s,\chi)$.

We study the poles of Igusa's local zeta function for a special but important class of polynomials, namely the polynomials that are non-degenerated with respect to their Newton polyhedron. This study was first started by Lichtin ans Meuser [LM85] for polynomials in two variables.

Definition 1.2. *Let $f(x) = f(x_1,\ldots,x_n) = \sum_{\omega\in\mathbb{N}^n} a_\omega x_1^{\omega_1}\cdots x_n^{\omega_n}$ be a non-zero polynomial over $\mathbb{Z}_p$ with $f(0) = 0$. Let $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathrm{supp}\,(f) = \{\omega \in \mathbb{N}^n \mid a_\omega \neq 0\}$ the support of f. The* Newton polyhedron $\Gamma(f)$ *of f is defined as the convex hull in $(\mathbb{R}^+)^n$ of the set*

$$\bigcup_{\omega\in\mathrm{supp}\,(f)} \omega + (\mathbb{R}^+)^n.$$

The global Newton polyhedron $\Gamma_{\mathrm{gl}}(f)$ *of f is defined as the convex hull of* $\mathrm{supp}\,(f)$*. It is easy to verify that $\Gamma(f) = \Gamma_{\mathrm{gl}}(f) + (\mathbb{R}^+)^n$.*

Because a Newton polyhedron is a polyhedron, every proper face is an exposed face.[1] So, every proper face of $\Gamma(f)$ is the intersection of $\Gamma(f)$ with a supporting hyperplane [Roc70, pp. 99–100]. **By the faces of $\Gamma(\mathbf{f})$ we mean the proper faces of $\Gamma(\mathbf{f})$ and the Newton polyhedron $\Gamma(\mathbf{f})$ itself.**

Definition 1.3. *Let p be a prime number. Let f be as in Definition 1.2. For each face τ of the Newton polyhedron $\Gamma(f)$ of f, we define*

$$f_\tau(x) = \sum_{\omega\in\tau} a_\omega x^\omega,$$

and the polynomial $\bar{f}_\tau(x)$ with coefficients in $\mathbb{F}_p$ by reducing each coefficient a_ω of f_τ modulo $p\mathbb{Z}_p$.

[1] Face and exposed face in the sense of [Roc70, p. 162].

We say f is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron $\Gamma(f)$ *if for every*[2] *face τ of* $\Gamma(f)$ *the locus of the polynomial $\bar{f}_\tau$ has no singularities in $(\mathbb{F}_p^\times)^n$ or equivalently the set of congruences*

$$\begin{cases} f_\tau(x) & \equiv & 0 \bmod p \\ \frac{\partial f_\tau}{\partial x_i}(x) & \equiv & 0 \bmod p, \quad i = 1, \ldots, n \end{cases}$$

has no solution in $(\mathbb{Z}_p^\times)^n$.

We say f is non-degenerated over $\mathbb{F}_p$ with respect to the compact faces of its Newton polyhedron*, if we have the same condition, but only for the compact faces of* $\Gamma(f)$*.*

See [DH01] for some remarks about the definition of non-degenerated. In the same paper we can find an explicit formula for $Z_f(s)$ that holds if f is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron. The actors in this formula are the cones Δ_τ associated to the faces τ of $\Gamma(f)$ (see Definition 2.3) and the numbers of elements N_τ of the sets $\{x \in (\mathbb{F}_p^\times)^n \mid \bar{f}_\tau(x) = 0\}$. In Theorem 3.4 below we extend this result to $Z_f(s, \chi)$, where χ is a character of $\mathbb{Z}_p$ with conductor $c_\chi = 1$. The conductor c_χ is the smallest $c \in \mathbb{N} \setminus \{0\}$ such that χ is trivial on $1 + p^c\mathbb{Z}_p$. So, $c_{\chi_{\text{triv}}}$ is also equal to 1. It si well known (see [Den91, Theorem 3.3]) that $Z_f(s, \chi)$ is constant as a function of s for $p \gg 0$, if the conductor $c_\chi > 1$ and f is a polynomial over $\mathbb{Z}$. For the exact condition of a similar result for f a polynomial aver $\mathbb{Z}_p$ and p a fixed prime number, we refer to the same reference. So, we have an explicit formula for $Z_f(s, \chi)$ for the relevant cases. Zúñiga-Galindo [Zn99] obtained the same formula, as a special case of a more general result. For the related work of other authors we refer to [DH01] and the references therein.

Using the formula from Theorem 3.4 we obtain a set of candidate poles for $Z_f(s, \chi)$ together with their expected orders, i.e., upperbounds for the orders if these candidate poles are actual poles (see Proposition 4.1, Definition 4.2, Proposition 4.6 and Proposition 4.9). This formula also enables us to prove Theorem 4.10, which states that $Z_f(s)$ has always at least one real pole. In particular, this theorem gives more information on the largest real pole of $Z_f(s)$: it is either -1 or $-1/t_0$, where $(t_0, t_0, \ldots, t_0)$ is the intersection point of the diagonal $D = \{(t, t, \ldots, t) \in \mathbb{R}^n\}$ with $\Gamma(f)$.

2 More about Newton polyhedra

In Definition 2.3, we will give a partition of $(\mathbb{R}^+)^n$ in sets that are closely related to the Newton polyhedron of a polynomial f as in Definition 1.2. In order to define these sets Δ_τ and to know more about them, we will present a selection of some well-known definitions and properties. Although all results in this section are well-known, we provided some of them with a proof to make this material more easily accessible.

Definition 2.1. *Let f be as in Definition 1.2. For $a \in (\mathbb{R}^+)^n$, we define*

$$m(a) = \inf_{x \in \Gamma(f)} \{a \cdot x\}$$

[2] Thus also for $\Gamma(f)$.

and we define the first meet locus of a *as*

$$F(a) = \{x \in \Gamma(f) \mid a \cdot x = m(a)\}.$$

where $a \cdot x$ denotes the scalar product of a and x.

Note that the infimum in the definition of $m(a)$ is attained and so $m(a) = \min_{x\in\Gamma(f)} \{a \cdot x\}$. This is clear, because we can take the infimum over the elements in the closed and bounded set $\Gamma_{\text{gl}}(f)$ instead of $\Gamma(f) = \Gamma_{\text{gl}}(f) + (\mathbb{R}^+)^n$. Moreover, we can take the minimum over the elements in $\operatorname{supp}(f)$ instead of $\Gamma(f)$ or $\Gamma_{\text{gl}}(f)$, because $\Gamma_{\text{gl}}(f)$ is the convex hull of the (finite) set $\operatorname{supp}(f)$.

Property 2.2. *Let f be as in Definition 1.2 and $a \in (\mathbb{R}^+)^n$. Then $F(a)$, the first meet locus of a, is a face of $\Gamma(f)$. In particular $F(0) = \Gamma(f)$ and $F(a)$ is a proper face of $\Gamma(f)$, if $a \neq 0$. Moreover, $F(a)$ is a compact face if and only if $a \in (\mathbb{R}^+ \setminus \{0\})^n$.*

Definition 2.3. *We define an equivalence relation on $(\mathbb{R}^+)^n$ by*

$$a \sim a' \qquad \text{if and only if} \qquad F(a) = F(a').$$

If τ is a face of $\Gamma(f)$, we define the cone associated to τ *as*

$$\Delta_\tau = \{a \in (\mathbb{R}^+)^n \mid F(a) = \tau\}.$$

Note that $\Delta_{\Gamma(f)} = \{0\}$. We will now study the other equivalence classes Δ_τ in order to give an interesting description of them in Proposition 2.8.

Lemma 2.4. *Let f be as in Definition 1.2 and τ be a proper face of $\Gamma(f)$. Then*

(i) Δ_τ is a relatively open subset of $(\mathbb{R}^+)^n$.

(ii) $\bar{\Delta}_\tau = \{a \in (\mathbb{R}^+)^n \mid F(a) \supset \tau\}$ and is a polyhedral cone.

(iii) The function m is linear on $\bar{\Delta}_\tau$.

Proof. Suppose that $\Gamma(f)$ is the convex hull of the points $P_1, \ldots, P_s$ and the directions of recession $e_1, \ldots, e_n$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. By [Roc70, Theorem 18.3], we may suppose that τ is the convex hull of the points $\{P_1, \ldots, P_r\} = \tau \cap \{P_1, \ldots, P_r, P_{r+1}, \ldots, P_s\}$ and the directions of recession $e_1, \ldots, e_k$ with $r \leq s$ and $k \leq n$. Now, one easily proves that $\Delta_\tau = \{a = (a_1, \ldots, a_n) \in \mathbb{R}^n \mid a \cdot P_1 = a \cdot P_2 = \cdots = a \cdot P_r, a \cdot P_1 < a \cdot P_{r+1}, \ldots, a \cdot P_1 < a \cdot P_s,\ a_i = 0 \text{ for } i = 1, \ldots, k,\ a_j > 0 \text{ for } j = k+1, \ldots, n\}$, which is clearly a relatively open set. Now, it follows that $\bar{\Delta}_\tau = \{a \in \mathbb{R}^n \mid a \cdot P_1 = a \cdot P_2 = \cdots = a \cdot P_r, a \cdot P_1 \leq a \cdot P_{r+1}, \ldots, a \cdot P_1 \leq a \cdot P_s,\ a_i = 0 \text{ for } i = 1, \ldots, k,\ a_j \geq 0 \text{ for } j = k+1, \ldots, n\} = \{a \in (\mathbb{R}^+)^n \mid F(a) \supset \tau\}$. Finally, (ii) implies that $m(a) = a \cdot x_\tau$ for every $a \in \bar{\Delta}_\tau$, where x_τ is a fixed element of τ. This shows that m is linear on $\bar{\Delta}_\tau$. ∎

Definition 2.5. *A* fan $\mathcal{F}$ *is a finite set of rational polyhedral cones such that*

1. *every (non-empty) face of a cone of* $\mathcal{F}$ *is contained in* $\mathcal{F}$ *and*
2. *the intersection of two arbitrary cones* C_i *and* C_j *in* $\mathcal{F}$ *is a face of both* C_i *and* C_j.

Lemma 2.6. *Let* f *be as in Definition 1.2. The closures* $\bar{\Delta}_\tau$ *of the cones associated to the faces of* $\Gamma(f)$ *form a fan in* $(\mathbb{R}^+)^n$. *Moreover, we have the following.*

(i) Let τ *be a proper face of* $\Gamma(f)$. *Then the order-reversing map*

$$\{\textit{faces of } \Gamma(f) \textit{ that contain } \tau\} \to \{\textit{(non-empty) faces of } \bar{\Delta}_\tau\} : \sigma \mapsto \bar{\Delta}_\sigma$$

is one to one and onto.

(ii) Let τ_1, τ_2 *be faces of* $\Gamma(f)$. *Suppose that* τ_1 *is a facet*[3] *of* τ_2. *Then* $\bar{\Delta}_{\tau_2}$ *is a facet of* $\bar{\Delta}_{\tau_1}$.

Proof. We first prove (i) and that the cones $\bar{\Delta}_\tau$ form a fan. Fix a face $\sigma \neq \tau$ of $\Gamma(f)$ that contains the face τ. Suppose that $\Gamma(f)$ is the convex hull of the points

$$P_1, \ldots, P_k, P_{k+1}, \ldots, P_{k+t}, \ldots, P_l \tag{1}$$

and the directions of recession

$$e_1, \ldots, e_d, e_{d+1}, \ldots, e_{d+s}, \ldots, e_n, \tag{2}$$

with $P_1, \ldots, P_k$ (resp. $P_1, \ldots, P_{k+t}$) the only points from (1) that belong to τ (resp. σ) and with $e_1, \ldots, e_d$ (resp. $e_1, \ldots, e_{d+s}$) the only vectors from (2) that are directions of recession of τ (resp. σ). Now one proves that $\bar{\Delta}_\sigma$ is a face of $\bar{\Delta}_\tau$: the equation of the supporting hyperplane is

$$x \cdot ((P_{k+1} - P_1) + \cdots + (P_{k+t} - P_1) + e_{d+1} + \cdots + e_{d+s}) = 0.$$

It easily follows from Lemma 2.4(ii) that the mentioned map in the assertion is one to one and order-reversing. We now prove that it is also onto. Fix an arbitrary face γ of $\bar{\Delta}_\tau$ and an x in the relative interior of γ. Put $\sigma = F(x)$. Then σ is a face of $\Gamma(f)$ that contains τ, because $x \in \gamma \subset \bar{\Delta}_\tau = \{a \in (\mathbb{R}^+)^n \mid F(a) \supset \tau\}$. Now, the faces γ and $\bar{\Delta}_\sigma$ of $\bar{\Delta}_\tau$ must be equal, because their relative interiors have a common element, x (see [Roc70, Corollary 18.1.2]).

Finally, suppose that τ_1, τ_2 are faces of $\Gamma(f)$. Then $\bar{\Delta}_{\tau_1} \cap \bar{\Delta}_{\tau_2} = \bar{\Delta}_{\tau_3}$ with τ_3 the smallest face of $\Gamma(f)$ that contains $\tau_1 \cup \tau_2$, i.e., the intersection of all the faces of $\Gamma(f)$ that contain $\tau_1 \cup \tau_2$.

Assertion (ii) follows from (i). The cone $\bar{\Delta}_{\tau_2}$ is clearly a (proper) face of $\bar{\Delta}_{\tau_1}$. So, if it is not a facet, there exists a facet F of $\bar{\Delta}_{\tau_1}$ such that $\bar{\Delta}_{\tau_2} \subsetneq F \subsetneq \bar{\Delta}_{\tau_1}$. Then there exists a face γ of $\Gamma(f)$ that contains τ_1 and satisfies $F = \bar{\Delta}_\gamma$. But then $\tau_2 \supsetneq \gamma \supsetneq \tau_1$, which is not possible. ∎

[3] Recall that a facet is an $(n-1)$-dimensional face.

Definition 2.7. *A vector $a \in \mathbb{R}^n$ is called* primitive *if the components of a are integers whose greatest common divisor is* 1.

Recall that $\Delta_{\Gamma(f)} = \{0\}$. The following proposition gives the geometry of the other equivalence classes Δ_τ. Note that because a Newton polyhedron is a polyhedron, we can prove that every proper face τ of $\Gamma(f)$ is contained in a facet of $\Gamma(f)$. Moreover, every proper face τ of $\Gamma(f)$ is the (finite) intersection of the facets of $\Gamma(f)$ that contain τ. One can also prove that for every facet of $\Gamma(f)$ there exists a unique primitive vector in $\mathbb{N}^n \setminus \{0\}$ that is perpendicular to that facet.

Proposition 2.8. *Let f be as in Definition 1.2. Let τ be a proper face $\Gamma(f)$ and $\gamma_1, \ldots, \gamma_r$ be the facets of $\Gamma(f)$ that contain τ. Let $a_1, \ldots, a_r$ be the unique primitive vectors in $\mathbb{N}^n \setminus \{0\}$ that are perpendicular to respectively $\gamma_1, \ldots, \gamma_r$. Then the cones $\bar{\Delta}_\tau$ and Δ_τ are the following convex cones:*

$$\bar{\Delta}_\tau = \left\{ \sum_{i=1}^{r} \lambda_i a_i \mid \lambda_i \in \mathbb{R}, \lambda_i \geq 0 \right\} \quad \textit{and} \quad \Delta_\tau = \left\{ \sum_{i=1}^{r} \lambda_i a_i \mid \lambda_i \in \mathbb{R}, \lambda_i > 0 \right\}.$$

Moreover, $\dim \Delta_\tau = \dim \bar{\Delta}_\tau = \operatorname{codim} \tau = n - \dim \tau$.

Proof. The fact that $\dim \bar{\Delta}_\tau = \operatorname{codim} \tau = n - \dim \tau$ follows from Lemma 2.6. Indeed, suppose that $\dim \tau = k$. Then there exist faces $\tau = \tau^k, \tau^{k+1}, \ldots, \tau^n = \Gamma(f)$ of $\Gamma(f)$ with $\dim \tau^i = i$, such that

$$\tau = \tau^k \subsetneq \tau^{k+1} \subsetneq \cdots \subsetneq \tau^n = \Gamma(f).$$

Therefore,

$$\bar{\Delta}_\tau = \bar{\Delta}_{\tau^k} \supsetneq \bar{\Delta}_{\tau^{k+1}} \supsetneq \cdots \supsetneq \bar{\Delta}_{\Gamma(f)} = \{0\}.$$

So, by Lemma 2.6(ii), we know that $\dim \tau + \dim \bar{\Delta}_\tau = \dim \tau^{k+1} + \dim \bar{\Delta}_{\tau^{k+1}} = \cdots = \dim \Gamma(f) + \dim\{0\} = n$. The assertion for $\bar{\Delta}_\tau$ follows now from Lemma 2.9, Lemma 2.6 and the fact that $\bar{\Delta}_{\gamma_i} = \mathbb{R}^+ a_i$ for $i = 1, \ldots, r$. The assertion for Δ_τ follows from the assertion for $\bar{\Delta}_\tau$, by Lemma 2.4(i), [Roc70, Theorem 6.2] and [Roc70, Theorem 6.9]. ∎

Lemma 2.9. *Let C be a polyhedral cone in $\mathbb{R}^n$ such that the origin is a face of C. Then the rays generated by a minimal set of generators are exactly the one-dimensional faces of C.*

Proof. [Ful93, Sect. 1.2]. ∎

Definition 2.10. *If $a_1, \ldots, a_r \in \mathbb{R}^n \setminus \{0\}$, we call* $\operatorname{cone}(a_1, \ldots, a_r) = \{\sum_{i=1}^r \lambda_i a_i \mid \lambda_i \in \mathbb{R}, \lambda_i > 0\}$ *the* cone strictly positively spanned by the vectors $a_1, \ldots, a_r$. *Fix a cone Δ. If there exist linearly independent vectors $a_1, \ldots, a_r \in \mathbb{R}^n \setminus \{0\}$ such that $\Delta = \operatorname{cone}(a_1, \ldots, a_r)$, then Δ is called a* simplicial cone. *If moreover one can choose $a_1, \ldots, a_r \in \mathbb{Z}^n$ (or in $\mathbb{Q}^n$), we say Δ is a* rational simplicial cone.

Let τ be a proper face of $\Gamma(f)$ and let $a_1, \ldots, a_r$ be as in Proposition 2.8. Following from Proposition 2.8 and Lemma 2.11, one can partition the cone Δ_τ associated to the face τ into a finite number of rational simplicial cones such that each Δ_i is spanned by vectors from the set $\{a_1, \ldots, a_r\}$. We call such a decomposition a *rational simplicial decomposition of Δ_τ without introducing new rays.*

Lemma 2.11. *Let Δ be the cone strictly positively spanned by the vectors $a_1, \dots, a_r \in (\mathbb{R}^+)^n \setminus \{0\}$. Then there exists a finite partition of Δ into cones Δ_i, such that each Δ_i is strictly positively spanned by some vectors from the set $\{a_1, \dots, a_r\}$ that are linearly independent over $\mathbb{R}$.*

Proof. [Den95, Lemma 2]. We cite the proof here, because the algorithm is used in the proof of Proposition 4.9. Let $\bar{\Delta}$ be the closure of Δ. Thus $\bar{\Delta}$ is a closed convex polyhedral cone. If γ is a closed convex cone, we denote by $\operatorname{ri}\gamma$ the relative interior of γ (i.e., the interior in the linear space generated by γ). The proof is by induction on $\dim(\mathbb{R}a_1 + \cdots + \mathbb{R}a_r)$. We may suppose that $\mathbb{R}^+ a_1$ is a face of $\bar{\Delta}$. It is easy to see that Δ is the disjoint union of cones of the form

$$W = \{\lambda_1 a_1 + b \mid \lambda_1 \in \mathbb{R}, \lambda_1 > 0, b \in \operatorname{ri}\gamma\},$$

where γ is a proper face of $\bar{\Delta}$ such that W is not contained in a facet of $\bar{\Delta}$. By induction, we have that $\operatorname{ri}\gamma$ is the disjoint union of cones w_i with the required property. But then, W is the disjoint union of cones

$$W_i = \{\lambda_1 a_1 + b \mid \lambda_1 \in \mathbb{R}, \lambda_1 > 0, b \in w_i\}.$$

■

Definition 2.12. *Let $a_1, \dots, a_r$ be vectors in $\mathbb{Z}^n$ that are linearly independent over $\mathbb{Q}$. We define the* multiplicity of $a_1, \dots, a_r$ *as the index of the lattice $\mathbb{Z}a_1 + \cdots + \mathbb{Z}a_r$ in the group of the points with integral coordinates of the vector space generated by $a_1, \dots, a_r$.*

Remark: It is easy to verify that the multiplicity of $a_1, \dots, a_r$ equals the number of elements in the set

$$\mathbb{Z}^n \cap \left\{ \sum_{i=1}^{r} \lambda_i a_i \mid 0 \leq \lambda_i < 1 \text{ for } i = 1, \dots, r \right\}.$$

Proposition 2.13. *Let $a_1, \dots, a_r$ be vectors in $\mathbb{Z}^n$ that are linearly independent over $\mathbb{Q}$.*

(i) The multiplicity of $a_1, \dots, a_r$ equals the greatest common divisor of the determinants of all $r \times r$-matrices obtained by omitting columns from the matrix with rows $a_1, \dots, a_r$.

(ii) The multiplicity of $a_1, \dots, a_r$ equals the volume of the parallelepiped spanned by $a_1, \dots, a_r$ with respect to ω_A, where ω_A is the volume form on the vector space $\operatorname{vct} A$ generated by $A = \{a_1, \dots, a_r\}$, normalized such that the parallelepiped spanned by a lattice basis of $\mathbb{Z}^n \cap \operatorname{vct} A$ has volume 1.

3 An explicit formula for $Z_f(s,\chi)$

In Theorem 3.4, we will give a formula for $Z_f(s,\chi)$ that holds if f is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron. In [DH01], we already obtained the result for $\chi = \chi_{\text{triv}}$ and we extend it now to the case where the conductor c_χ of χ is equal to 1. The conductor c_χ of the character χ of $\mathbb{Z}_p^\times$ is defined as the smallest $c \in \mathbb{N}\setminus\{0\}$ such that χ is trivial on $1+p^c\mathbb{Z}_p$. So, $c_{\chi_{\text{triv}}}$ is also equal to 1. Lemma 3.1 and Proposition 3.2 are special cases of more general results [Den91, Theorem 3.4], but for convenience of the reader we will give a direct proof.

Lemma 3.1. *Let p be a prime number and $f(x) = f(x_1,\dots,x_n) \in \mathbb{Z}_p[x_1,\dots,x_n]$. Let $\chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ be a non-trivial character of $\mathbb{Z}_p^\times$. Suppose that $a \in \mathbb{Z}_p^n$ such that $f(a) \equiv 0 \bmod p$ and such that the set of congruences $(\partial f/\partial x_i)(x) \equiv 0 \bmod p$, $i = 1,\dots,n$ has no solution in $a + (p\mathbb{Z}_p)^n$. Then*

$$\int\limits_{a+(p\mathbb{Z}_p)^n} \chi(\operatorname{ac} f(x))|f(x)|^s|dx| = 0.$$

Proof. From the condition in the statement of Lemma 3.1, it follows that there exists an $i \in \{1,\dots,n\}$ such that $(\partial f/\partial x_i)(a) \not\equiv 0 \bmod p$. Put

$$\begin{array}{llll}\psi: & \mathbb{Z}_p^n & \to & \mathbb{Z}_p^n \\ & (x_1,\dots,x_n) & \mapsto & (v_1,\dots,v_n),\end{array}$$

where

$$v_j = \begin{cases} p^{-1}(f(a+px)-f(a)) & \text{for } j = i \\ x_j & \text{for } j \neq i. \end{cases}$$

Then ψ is a measure-preserving bi-analytic map of $\mathbb{Z}_p^n$ to itself. Therefore, we get

$$\begin{aligned}\int\limits_{a+(p\mathbb{Z}_p)^n} \chi(\operatorname{ac} f(x))|f(x)|^s|dx| &= p^{-n}\int\limits_{\mathbb{Z}_p^n} \chi(\operatorname{ac} f(a+px))|f(a+px)|^s|dx| \\ &= p^{-n}\int\limits_{\mathbb{Z}_p^n} \chi(\operatorname{ac}(f(a)+pv_i))|f(a)+pv_i|^s|dv| \\ &= p^{-n}\int\limits_{\mathbb{Z}_p} \chi(\operatorname{ac}(f(a)+pv_i))|f(a)+pv_i|^s|dv_i| \\ &= p^{-n-s}\int\limits_{\mathbb{Z}_p} \chi(\operatorname{ac}(v_i))|v_i|^s|dv_i|,\end{aligned}$$

because the translation over $f(a)/p \in \mathbb{Z}_p$ is a measure-preserving bi-analytic map of $\mathbb{Z}_p$ to itself. The last integral is equal to zero. Indeed, because χ is a non-trivial character, there exists an $u \in \mathbb{Z}_p^\times$, such that $\chi(u) \neq 1$. Hence

$$\begin{aligned}\chi(u)\int\limits_{\mathbb{Z}_p} \chi(\operatorname{ac}(v_i))|v_i|^s|dv_i| &= \int\limits_{\mathbb{Z}_p} \chi(\operatorname{ac}(uv_i))|uv_i|^s|dv_i| \\ &= \int\limits_{\mathbb{Z}_p} \chi(\operatorname{ac}(y_i))|y_i|^s|dy_i|.\end{aligned}$$

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Proposition 3.2. *Let p be a prime number and $f(x) = f(x_1,\dots,x_n) \in \mathbb{Z}_p[x_1,\dots,x_n]$. Let $\chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ be a non-trivial character of $\mathbb{Z}_p^\times$ and let c_χ be the conductor of χ. Suppose that the set of congruences*

$$\begin{cases} f(x) \equiv 0 \bmod p \\ \frac{\partial f}{\partial x_i}(x) \equiv 0 \bmod p, \quad i = 1,\dots,n \end{cases}$$

has no solution in $(\mathbb{Z}_p^\times)^n$*. Then*

$$\int\limits_{(\mathbb{Z}_p^\times)^n} \chi(\operatorname{ac} f(x))|f(x)|^s|dx| = p^{-nc_\chi} \sum_{\substack{a \in (\mathbb{Z}_p^\times)^n \\ a \bmod p^{c_\chi} \\ f(a) \not\equiv 0 \bmod p}} \chi(f(a)).$$

Proof. It is clear that

$$\begin{aligned}\int\limits_{(\mathbb{Z}_p^\times)^n} \chi(\operatorname{ac} f(x))|f(x)|^s|dx| &= \sum_{\substack{a \in (\mathbb{Z}_p^\times)^n \\ a \bmod p^{c_\chi} \\ f(a) \not\equiv 0 \bmod p}} \int\limits_{a+(p^{c_\chi}\mathbb{Z}_p)^n} \chi(\operatorname{ac} f(x))|f(x)|^s|dx| \\ &+ \sum_{\substack{a \in (\mathbb{Z}_p^\times)^n \\ a \bmod p \\ f(a) \equiv 0 \bmod p}} \int\limits_{a+(p\mathbb{Z}_p)^n} \chi(\operatorname{ac} f(x))|f(x)|^s|dx|.\end{aligned}$$

By Lemma 3.1, we know that the integral $\int_{a+(p\mathbb{Z}_p)^n} \chi(\operatorname{ac} f(x))|f(x)|^s|dx|$ is zero for every $a \in \mathbb{Z}_p^\times$ with $f(a) \equiv 0 \bmod p$, which implies the assertion. ■

Definition 3.3. *For* $k = (k_1, \dots, k_n) \in \mathbb{R}^n$*, we define*

$$\sigma(k) = \sum_{i=1}^n k_i.$$

Theorem 3.4. *Let* p *be a prime number. Let* f *be as in Definition 1.2. Suppose that* f *is non-degenerated over the finite field* $\mathbb{F}_p$ *with respect to all the faces of its Newton polyhedron* $\Gamma(f)$*. Let* χ *be a character of* $\mathbb{Z}_p^\times$ *with conductor* $c_\chi = 1$*. Denote for each face* τ *of* $\Gamma(f)$ *by* N_τ *the number of elements in the set*

$$\{a \in (\mathbb{F}_p^\times)^n \mid \bar{f}_\tau(a) = 0\}.$$

Let s *be a complex variable with* $\operatorname{Re}(s) > 0$*. Then*

$$Z_f(s,\chi) = \sum_{\substack{\tau \text{ face} \\ \text{of } \Gamma(f)}} L_\tau S_{\Delta_\tau},$$

with

$$L_\tau = \begin{cases} p^{-n}\left((p-1)^n - pN_\tau \frac{p^s-1}{p^{s+1}-1}\right) & \text{for } \chi = \chi_{\text{triv}}, \\ p^{-n} \sum\limits_{a \in (\mathbb{F}_p^\times)^n} \chi\left(f_\tau(a)\right) & \text{for } \chi \neq \chi_{\text{triv}}, \end{cases}$$

and

$$S_{\Delta_\tau} = \sum_{k \in \mathbb{N}^n \cap \Delta_\tau} p^{-\sigma(k) - m(k)s},$$

for each face τ *of* $\Gamma(f)$ *(including* $\tau = \Gamma(f)$*).*

It is clear that $S_{\Delta_{\Gamma(f)}} = 1$*. The other* S_{Δ_τ} *can be calculated as follows. Take a partition of the cone* Δ_τ *associated to the proper face* τ *into rational simplicial cones* Δ_i*. Then clearly* $S_{\Delta_\tau} = \sum_i S_{\Delta_i}$*, where the summation is over the rational simplicial cones* Δ_i *and*

$$S_{\Delta_i} = \sum_{k \in \mathbb{N}^n \cap \Delta_i} p^{-\sigma(k) - m(k)s}.$$

Let Δ_i be the cone strictly positively spanned by the linearly independent vectors $a_1, \ldots, a_r \in \mathbb{N}^n \setminus \{0\}$. Then

$$S_{\Delta_i} = \frac{\sum_h p^{\sigma(h)+m(h)s}}{(p^{\sigma(a_1)+m(a_1)s}-1)\cdots(p^{\sigma(a_r)+m(a_r)s}-1)},$$

where h runs through the elements of the set

$$\mathbb{Z}^n \cap \left\{ \sum_{j=1}^{r} \lambda_j a_j \mid 0 \leq \lambda_j < 1 \text{ for } j = 1, \ldots, r \right\}.$$

(We recall that $m(\cdot)$ and $\sigma(\cdot)$ are defined as in Definition 2.1 and Definition 3.3).

Remarks:

1. To obtain the formula for S_{Δ_τ} we only need f to be a non-zero polynomial in n variables over an arbitrary commutative ring with $f(0) = 0$, $p > 1$ a real number and $\mathrm{Re}\,(s) > 0$. Remark that S_{Δ_τ} is a rational function in p^{-s}. We also denote its meromorphic continuation by S_{Δ_τ}.

2. By a similar argument as in the proof below, we can prove that

$$Z_{f,0}(s,\chi) = \sum_{\substack{\tau \text{ compact face} \\ \text{of } \Gamma(f)}} L_\tau S_{\Delta_\tau},$$

   with L_τ and S_{Δ_τ} as in the statement of the theorem above. For this formula, we only need f to be non-degenerated over the finite field $\mathbb{F}_p$ with respect to the *compact* faces of $\Gamma(f)$.

Proof. For the case of $\chi = \chi_{\text{triv}}$, we refer to [DH01, Theorem 4.2]. The proof for the case of $\chi \neq \chi_{\text{triv}}$ is completely similar by using Proposition 3.2 instead of [DH01, Corollary 3.2]. Remark that Proposition 3.2 implies that the integral $\int_{(\mathbb{Z}_p^\times)^n} \chi(\mathrm{ac}\,(f_\tau(u) + p\tilde{f}_{\tau,k}(u))|f_\tau(u) + p\tilde{f}_{\tau,k}(u)|^s|du|$ is independent of $\tilde{f}_{\tau,k}$ for $c_\chi = 1$, where $\tilde{f}_{\tau,k}(u)$ is the polynomial as defined in the proof of [DH01, Theorem 4.2]. ∎

4 The candidate poles of $Z_f(s,\chi)$ and their expected orders

Proposition 4.1. *Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Theorem 3.4. Let $\gamma_1, \ldots, \gamma_r$ be all the facets of $\Gamma(f)$ and let $a_1, \ldots, a_r$ be the unique primitive vectors in $\mathbb{N}^n \setminus \{0\}$ that are perpendicular to respectively $\gamma_1, \ldots, \gamma_r$. Then the following holds:*

(i) If s_1 is a pole of $Z_f(s)$, then

$$\begin{aligned} s_1 &= -1 + i\frac{k2\pi}{\log_e p} \quad \text{with } k \in \mathbb{Z}, \text{ or} \\ s_1 &= -\frac{\sigma(a_j)}{m(a_j)} + i\frac{k2\pi}{m(a_j)\log_e p}, \end{aligned}$$

with $k \in \mathbb{Z}$ and $j \in \{1, \ldots, r\}$ such that $m(a_j) \neq 0$.

(ii) If s_1 is a pole of $Z_f(s)$, then $\mathrm{Re}\,(s_1)$ is -1 or $\mathrm{Re}\,(s_1)$ is of the form $-1/t_1$, where $(t_1, t_1, \ldots, t_1)$ is the intersection point of the diagonal $D = \{(t, t, \ldots, t) \in \mathbb{R}^n\}$ with the supporting hyperplane of a facet of $\Gamma(f)$.

(iii) Suppose that s_1 is a pole of $Z_f(s, \chi)$ with $\chi \neq \chi_{\mathrm{triv}}$, then

$$s_1 = -\frac{\sigma(a_j)}{m(a_j)} + i\frac{k2\pi}{m(a_j)\log_e p},$$

with $k \in \mathbb{Z}$ and $j \in \{1, \ldots, r\}$ such that $m(a_j) \neq 0$. Moreover, $\mathrm{Re}\,(s_1)$ is of the form $-1/t_1$, where $(t_1, t_1, \ldots, t_1)$ is the intersection point of the diagonal $D = \{(t, t, \ldots, t) \in \mathbb{R}^n\}$ with the supporting hyperplane of a facet of $\Gamma(f)$.

Remarks:

1. The same is true for $Z_{f,0}(s)$ and $Z_{f,0}(s, \chi)$. Of course, then we only need f to be non-degenerated over $\mathbb{F}_p$ with respect to all the *compact* faces of $\Gamma(f)$ (see Remark 2 after Theorem 3.4).
2. We call the complex numbers in Proposition 4.1(i) and (iii) the *candidate poles* of $Z_f(s)$ and $Z_f(s, \chi)$ respectively. The first candidate poles in (i) (with real part equal to -1) come from the L_τ, the second ones in (i) and the ones in (iii) come from the S_{Δ_τ}. In the next section, we will give more information about the largest real candidate pole coming from the S_{Δ_τ}.

Proof. This result can be derived from the material in Section 2 and Section 3, see [DH01, Proposition 5.1]. It is also a special case of [Zn99, Theorem A]. The case of $Z_f(s)$ is a direct consequence of [Den95]. ■

Looking at the formula from Theorem 3.4, we can give upperbounds for the orders of the poles of $Z_f(s, \chi)$.

Definition 4.2. *Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Theorem 3.4. Suppose that s_1 is a candidate pole of $Z_f(s, \chi)$, i.e., a number from the set described in Proposition 4.1. We define* the expected order of the candidate pole s_1 *for $Z_f(s, \chi)$ (according to the formula for $Z_f(s, \chi)$ in Theorem 3.4) as*

$$\mathrm{Max}\,\{\textit{order of the pole } s_1 \textit{ for } L_\tau S_{\Delta_\tau} \mid \tau \textit{ a face of } \Gamma(f)\},$$

where the order of the pole s_1 is defined as 0, if it is no pole, and where L_τ and S_{Δ_τ} are defined as in Theorem 3.4.

Remarks:

1. If $\mathrm{Re}(s_1) \neq -1$ or $\chi \neq \chi_{\mathrm{triv}}$, we omit L_τ in $L_\tau S_{\Delta_\tau}$.
2. It is clear that the actual order of the pole s_1 for $Z_f(s, \chi)$ will not be larger than the expected order of s_1.
3. We can give a similar definition for the expected order of a candidate pole s_1 for $Z_{f,0}(s, \chi)$. Then τ runs through the *compact* faces of $\Gamma(f)$.

4.1 The largest real candidate pole coming from the S_{Δ_τ}

Definition 4.3. *Let $f(x_1,\ldots,x_n)$ be a non-zero polynomial with $f(0)=0$. We denote the unique intersection point of the diagonal $D=\{(t,t,\ldots,t)\in\mathbb{R}^n\}$ with the boundary of the Newton polyhedron $\Gamma(f)$ of f by $(t_0,t_0,\ldots,t_0)$. Denote the smallest face of $\Gamma(f)$ that contains $(t_0,t_0,\ldots,t_0)$ by τ_0 and its codimension in $\mathbb{R}^n$ by κ. Put $\rho=-1/t_0$.*

Note that $\kappa\geq 1$, because $\dim\tau_0\leq n-1$. Moreover, $\rho\in\mathbb{Q}$.

We search for more information about the largest real candidate pole of $Z_f(s)$ and $Z_f(s,\chi)$ coming from the S_{Δ_τ}. The next lemma implies that this is ρ and Proposition 4.6 shows that the expected order of ρ for $Z_f(s)$ and $Z_f(s,\chi)$ is either κ or $\kappa+1$.

Lemma 4.4. *Let $f(x)=f(x_1,\ldots,x_n)$ be a non-zero polynomial with $f(0)=0$. For every $a\in(\mathbb{R}^+)^n$ it holds that $\sigma(a)-m(a)(1/t_0)\geq 0$ with equality if and only if $\tau_0\subset F(a)$ (or equivalently $a\in\bar{\Delta}_{\tau_0}$).*

Proof. See [DH01, Lemma 5.3]. ∎

Lemma 4.5. *Let $f(x_1,\ldots,x_n)$ be a non-zero polynomial with $f(0)=0$. Suppose that τ is a proper face of $\Gamma(f)$ and let S_{Δ_τ} be as in Theorem 3.4. Then we have the following:*

(i) $\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau}\geq 0$.

(ii) If $\tau\not\subset\tau_0$, then, if ρ is a pole of S_{Δ_τ}, its order is $<\kappa$. Otherwise, ρ is a pole of S_{Δ_τ} of order κ.

Proof. See [DH01, Lemma 5.4]. ∎

Proposition 4.6. *Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Theorem 3.4. Then for every pole s_1 of $Z_f(s,\chi)$ one has* $\mathrm{Re}\,(s_1)\leq\rho$ *if $\chi\neq\chi_{\mathrm{triv}}$; otherwise,* $\mathrm{Re}\,(s_1)=-1$ *or* $\mathrm{Re}\,(s_1)\leq\rho$. *Moreover, the expected order of the candidate pole ρ for $Z_f(s,\chi)$ equals $\kappa+1$ if $\chi=\chi_{\mathrm{triv}}$, $\rho=-1$ and there is a face $\tau\subset\tau_0$ such that $N_\tau\neq 0$; otherwise, its expected order will be κ. Remember that the actual order is not larger than the expected order, which is defined in Definition 4.2.*

Proof. See [DH01, Proposition 5.5] or [Zn99]. The case of trivial character is also a direct consequence of the material in [Den95]. ∎

4.2 The expected order of an arbitrary pole of $Z_f(s,\chi)$

Definition 4.7. *Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Theorem 3.4. Suppose that s_1 is a candidate pole of $Z_f(s,\chi)$, i.e., a number from the set described in Proposition 4.1. We say that a vector $a\in(\mathbb{R}^+)^n\setminus\{0\}$ or a facet $F(a)$* contributes to the candidate pole s_1 *if $p^{\sigma(a)+m(a)s_1}=1$, or equivalently,* $\mathrm{Re}\,(s_1)=-\sigma(a)/m(a)$ *and there exists a $k\in\mathbb{Z}$*

such that $\mathrm{Im}\,(s_1) = k2\pi/(m(a)\log_e p)$. *We say a face of* $\Gamma(f)$ *is* a face of pure contribution to the candidate pole s_1 *if every facet that contains* τ *contributes to this candidate pole* s_1. *We define* $\kappa(s_1)$ *as the maximum of the codimensions of such faces. We call such a face of codimension* $\kappa(s_1)$ a face of $\Gamma(f)$ of maximal pure contribution to the candidate pole.

Remarks:

1. It can occur that not all faces of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1 are disjoint. See Section 5.
2. If $s_1 = \rho$, then $\kappa(s_1) = \kappa$ and τ_0 is the only face of $\Gamma(f)$ of maximal pure contribution to the candidate pole ρ of $Z_f(s,\chi)$.
3. It is easy to see that $\kappa(s_1) \leq \kappa(\mathrm{Re}\,(s_1))$.

Lemma 4.8. *Suppose that the prime number* p, *the polynomial* f *and the character* χ *satisfy the conditions of Theorem 3.4. Suppose that* s_1 *is a candidate pole of* $Z_f(s,\chi)$, *i.e., a number from the set described in Proposition 4.1. Suppose that* μ *is a face of* $\Gamma(f)$ *of pure contribution to the candidate pole* s_1 *and suppose that its codimension is* k.

(i) If $a \in \bar{\Delta}_\mu$, *then* a *contributes to the pole* s_1, *i.e.,* $p^{\sigma(a)+m(a)s_1} = 1$.

(ii) There exists a constant $c_\mu > 0$, *independent of* p, *such that*

$$\lim_{s\to s_1}\left(p^{s-s_1}-1\right)^k S_{\Delta_\mu} = c_\mu.$$

Define the half-space $H_1^- = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i \leq 1\}$ *and define* $\omega_{\bar{\Delta}_\mu}$ *as the volume form on the vector space* $\mathrm{vct}\,\bar{\Delta}_\mu$ *generated by* $\bar{\Delta}_\mu$, *normalized such that the parallelepiped spanned by a lattice basis of* $\mathbb{Z}^n \cap \mathrm{vct}\,\bar{\Delta}_\mu$ *has volume* 1. *Then*

$$c_\mu = k!|\mathrm{Re}\,(s_1)|^k \mathrm{Vol}_{\omega_{\bar{\Delta}_\mu}}(\bar{\Delta}_\mu \cap H_1^-),$$

where the volume is taken with respect to $\omega_{\bar{\Delta}_\mu}$.

Proof. Assertion (i) follows by the fact that all generators of $\bar{\Delta}_\mu$ contribute to the pole s_1 and the fact that σ and m are linear on $\bar{\Delta}_\mu$.

We now prove Assertion (ii). Suppose that $\cup_i\Delta_i$ is a finite partition of Δ_μ into rational simplicial cones. Then $S_{\Delta_\mu} = \sum_i S_{\Delta_i}$. From the formula for S_{Δ_i} in Theorem 3.4 and (i), it follows that $\lim_{s\to s_1}(p^{s-s_1}-1)^k S_{\Delta_i} > 0$ if and only if k generators of Δ_i belong to $\bar{\Delta}_\mu$, or equivalently, $\dim\Delta_i = k$; otherwise this limit is zero. Moreover, if Δ_i is a cone from this partition with $\dim\Delta_i = k$, then $\mathrm{vct}\,\bar{\Delta}_i = \mathrm{vct}\,\bar{\Delta}_\mu$ and hence $\omega_{\bar{\Delta}_i} = \omega_{\bar{\Delta}_\mu}$. So, it suffices to prove that for such a cone Δ_i one has

$$\lim_{s\to s_1}(p^{s-s_1}-1)^k S_{\Delta_i} = k!|\mathrm{Re}\,(s_1)|^k \mathrm{Vol}_{\omega_{\bar{\Delta}_i}}(\bar{\Delta}_i \cap H_1^-),$$

where the volume is taken with respect to $\omega_{\bar{\Delta}_i}$. Suppose that Δ_i is the cone strictly positively spanned by $a_1,\dots,a_k$, linearly independent vectors in $\mathbb{Z}^n \cap \bar{\Delta}_\mu$. Then we know from the formula for S_{Δ_i} in Theorem 3.4 that

$$S_{\Delta_i} = \frac{\sum_h p^{\sigma(h)+m(h)s}}{(p^{\sigma(a_1)+m(a_1)s}-1)\cdots(p^{\sigma(a_k)+m(a_k)s}-1)},$$

where h runs through the elements of the set

$$\mathbb{Z}^n \cap \left\{ \sum_{j=1}^{k} \lambda_j a_j \mid 0 \leq \lambda_j < 1 \text{ for } j = 1, \dots, k \right\}. \tag{3}$$

We know that $p^{\sigma(h)+m(h)s_1} = 1$, for every h of the set (3), because every h belongs to $\bar{\Delta}_\mu$. Consequently,

$$\begin{aligned} \lim_{s \to s_1} (p^{s-s_1} - 1)^k S_{\Delta_i} &= \frac{\text{multiplicity of } a_1, \dots, a_k}{m(a_1) \cdots m(a_k)} \\ &= |\mathrm{Re}\,(s_1)|^k \frac{\text{multiplicity of } a_1, \dots, a_k}{\sigma(a_1) \cdots \sigma(a_k)}. \end{aligned}$$

It follows by Proposition 2.13 that this equals $|\mathrm{Re}\,(s_1)|^k$ times the volume of the parallelepiped spanned by $a_1/\sigma(a_1), \dots, a_k/\sigma(a_k)$ with respect to $\omega_{\bar{\Delta}_i}$. Moreover, $\bar{\Delta}_i \cap H_1^-$ is the convex hull of $\{0, a_1/\sigma(a_1), \dots, a_k/\sigma(a_k)\}$. So, $\lim_{s \to s_1} (p^{s-s_1} - 1)^k S_{\Delta_i}$ also equals $k! |\mathrm{Re}\,(s_1)|^k \mathrm{Vol}_{\omega_{\bar{\Delta}_i}}(\bar{\Delta}_i \cap H_1^-)$. ■

Proposition 4.9. *Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Theorem 3.4. Suppose that s_1 is a candidate pole of $Z_f(s, \chi)$, i.e., a number from the set described in Proposition 4.1.*

(i) If τ is a face of $\Gamma(f)$ that is not contained in any face of $\Gamma(f)$ of maximal pure contributions to the candidate pole s_1, then, if s_1 is a pole of S_{Δ_τ}, its order is $< \kappa(s_1)$.

(ii) If $p^{s_1+1} \neq 1$ or $\chi \neq \chi_{\text{triv}}$, then the expected order of the candidate pole s_1 of $Z_f(s, \chi)$ is $\kappa(s_1)$. Otherwise the expected order is either $\kappa(s_1)$ or $\kappa(s_1) + 1$.

(iii) If s_1 is a pole of $Z_f(s, \chi)$ of order k with $p^{s_1+1} \neq 1$ or $\chi \neq \chi_{\text{triv}}$, then it is necessary that there exists a face of codimension k that is a face of $\Gamma(f)$ of pure contribution to the candidate pole s_1. If s_1 is a pole of $Z_f(s)$ of order k with $p^{s_1+1} = 1$, then it is necessary that there exists a face of codimension $(k-1)$ that is a face of $\Gamma(f)$ of pure contribution to the candidate pole s_1.

Remark: Result (iii) also holds for $Z_{f,0}(s, \chi)$. We have to be more careful for an analogue of (ii) for the case of $Z_{f,0}(s, \chi)$. If s_1 is a candidate pole of $Z_{f,0}(s)$ with $p^{s_1+1} \neq 1$ and one of the faces of $\Gamma(f)$ of maximal pure contributions to the candidate pole s_1 is compact, then its expected order is $\kappa(s_1)$; otherwise we can only state that its expected order is $\leq \kappa(s_1)$. (A similar remark holds if $p^{s_1+1} = 1$ and $\chi = \chi_{\text{triv}}$.) Indeed, if for example, s_1 is a real number that is not the largest real candidate pole of an S_{Δ_τ}, it can occur that $\lim_{s \to s_1} (p^{s-s_1} - 1)^{\kappa(s_1)} S_{\Delta_i} > 0$, for some cone Δ_i in a partition of Δ_τ into rational simplicial cones and $\lim_{s \to s_1} (p^{s-s_1} - 1)^{\kappa(s_1)} S_{\Delta_j} < 0$ for another cone Δ_j in this partition. So, in contrast to the case of ρ, it is not certain that s_1 is a pole of S_{Δ_τ} of order $\kappa(s_1)$, if τ is a face contained in one of the faces of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1.

Proof. Because Lemma 4.8(ii) is true for μ a face of maximal pure contribution to the candidate pole s_1, for Assertion (ii), it suffices to verify that, if s_1 is a pole of

S_{Δ_τ} with τ a face of $\Gamma(f)$, then its order is $\leq \kappa(s_1)$. Or it is sufficient to prove that there exists a partition of Δ_τ into rational simplicial cones Δ_i such that at most $\kappa(s_1)$ generators of Δ_i contribute to the candidate pole s_1. To prove Assertion (i), we have to verify that, if τ is not contained in a face of maximal pure contribution, there exists a partition of Δ_τ into rational simplicial cones Δ_i such that $< \kappa(s_1)$ generators of Δ_i contribute to s_1. We prove by induction on $\dim \Delta_\tau$ that we can even do this without introducing new rays. It is clear that the property is true if $\dim \Delta_\tau = 1$. Suppose now that $\dim \Delta_\tau > 1$ and that the property is true for cones of smaller dimension. If all generators of Δ_τ contribute to the pole s_1, then it is easy to see that a partition without introducing new rays is an appropriate partition of Δ_τ. If there exists a generator that does not contribute to the pole s_1, we then use this vector as a_1 in the algorithm described in the proof of Lemma 2.11. Then the property follows by the fact that it is true for cones of smaller dimension.

By the same argument, it is clear that we can find a partition of Δ_τ into rational simplicial cones Δ_i such that the set of generators of Δ_i that contribute to the candidate pole s_1 belong to some cone $\bar{\Delta}_{\mu_i}$, where μ_i is a face of $\Gamma(f)$ of pure contribution to the candidate pole s_1. This implies (iii). ∎

4.3 $Z_f(s)$ has always at least one real pole

The following proposition shows that $Z_f(s)$ has always a real pole, if f satisfies the conditions to use the formula for $Z_f(s)$ from Theorem 3.4. In particular, it gives more information on the largest real pole of $Z_f(s)$ and its order.

Theorem 4.10. *Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Theorem 3.4.*

(i) Suppose that $\rho > -1$.
Then ρ is the largest real pole of $Z_f(s)$ and its order is κ.

(ii) Suppose that $\rho < -1$.
If there exists a face τ of $\Gamma(f)$ such that $N_\tau \neq 0$, then -1 will be the largest real pole of $Z_f(s)$ and its order will be 1. Otherwise, ρ will be the largest real pole of $Z_f(s)$ and its order will be κ.

(iii) Suppose that $\rho = -1$.
Then $\rho = -1$ will be the largest real pole of $Z_f(s)$. If there exists a face $\tau \subset \tau_0$ of $\Gamma(f)$ such that $N_\tau \neq 0$, then its order will be $\kappa + 1$. Otherwise, its order will be κ.

We recall that ρ, κ and τ_0 are defined in Definition 4.3.

Remark: A similar result holds for $Z_{f,0}(s)$: one has to replace "face" everywhere by "compact face".

Proof. Case (i) is well known, see e.g., [DH01, Proposition 5.5].

First note that we can use the formula for $Z_f(s)$ from Theorem 3.4. Thus

$$Z_f(s) = \sum_{\substack{\tau \text{ face} \\ \text{of } \Gamma(f)}} L_\tau S_{\Delta_\tau},$$

with L_τ and S_{Δ_τ} as in this theorem.

It is easy to see that

$$\begin{array}{rcll} L_\tau & = & p^{-n}(p-1)^n > 0, & \text{if } N_\tau = 0, \\ \lim_{s\to -1}(p^{s+1}-1)L_\tau & = & p^{-n}(1-\frac{1}{p})pN_\tau > 0, & \text{if } N_\tau \neq 0. \end{array} \quad (4)$$

We know by Lemma 4.5 that $\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau} > 0$ for faces $\tau \subset \tau_0$. Otherwise, this limit will be equal to 0. By a similar argument as in the proof of Lemma 4.5(i), we can prove that $\lim_{s\to\rho}(p^{s-\rho}-1)^{\kappa-1} S_{\Delta_\tau} \geq 0$ for faces $\tau \not\subset \tau_0$. The second assertion for Case (ii) and the assertion for Case (iii) follows now from (4).

Suppose that $\rho < -1$. Fix a proper face τ of $\Gamma(f)$ and fix a partition of the cone Δ_τ into rational simplicial cones Δ_i. Then

$$\lim_{s\to -1} S_{\Delta_\tau} = \sum_i \lim_{s\to -1} S_{\Delta_i} > 0. \quad (5)$$

Indeed, let Δ_i be the cone strict positively spanned by linearly independent vectors $a_1, \ldots, a_r \in \mathbb{N}^n \setminus \{0\}$. Then we know that

$$S_{\Delta_i} = \frac{\sum_h p^{\sigma(h)+m(h)s}}{(p^{\sigma(a_1)+m(a_1)s}-1)\ldots(p^{\sigma(a_r)+m(a_r)s}-1)},$$

where h runs through the elements of the set

$$\mathbb{Z}^n \cap \left\{ \sum_{j=1}^r \lambda_j a_j \mid 0 \leq \lambda_j < 1 \text{ for } j = 1, \ldots, r \right\}.$$

Hence,

$$\lim_{s\to -1} S_{\Delta_i} = \frac{\sum_h p^{\sigma(h)-m(h)}}{(p^{\sigma(a_1)-m(a_1)}-1)\ldots(p^{\sigma(a_r)-m(a_r)}-1)} > 0,$$

because $\sigma(a_i) - m(a_i) > 0$ by Lemma 4.4 and the fact that $-1 > \rho$. The first assertion for Case (ii) follows now from (4) and (5). ∎

5 Examples

We use Theorem 3.4 to calculate Igusa's local zeta function in the following example. To find the cones associated to the proper faces of the Newton polyhedron, we use Lemma 2.8. More examples can be found in [HL00] and [DH01].

Let $f(x,y,z) = xy + xz^2 + yz^2$. The Newton polyhedron $\Gamma(f)$ of f is defined by the system of linear inequalities

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ z \geq 0 \\ 2x + 2y + z \geq 4 \\ 2y + z \geq 2 \\ 2x + z \geq 2 \\ x + y \geq 1. \end{cases}$$

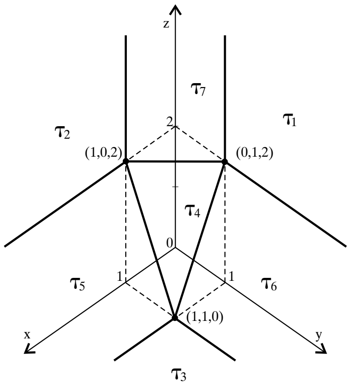

Figure 1: Newton polyhedron of $xy + xz^2 + yz^2$

We denote by $\tau_1, \dots, \tau_7$ the facets with supporting hyperplanes given by the corresponding equalities in the inequalities above. A picture of the Newton polyhedron of f is given in Figure 1.

There are 19 proper faces. One easily verifies that f is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron, for every prime $p \neq 2$. So, we can use the formula for $Z_f(s)$ from Theorem 3.4 for those primes. According to this formula, the candidate real poles of $Z_f(s)$ are -1, $-5/4$, $-3/2$ and -2, all with expected order 1. The set of faces of $\Gamma(f)$ of maximal pure contribution to the candidate poles $-5/4$, $-3/2$, -2 are $\{\tau_4\}$, $\{\tau_5, \tau_6\}$ and $\{\tau_7\}$ respectively (see Definition 4.2, Definition 4.7 and Proposition 4.9). Remark that this example shows that two faces of $\Gamma(f)$ of maximal pure contribution to the same candidate pole don't have to be disjoint. However $-3/2$ and -2 are not actual poles of $Z_f(s)$. Indeed, by using the formula for $Z_f(s)$ from Theorem 3.4 (see [HL00]), we get

$$Z_f(s) = \frac{p^{2s}(p-1)(p^{5+3s} + p^{2+s} - p^{2+2s} - p^{1+s} + p - 1)}{(p^{1+s} - 1)(p^{5+4s} - 1)}.$$

In [Hoo01] we explain why this happens.

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