Self-dual Centroaffine Surfaces of Codimension Two with Constant Affine Mean Curvature*

Hitoshi Furuhata Takashi Kurose

Abstract

We explicitly determine the self-dual centroaffine surfaces of codimension two with constant affine mean curvature and indefinite affine fundamental form by giving representation formulas.

1 Introduction

An immersion f of an n-dimensional manifold M into $\mathbb{R}^{n+2} \setminus \{0\}$ is called a *centroaffine immersion of codimension two* if the position vector $f(x)$ is transversal to $f_* T_x M$ at each point x of M. Properties of such immersions invariant under special linear transformations were first studied by Walter [5], and later by Nomizu and Sasaki [3] in a more general and systematic way, particularly from the viewpoint of its closed connection with projective hypersurface theory.

For a given centroaffine immersions f of codimension two, one of the most fundamental results established by them is: if f is non-degenerate, then f uniquely determines a pseudo-Riemannian metric on M, called the *affine fundamental form* of f; moreover, the affine fundamental form is invariant under the change $f \mapsto Af$ of centroaffine immersions by an element A of $SL(n+2;\mathbb{R})$.

In [2], the first author considered a certain area-variational problem with respect to the affine fundamental form and studied its extremals, which he called *minimal centroaffine immersions*. Furthermore, he showed that the space of the $SL(4;\mathbb{R})$-congruence classes of minimal ISDC immersions $\mathbb{R}^2 \to \mathbb{R}^4 \setminus \{0\}$ is in one-to-one

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correspondence with the space of solutions for a wave equation on $\mathbb{R}^2$. Here, we refer to 'self-dual centroaffine immersions with indefinite affine fundamental form' as 'ISDC immersions'.

In this paper, our main goal is to give a representation formula for the minimal ISDC surfaces (Theorem 3.2), which shows that any minimal ISDC surface is locally represented as the tensor product of two centroaffine curves in $\mathbb{R}^2 \setminus \{0\}$. The Clifford torus is a typical example of such surfaces. In Section 4, we give a representation formula for ISDC surfaces with non-zero constant affine mean curvature (Theorem 4.4), which shows that such a surface is also constructed by two centroaffine curves. We remark that an ISDC surface with non-zero constant affine mean curvature corresponds to a solution of Liouville's equation (4.11).

As an appendix, we give the affine mean curvature formula for a non-parametric centroaffine immersion in Section 5. Section 2 is devoted to basic definitions and facts on centroaffine immersions.

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2 Preliminaries

In this section, we briefly review geometry of centroaffine immersions of a simply-connected, oriented, n-dimensional manifold M into $\mathbb{R}^{n+2} \setminus \{0\}$. For further detail, we refer the reader to [3].

Let $f : M \to \mathbb{R}^{n+2} \setminus \{0\}$ be a centroaffine immersion. We denote by D the standard flat affine connection of $\mathbb{R}^{n+2}$, and by Det the standard parallel volume form on $\mathbb{R}^{n+2}$. A vector field ξ along f is called a *normal vector field* if it satisfies at each point x of M the tangent space $T_{f(x)}\mathbb{R}^{n+2}$ is decomposed as

$$T_{f(x)}\mathbb{R}^{n+2} = f_* T_x M \oplus \mathbb{R}\xi_x \oplus \mathbb{R}f(x), \tag{2.1}$$

and that the volume form θ defined by

$$\theta(X_1, \dots, X_n) := \mathrm{Det}(f_* X_1, \dots, f_* X_n, \xi, f) \tag{2.2}$$

for $X_1, \dots, X_n \in \Gamma(TM)$ is compatible with the orientation of M.

When we choose a normal vector field ξ, we determine a torsion-free affine connection ∇, two symmetric $(0,2)$-tensor fields h and T, a $(1,1)$-tensor field S, and two 1-forms τ and P by

$$\begin{aligned} D_X f_* Y &= f_* \nabla_X Y + h(X,Y)\xi + T(X,Y)f, \\ D_X \xi &= -f_* SX + \tau(X)\xi + P(X)f, \end{aligned} \tag{2.3}$$

according to the decomposition (2.1).

It is easily shown that the conformal class of h does not depend on the choice of ξ. When h is non-degenerate (resp. definite, indefinite), f is said to be *non-degenerate* (resp. *definite*, *indefinite*). If f is non-degenerate, there is a normal vector field ξ satisfying that

$$\begin{aligned} &\tau = 0, \\ &\theta = \mathrm{Vol}_h, \quad \text{where } \mathrm{Vol}_h \text{ is the volume form with respect to } h, \\ &\mathrm{tr}_h\{(X,Y) \mapsto T(X,Y) + h(SX,Y)\} = 0. \end{aligned} \tag{2.4}$$

Moreover, such a ξ is uniquely determined. We call ξ the *Blaschke normal vector field* of f. From now on, we always choose the Blaschke normal vector field as a normal vector field of f. In this case, we call the non-degenerate tensor h the *affine fundamental form* of f.

For an element A of $SL(n+2;\mathbb{R})$, both D and Det are invariant under a transformation $v \mapsto Av$ of $\mathbb{R}^{n+2}$. Hence, Af is also a non-degenerate centroaffine immersion and its Blaschke normal vector field is $A\xi$; moreover, f and Af induce the same set of the geometric quantities ∇, h, T, S and P. Conversely, if two non-degenerate centroaffine immersions f_1, f_2 induce completely the same quantities, f_1 and f_2 are *congruent*, that is, there exists an element A of $SL(n+2;\mathbb{R})$ such that $f_2 = Af_1$.

For later use, we recall the equations of Gauss, of Codazzi, and of Ricci for a centroaffine immersion $f : M \to \mathbb{R}^{n+2} \setminus \{0\}$: let ∇, h, T, S and P be the objects determined by (2.3). Then they satisfy

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY - T(Y,Z)X + T(X,Z)Y, \tag{2.5}$$
$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z), \tag{2.6}$$
$$(\nabla_X T)(Y,Z) + h(Y,Z)P(X) = (\nabla_Y T)(X,Z) + h(X,Z)P(Y), \tag{2.7}$$
$$(\nabla_X S)Y + P(X)Y = (\nabla_Y S)X + P(Y)X, \tag{2.8}$$
$$h(X,SY) = h(Y,SX), \tag{2.9}$$
$$T(X,SY) - T(Y,SX) = dP(X,Y), \tag{2.10}$$
$$\mathrm{tr}_h\{(Y,Z) \mapsto (\nabla_X h)(Y,Z)\} = 0, \tag{2.11}$$
$$\mathrm{tr}_h\{(X,Y) \mapsto T(X,Y) + h(SX,Y)\} = 0, \tag{2.12}$$

where R denotes the curvature tensor field of the induced connection ∇.

Conversely, if a torsion-free affine connection ∇ and tensor fields h, T, S, P are given on M, and if they satisfy the relations (2.5)–(2.12), then we can construct a non-degenerate centroaffine immersion f of M into $\mathbb{R}^{n+2}\setminus\{0\}$ with Blaschke normal vector field ξ such that decomposition (2.3) of $D_X f_* Y$ and $D_X\xi$ holds.

Let $\mathbb{R}_{n+2}$ denote the dual space of $\mathbb{R}^{n+2}$ endowed with the volume form induced by Det. For a given centroaffine immersion $f : M \to \mathbb{R}^{n+2} \setminus \{0\}$, we define the *dual map* $f^* : M \to \mathbb{R}_{n+2} \setminus \{0\}$ by

$$f^*(x)(f_*X) = 0, \quad f^*(x)(\xi(x)) = 1, \quad \text{and} \quad f^*(x)(f(x)) = 0, \tag{2.13}$$

for each $x \in M$ and $X \in T_xM$.

Definition 2.1. A centroaffine immersion $f : M \to \mathbb{R}^{n+2}\setminus\{0\}$ is said to be *self-dual* if there exists a volume-preserving linear map $L : \mathbb{R}^{n+2} \to \mathbb{R}_{n+2}$ such that $f^* = Lf$.

Fact 2.2 ([3]). *For a centroaffine immersion $f : M \to \mathbb{R}^{n+2} \setminus \{0\}$, the following three conditions are mutually equivalent:*

(1) f is self-dual;

(2) the image of f is contained in a non-degenerate quadratic cone;

(3) $\nabla h = 0$.

Definition 2.3. For a non-degenerate centroaffine immersion $f : M \to \mathbb{R}^{n+2} \setminus \{0\}$, the *affine mean curvature* H is defined to be $(1/n)\,\mathrm{tr}\, S$. A non-degenerate centroaffine immersion f is said to be *minimal* if the affine mean curvature H vanishes everywhere.

We abbreviate the phrase 'self-dual centroaffine immersion with indefinite affine fundamental form' to 'ISDC immersion'.

Example 2.4 ([2]). The Clifford torus $f_C : \mathbb{R}^2 \to \mathbb{R}^4 \setminus \{0\}$ and a quadric $f_{Q_1} : \mathbb{R}^2 \to \mathbb{R}^4 \setminus \{0\}$ defined below are minimal ISDC immersions.

$$f_C(u^1,u^2) = \begin{bmatrix} \cos u^1 \cos u^2 \\ \cos u^1 \sin u^2 \\ \sin u^1 \cos u^2 \\ \sin u^1 \sin u^2 \end{bmatrix}, \quad f_{Q_1}(u^1,u^2) = \begin{bmatrix} 1 \\ u^2 \\ u^1 \\ u^1u^2 \end{bmatrix}, \quad (u^1,u^2) \in \mathbb{R}^2.$$

They have the same induced connection and affine fundamental form

$$\nabla \partial_i = 0 \quad \text{and} \quad h = 2du^1du^2. \tag{2.14}$$

Example 2.5. The immersion $f_{Q_2} : \mathbb{R}^2 \to \mathbb{R}^4 \setminus \{0\}$ defined by

$$f_{Q_2}(u^1,u^2) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 + (u^1)^2 + (u^2)^2 \\ 1 - (u^1)^2 - (u^2)^2 \\ 2u^1 \\ 2u^2 \end{bmatrix}, \quad (u^1,u^2) \in \mathbb{R}^2,$$

is a minimal self-dual centroaffine surface with flat induced affine connection and definite affine fundamental form $h = 2(du^1du^1 + du^2du^2)$.

Remark 2.6. Nomizu and Sasaki [3] proved that the image of a centroaffine immersion lies in an affine hyperplane which does not contain the origin if T determined by (2.3) vanishes identically. In this case, a minimal centroaffine surface is reduced to a minimal affine surface (or, one may say, an affine maximal surface) in $\mathbb{R}^3$.

3 Representation Formula for Minimal Self-Dual Centroaffine Surfaces

Throughout Sections 3 and 4, we discuss problems locally and always identify two centroaffine immersions $M \to \mathbb{R}^4 \setminus \{0\}$ if they are congruent.

Lemma 3.1. *For an indefinite centroaffine surface $f : M \to \mathbb{R}^4 \setminus \{0\}$, we choose local asymptotic coordinates u^1, u^2 on M such that $h = 2\rho du^1du^2$. Then we have*

$$\nabla_{\partial_1}\partial_2 = \nabla_{\partial_2}\partial_1 = 0, \tag{3.1}$$
$$h(\nabla_{\partial_1}\partial_1, \partial_2) = \partial_1\rho, \quad h(\partial_1, \nabla_{\partial_2}\partial_2) = \partial_2\rho, \tag{3.2}$$
$$T(\partial_1,\partial_2) + \rho H = 0, \tag{3.3}$$
$$h(\partial_1, S\partial_2) = h(\partial_2, S\partial_1) = \rho H. \tag{3.4}$$

Moreover, if f is minimal and self-dual, then we can choose local asymptotic coordinates such that $\rho = 1$, and in these coordinates we have

$$\nabla \partial_1 = \nabla \partial_2 = 0, \tag{3.5}$$
$$T = \tau_1(u^1) du^1 du^1 + \tau_2(u^2) du^2 du^2, \tag{3.6}$$
$$S = -\tau_1(u^1) du^1 \otimes \partial_2 - \tau_2(u^2) du^2 \otimes \partial_1, \tag{3.7}$$
$$P = 0, \tag{3.8}$$

where τ_i, $i = 1, 2$, are functions in one variable.

Proof. By equation (2.11), we obtain

$$(\nabla_{\partial_i} h)(\partial_1, \partial_2) = \frac{\rho}{2} \operatorname{tr}_h \{(Y, Z) \mapsto (\nabla_{\partial_i} h)(Y, Z)\} = 0, \quad i = 1, 2. \tag{3.9}$$

Using (2.6), we have

$$\begin{aligned} h(\nabla_{\partial_1} \partial_2, \partial_1) &= h(\nabla_{\partial_2} \partial_1, \partial_1) \\ &= -\frac{1}{2} (\nabla_{\partial_2} h)(\partial_1, \partial_1) = -\frac{1}{2} (\nabla_{\partial_1} h)(\partial_1, \partial_2) = 0, \\ h(\nabla_{\partial_1} \partial_2, \partial_2) &= -\frac{1}{2} (\nabla_{\partial_1} h)(\partial_2, \partial_2) = -\frac{1}{2} (\nabla_{\partial_2} h)(\partial_1, \partial_2) = 0. \end{aligned}$$

Then we have (3.1). Moreover, we have (3.2) because

$$\begin{aligned} \partial_1 \rho &= \partial_1 h(\partial_1, \partial_2) = h(\nabla_{\partial_1} \partial_1, \partial_2), \\ \partial_2 \rho &= \partial_2 h(\partial_1, \partial_2) = h(\partial_1, \nabla_{\partial_2} \partial_2). \end{aligned}$$

Since

$$H = \frac{1}{2} \operatorname{tr} S = \frac{1}{2\rho} \{h(\partial_1, S\partial_2) + h(\partial_2, S\partial_1)\},$$

we obtain Equation (3.4) by virtue of (2.9). Equation (3.3) follows from (3.4) and (2.9).

Now we assume that f is minimal and self-dual. By the self-duality, we have

$$h(\nabla_{\partial_i} \partial_i, \partial_i) = -\frac{1}{2} (\nabla_{\partial_i} h)(\partial_i, \partial_i) = 0, \quad i = 1, 2. \tag{3.10}$$

Hence $\nabla_{\partial_2} \partial_2 = \rho^{-1} \partial_2 \rho \cdot \partial_2 = \partial_2 \log |\rho| \cdot \partial_2$ and

$$R(\partial_1, \partial_2) \partial_2 = \nabla_{\partial_1} \nabla_{\partial_2} \partial_2 = \partial_1 \partial_2 \log |\rho| \cdot \partial_2.$$

Using (2.5), (3.3) and (3.4) with $H = 0$, we obtain

$$\partial_1 \partial_2 \log |\rho| = \frac{1}{\rho} h(\partial_1, R(\partial_1, \partial_2) \partial_2) = -h(\partial_1, S\partial_2) + T(\partial_1, \partial_2) = 0.$$

Hence there exist two functions $\phi(t)$ and $\psi(t)$ in one variable such that $\rho(u^1, u^2) = \phi(u^1) \psi(u^2)$. Setting $\tilde{u}^1 = \int^{u^1} \phi(t) dt$ and $\tilde{u}^2 = \int^{u^2} \psi(t) dt$, we have $h = 2 d\tilde{u}^1 d\tilde{u}^2$. Thus we may assume $\rho = 1$. Equations (3.1), (3.2) and (3.10) imply $\nabla \partial_1 = \nabla \partial_2 = 0$, that is, ∇ is a flat affine connection with affine coordinates u^1, u^2.

From (2.5), we see that

$$0 = h(\partial_1, R(\partial_1, \partial_2)\partial_1) = h(\partial_1, S\partial_1) + T(\partial_1, \partial_1).$$

On the other hand, by (3.3) and (2.7) we have

$$\partial_2 T(\partial_1, \partial_1) = (\nabla_{\partial_2} T)(\partial_1, \partial_1) = P(\partial_1),$$

and by (3.4) and (2.8) we have

$$\partial_2 h(\partial_1, S\partial_1) = h(\partial_1, (\nabla_{\partial_2} S)\partial_1) = P(\partial_1).$$

Therefore, $P(\partial_1) = 0$ and $\tau_1 = T(\partial_1, \partial_1) = -h(\partial_1, S\partial_1)$ is independent of u^2. In the same way, we can prove $P(\partial_2) = 0$ and $\tau_2 = T(\partial_2, \partial_2) = -h(\partial_2, S\partial_2)$ is independent of u^1, thereby completing the proof. ∎

A curve $\gamma : I \to \mathbb{R}^2 \setminus \{0\}$ is called a *centroaffine curve* if γ, γ' are linearly independent, and a centroaffine curve is said to be centroaffine arclength parametrized if the areal velocity $(1/2)\,\mathrm{Det}(\gamma, \gamma')$ is constant. In this case, γ'' is proportional to γ because

$$0 = (\mathrm{Det}(\gamma, \gamma'))' = \mathrm{Det}(\gamma, \gamma'').$$

In the remainder, we will always assume that on a centroaffine curve a centroaffine arclength parameter is chosen.

For two centroaffine curves $\gamma_i : I_i \to \mathbb{R}^2$, $i = 1, 2$, we define their tensor product $\gamma_1 \otimes \gamma_2 : I_1 \times I_2 \to \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$ by

$$(\gamma_1 \otimes \gamma_2)\,(u^1, u^2) = \gamma_1(u^1) \otimes \gamma_2(u^2), \quad (u^1, u^2) \in I_1 \times I_2.$$

By identifying $(x^1, x^2) \otimes (y^1, y^2) \in \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$ with $(x^1y^1, x^1y^2, x^2y^1, x^2y^2) \in \mathbb{R}^4$, the map $\gamma_1 \otimes \gamma_2$ is regarded as a centroaffine surface in $\mathbb{R}^4 \setminus \{0\}$. Now we prove the following:

Theorem 3.2. *For two centroaffine curves $\gamma_i : I_i \to \mathbb{R}^2 \setminus \{0\}$, $i = 1, 2$, the centroaffine surface $f = \gamma_1 \otimes \gamma_2 : I_1 \times I_2 \to \mathbb{R}^4 \setminus \{0\}$ is a minimal ISDC surface. Conversely, any minimal ISDC surface is locally represented in this form.*

Proof. Setting $\gamma_i'' = \kappa_i \gamma$, $i = 1, 2$, we have

$$\partial_1\partial_1 f(u^1, u^2) = \kappa_1(u^1) f(u^1, u^2),$$
$$\partial_2\partial_2 f(u^1, u^2) = \kappa_2(u^2) f(u^1, u^2).$$

Hence we have

$$\nabla_{\partial_1}\partial_1 = \nabla_{\partial_2}\partial_2 = 0, \tag{3.11}$$
$$h(\partial_1, \partial_1) = h(\partial_2, \partial_2) = 0. \tag{3.12}$$

Thus u^1, u^2 are asymptotic coordinates. By (3.1) in Lemma 3.1 and (3.11), we see that ∇ is a flat affine connection with affine coordinates u^1, u^2. Moreover, from (3.9) we have $\rho = h(\partial_1, \partial_2)$ is a non-zero constant. Hence $\nabla h = 0$ and the surface f is self-dual.

Since ∇ is flat, equation (2.5) with (3.3) and (3.4) implies that

$$0 = h(\partial_1, R(\partial_1, \partial_2)\partial_2) = -\rho^2 H + T(\partial_1, \partial_2)\rho = -2\rho^2 H.$$

Therefore f is a minimal ISDC surface.

We shall prove the converse. Let f be a minimal ISDC surface. By Lemma 3.1, we may choose local coordinates u^1, u^2 satisfying $h = 2du^1du^2$ and equations (3.5)–(3.8). Let $u_0 = (u_0^1, u_0^2)$ be a point of M. Changing f by an element of $SL(4;\mathbb{R})$ if necessary, we may assume that

$$x(u_0) = (0,0,0,1), \quad \partial_1 x(u_0) = (1,0,0,0), \quad \partial_2 x(u_0) = (0,1,0,0).$$

Since $\partial_1\partial_1 f = \tau_1(u^1)f$, the curve $t \mapsto x(t, u_0^2)$ lies in a plane spanned by $x(u^0)$ and $\partial_1 x(u_0)$. Hence there exists a curve $\gamma_1 = (\gamma_1^1, \gamma_1^2)$ of $\mathbb{R}^2$ such that

$$x(t, u_0^2) = (\gamma_1^1(t), 0, 0, \gamma_1^2(t)).$$

In the same way, we have another plane curve $\gamma_2 = (\gamma_2^1, \gamma_2^2)$ such that

$$x(u_0^1, t) = (0, \gamma_2^1(t), 0, \gamma_2^2(t)).$$

Since $\gamma_i'' = \tau_i\gamma_i$, we have $(\mathrm{Det}(\gamma_i, \gamma_i'))' = 0$. Then γ_i, $i = 1, 2$ are centroaffine curves. It is easily verified that the centroaffine surfaces f and $\gamma_1 \otimes \gamma_2$ have same centroaffine invariants ∇, h, T, S and P. Therefore f is congruent to $\gamma_1 \otimes \gamma_2$. ■

Example 3.3. The Clifford torus f_C and the quadric f_{Q_1} in Example 2.4 are given by tensor products of two centroaffine curves as follows:

$$\begin{aligned} f_C(u^1, u^2) &= (\cos u^1, \sin u^1) \otimes (\cos u^2, \sin u^2), \\ f_{Q_1}(u^1, u^2) &= (1, u^1) \otimes (1, u^2). \end{aligned}$$

Next, we shall consider definite centroaffine surfaces. The following lemma can be obtained in a similar way to Lemma 3.1.

Lemma 3.4. *For a definite centroaffine surface $f : M \to \mathbb{R}^4 \setminus \{0\}$, we choose local conformal coordinates u^1, u^2 for h such that $h = 2\rho dz d\bar{z}$, where $z = u^1 + \sqrt{-1}u^2$. Then we have*

$$\nabla_{\partial_z}\partial_{\bar{z}} = \nabla_{\partial_{\bar{z}}}\partial_z = 0, \tag{3.13}$$

$$h(\nabla_{\partial_z}\partial_z, \partial_{\bar{z}}) = \partial_z\rho, \quad h(\partial_z, \nabla_{\partial_{\bar{z}}}\partial_{\bar{z}}) = \partial_{\bar{z}}\rho, \tag{3.14}$$

$$T(\partial_z, \partial_{\bar{z}}) + \rho H = 0, \tag{3.15}$$

$$h(\partial_z, S\partial_{\bar{z}}) = h(\partial_{\bar{z}}, S\partial_z) = \rho H. \tag{3.16}$$

Moreover, if f is minimal and self-dual, then we can choose local conformal coordinates such that $\rho = 1$, and in these coordinates we have

$$\nabla\partial_z = \nabla\partial_{\bar{z}} = 0, \tag{3.17}$$

$$T = \tau(z)dzdz + \overline{\tau(z)}d\bar{z}d\bar{z}, \tag{3.18}$$

$$S = -\tau(z)dz \otimes \partial_{\bar{z}} - \overline{\tau(z)}d\bar{z} \otimes \partial_z, \tag{3.19}$$

$$P = 0, \tag{3.20}$$

where τ is a holomorphic function.

A holomorphic map Λ from a domain $U \subset \mathbb{C}$ to $\mathbb{C}^2 \setminus \{0\}$ is called a *holomorphic centroaffine curve* if $\det(\Lambda, \Lambda')$ is constant. In this case, there exists a holomorphic function κ such that $\Lambda'' = \kappa\Lambda$.

For a given holomorphic centroaffine curve $\Lambda : U \to \mathbb{C}^2 \setminus \{0\}$, we consider a map $U \ni z \mapsto \Lambda(z) \otimes \overline{\Lambda(z)} \in \mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^2$. By identifying $(x^1, x^2) \otimes (y^1, y^2) \in \mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^2$ with the element

$$\frac{1}{2}\left(x^1y^1 + x^2y^2, x^1y^1 - x^2y^2, x^1y^2 + x^2y^1, \sqrt{-1}(x^1y^2 - x^2y^1)\right)$$

of $\mathbb{C}^4$, the image of the map above lies in $\mathbb{R}^4 \subset \mathbb{C}^4$. Thus we regard the map $\Lambda \otimes \overline{\Lambda}$ as a surface of $\mathbb{R}^4 \setminus \{0\}$.

Theorem 3.5. *For a holomorphic centroaffine curve $\Lambda : M \to \mathbb{C}^2 \setminus \{0\}$, the centroaffine surface $f = \Lambda \otimes \overline{\Lambda} : M \to \mathbb{R}^4 \setminus \{0\}$ is a definite minimal self-dual centroaffine surface. Conversely, any definite minimal self-dual centroaffine surface is locally represented in this form.*

Proof. By virtue of Lemma 3.4, we can verify that $\Lambda \otimes \overline{\Lambda}$ is minimal and self-dual in a similar way to the proof of Theorem 3.2.

To prove the converse, let $f : M \to \mathbb{R}^4$ be an arbitrary definite minimal self-dual centroaffine surface. As in Lemma 3.4, we choose local coordinates u^1, u^2 on M satisfying $h = 2dzd\bar{z}$ and (3.17)–(3.20). Then the structure equation of the surface f is given by

$$\begin{aligned} &\partial_z\partial_z f = \tau f, \quad \partial_z\partial_{\bar{z}} f = \xi, \quad \partial_{\bar{z}}\partial_{\bar{z}} f = \bar{\tau} f, \\ &\partial_z \xi = -\tau \partial_{\bar{z}} f, \quad \partial_{\bar{z}} \xi = -\bar{\tau}\partial_z f. \end{aligned} \tag{3.21}$$

Let z_0 be a point of M. Changing f by a suitable element of $SL(4;\mathbb{R})$, we may assume that

$$\begin{aligned} &f(z_0) = \frac{1}{\sqrt{2}}(1,1,0,0), \quad \partial_z f(z_0) = \frac{1}{\sqrt{2}}\left(0,0,1,-\sqrt{-1}\right), \\ &\xi(z_0) = \frac{1}{\sqrt{2}}(1,-1,0,0). \end{aligned} \tag{3.22}$$

Let $\lambda = f^1 + f^2$ and $\mu = \xi^1 + \xi^2$. We shall prove that the complex function $\partial_z\lambda/\lambda$ is holomorphic. Using (3.21), we have

$$\begin{aligned} \partial_{\bar{z}}\left(\frac{\partial_z\lambda}{\lambda}\right) &= \frac{\mu}{\lambda} - \frac{\partial_{\bar{z}}\lambda\partial_z\lambda}{\lambda^2} \\ &= 0 \quad \text{at } z_0 \text{ by (3.22)}, \end{aligned}$$

and

$$\partial_z\left\{\lambda^2\partial_{\bar{z}}\left(\frac{\partial_z\lambda}{\lambda}\right)\right\} = 0.$$

Hence $\partial_{\bar{z}}(\partial_z\lambda/\lambda)$ is identically zero. Now we define holomorphic functions Λ^i, $i = 1, 2$, around z_0 by

$$\begin{aligned} \Lambda^1(z) &= 2^{1/4}\exp\int_{z_0}^{z}\frac{\partial_z\lambda(w)}{\lambda(w)}dw, \\ \Lambda^2(z) &= \sqrt{2}\Lambda^1(z)\int_{z_0}^{z}\frac{1}{(\Lambda^1(w))^2}dw. \end{aligned}$$

Then $\Lambda = (\Lambda^1, \Lambda^2)$ is a holomorphic curve in $\mathbb{C}^2 \setminus \{0\}$ satisfying $\Lambda'' = \tau\Lambda$. Hence Λ is a holomorphic centroaffine curve. It is easily verified that f and $\Lambda \otimes \overline{\Lambda}$ is congruent around z_0. ∎

Example 3.6. We define a holomorphic centroaffine curve Λ by $\Lambda(z) = 2^{1/4}(1, z)$, $z \in \mathbb{C}$. Then the minimal self-dual centroaffine surface given by $\Lambda \otimes \overline{\Lambda}$ is the quadric f_{Q_2} in Example 2.5.

4 Self-Dual Centroaffine Surfaces with Non-Zero Constant Affine Mean Curvature

In this section, we consider self-dual centroaffine surfaces with constant affine mean curvature $H \neq 0$. Since such a surface lies in a quadratic cone, it can be locally written as a graph on a quadric f_{Q_1} or f_{Q_2} (see Examples 2.4 and 2.5). For this reason we first study graphs on a given centroaffine immersion generally.

Lemma 4.1. *Let $f : M \to \mathbb{R}^{n+2} \setminus \{0\}$ be a centroaffine immersion of an n-dimensional manifold M. Suppose that we change f to $\tilde{f} = e^{\omega} f$, where ω is a function on M. Then the affine fundamental form h and the affine mean curvature H change as follows:*

$$\tilde{h}(X,Y) = e^{2\omega} h(X,Y), \tag{4.1}$$

$$\tilde{H} = e^{-2\omega} \left\{ H - \frac{n-2}{2n} h(d\omega, d\omega) - \frac{1}{n} \operatorname{tr}_h \nabla d\omega \right\}. \tag{4.2}$$

Proof. Let ξ and $\tilde{\xi}$ be the Blaschke normal vector fields of f and $\tilde{f}$, respectively. We choose a positive function ρ, a function a and a vector field U on M so that

$$\tilde{\xi} = \rho^{-1}(\xi + af + f_* U). \tag{4.3}$$

By the definition of $\tilde{f}$, we have

$$\tilde{f}_* X = e^{\omega}\Big(d\omega(X) f + f_* X\Big)$$

and

$$\begin{aligned} D_X \tilde{f}_* Y = f_*\Big\{e^{\omega}(\nabla_X Y + d\omega(X)Y + d\omega(Y)X)\Big\} + e^{\omega} h(X,Y)\xi \\ + e^{\omega}(T(X,Y) + d\omega(X)d\omega(Y) + XY\omega) f. \end{aligned} \tag{4.4}$$

On the other hand, from (4.3) we have

$$\begin{aligned} D_X \tilde{f}_* Y &= \tilde{f}_* \tilde{\nabla}_X Y + \tilde{h}(X,Y)\tilde{\xi} + \tilde{T}(X,Y)\tilde{f} \\ &= f_*(e^{\omega}\tilde{\nabla}_X Y + \rho^{-1}\tilde{h}(X,Y)U) + \rho^{-1}\tilde{h}(X,Y)\xi \\ &\quad + \left\{ e^{\omega}\Big(\tilde{T}(X,Y) + d\omega(\tilde{\nabla}_X Y)\Big) + a\rho^{-1}\tilde{h}(X,Y) \right\} f. \end{aligned} \tag{4.5}$$

Comparing (4.4) and (4.5), we get

$$\tilde{\nabla}_X Y + (\rho e^{\omega})^{-1}\tilde{h}(X,Y)U = \nabla_X Y + d\omega(X)Y + d\omega(Y)X, \tag{4.6}$$

$$\tilde{h}(X,Y) = \rho e^{\omega} h(X,Y), \tag{4.7}$$

$$\begin{aligned} \tilde{T}(X,Y) + d\omega(\tilde{\nabla}_X Y) + a(\rho e^{\omega})^{-1}\tilde{h}(X,Y) \\ = T(X,Y) + d\omega(X)d\omega(Y) + XY\omega. \end{aligned} \tag{4.8}$$

A similar calculation on $D_X\tilde{\xi}$ shows that

$$\rho e^{\omega}\tilde{S}X = SX - aX - \nabla_X U + d\log\rho(X)U, \tag{4.9}$$

$$0 = -d\log\rho(X) + h(X,U). \tag{4.10}$$

By the equation (4.7) and the second condition of (2.4), we have

$$\begin{aligned} \sqrt{\left|\det\Big(h(e_i,e_j)\Big)\right|} &= (\rho e^{\omega})^{-n/2}\sqrt{\left|\det\Big(\tilde{h}(e_i,e_j)\Big)\right|} \\ &= (\rho e^{\omega})^{-n/2}\,\mathrm{Det}(\tilde{f}_*e_1,\dots,\tilde{f}_*e_n,\tilde{\xi},\tilde{f}) \\ &= \Big(\rho^{-1}e^{\omega}\Big)^{(n+2)/2}\,\mathrm{Det}(f_*e_1,\dots,f_*e_n,\xi,f). \end{aligned}$$

This implies that $\rho = e^{\omega}$. Hence we obtain (4.1) from (4.7) and $U = \mathrm{grad}_h\,\omega$ from (4.10). Moreover, we have

$$\begin{aligned} \tilde{h}(\tilde{S}X,Y) &= h(e^{2\omega}\tilde{S}X,Y) \\ &= d\omega(X)d\omega(Y) + h(SX,Y) - ah(X,Y) - h(\nabla_X U,Y) \\ &= d\omega(X)d\omega(Y) + h(SX,Y) - ah(X,Y) + (\nabla_X h)(U,Y) \\ &\quad - XY\omega + d\omega(\nabla_X Y), \end{aligned}$$

and

$$\begin{aligned} \tilde{T}(X,Y) + \tilde{h}(\tilde{S}X,Y) = T(X,Y) &+ h(SX,Y) \\ &+ h(X,Y)h(d\omega,d\omega) - 2ah(X,Y) + (\nabla_X h)(Y,U). \end{aligned}$$

By (2.6) and (2.11), we have

$$\mathrm{tr}_h\{(X,Y)\mapsto(\nabla_X h)(U,Y)\} = \mathrm{tr}_h\{(X,Y)\mapsto(\nabla_U h)(X,Y)\} = 0.$$

Hence, from (2.12) we obtain

$$\begin{aligned} 0 &= \mathrm{tr}_{\tilde{h}}\{(X,Y)\mapsto\tilde{T}(X,Y) + \tilde{h}(\tilde{S}X,Y)\} \\ &= n\Big(h(d\omega,d\omega) - 2a\Big). \end{aligned}$$

Consequently, we have

$$\begin{aligned} \tilde{H} &= \frac{1}{n}\,\mathrm{tr}_{\tilde{h}}\{(X,Y)\mapsto\tilde{h}(\tilde{S}X,Y)\} \\ &= \frac{1}{n}e^{-2\omega}\,\mathrm{tr}_{\tilde{h}}\{(X,Y)\mapsto\tilde{h}(\tilde{S}X,Y)\} \\ &= e^{-2\omega}\left\{H - \frac{n-2}{2n}h(d\omega,d\omega) - \frac{1}{n}\,\mathrm{tr}_h\,\nabla d\omega\right\}, \end{aligned}$$

thereby completing the proof. ■

Remark 4.2. When M is a compact 2-dimensional manifold, Lemma 4.1 implies that $I = \int_M H \operatorname{Vol}_h$ is invariant by the change $f \mapsto e^\omega f$. Then I is a projective invariant of the immersion $\pi \circ f : M \to \mathbb{R}P^3$, where $\pi : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}P^3$ is the natural projection. It can be easily checked that $I = (1/2)\int_M K \operatorname{Vol}_h - (1/16)A$, where K is the Gaussian curvature of h, and A is the area of M measured by the projective volume element (cf. [1, p. 174]). Hence, when h is positive definite, $I \leq 2\pi$ and the equality holds if and only if f is self-dual.

Let $f : M \to \mathbb{R}^4 \setminus \{0\}$ be a self-dual centroaffine surface with affine mean curvature H. From Lemma 4.1, we see that if $f = e^\omega f_{Q_1}$, then

$$H = -e^{-2\omega}\partial_1\partial_2\omega; \tag{4.11}$$

and that if $f = e^\omega f_{Q_2}$, then

$$H = -e^{-2\omega}\partial_z\partial_{\bar{z}}\omega. \tag{4.12}$$

Hence, H is non-zero constant if and only if ω is a solution of Liouville's equation. For details on Liouville's equation we refer the reader to [4], and we note only the following fact:

Fact 4.3. *Assume that H is a non-zero constant. Then,*

(1) a function ω satisfies (4.11) *if and only if there exist two functions μ and ν in one variable such that*

$$\begin{aligned}\omega(u^1,u^2) &= -\mu(u^1) - \nu(u^2)\\ &\quad - \log\left\{\alpha\int^{u^1} e^{-2\mu(t)}dt + \beta\int^{u^2} e^{-2\nu(t)}dt + A\right\},\end{aligned} \tag{4.13}$$

where $\alpha, \beta, A \in \mathbb{R}$, $\alpha\beta = -H$;

(2) a (complex) function ω satisfies (4.12) *if and only if there exits two holomorphic functions h and k in $z = u^1 + \sqrt{-1}u^2$ such that*

$$\begin{aligned}\omega(u^1,u^2) &= -h(z) - \overline{k(z)}\\ &\quad - \log\left\{\alpha\int^{z} e^{-2h(w)}dw + \beta\left(\overline{\int^{z} e^{-2k(w)}dw}\right) + A\right\},\end{aligned} \tag{4.14}$$

where $\alpha, \beta, A \in \mathbb{C}$, $\alpha\beta = -H$.

As an application of this fact, we shall show that any ISDC surface can be constructed by two centroaffine curves:

Theorem 4.4. *For given two non-zero constants α, β and two centroaffine curves γ_1, γ_2 with $\operatorname{Det}(\gamma_1, \gamma_1') = \operatorname{Det}(\gamma_2, \gamma_2') = 1$, the surface*

$$f(u^1,u^2) = \frac{1}{\alpha\gamma_1^2(u^1)\gamma_2^1(u^2) + \beta\gamma_1^1(u^1)\gamma_2^2(u^2)} f_{Q_1}(u^1,u^2)$$

is an ISDC surface with affine mean curvature $H = -\alpha\beta$. Conversely, any ISDC surface with non-zero constant affine mean curvature is locally represented in this form.

Proof. By a direct calculation we can verify that

$$\omega(u^1,u^2) = -\log\Big\{\alpha\gamma_1^2(u^1)\gamma_2^1(u^2) + \beta\gamma_1^1(u^1)\gamma_2^2(u^2)\Big\} \tag{4.15}$$

satisfies Liouville's equation (4.11) with $H = -\alpha\beta$. Conversely, we suppose that ω is a solution of (4.11) given by (4.13). Then, setting

$$\gamma_1^1(t) = e^{\mu(t)}, \quad \gamma_1^2(t) = e^{\mu(t)}\left(\int^t e^{-2\mu(t)} + A_1\right),$$
$$\gamma_2^1(t) = e^{\nu(t)}, \quad \gamma_2^2(t) = e^{\nu(t)}\left(\int^t e^{-2\nu(t)} + A_2\right),$$

where A_1, A_2 are arbitrary constants satisfying $A_1+A_2 = A$, we have two centroaffine curve $\gamma_i = (\gamma_i^1,\gamma_i^2)$, $i = 1,2$, with $\mathrm{Det}(\gamma_i,\gamma_i') = 1$. It is easily checked that (4.15) holds. ∎

In a similar fashion, any (complex) solution for Liouville's equation of elliptic type (4.12) can be obtained by two holomorphic centroaffine curves:

Proposition 4.5. *For given two non-zero constants α,β and two holomorphic centroaffine curves Λ_1,Λ_2 with $\det(\Lambda_1,\Lambda_1') = \det(\Lambda_2,\Lambda_2') = 1$, the function*

$$\omega(z) = -\log\Big\{\alpha\Lambda_1^2(z)\overline{\Lambda_2^1(z)} + \beta\Lambda_1^1(z)\overline{\Lambda_2^2(z)}\Big\}$$

satisfies Liouville's equation (4.12) with $H = -\alpha\beta$. Conversely, any solution of (4.12) with non-zero constant H is given in this way.

However, in order to construct definite self-dual centroaffine surfaces, we need to find *real* solutions of (4.12), and hence our result is still partial.

Theorem 4.6. *For given two non-zero constants α,β and a holomorphic centroaffine curve $\Lambda = (\Lambda^1,\Lambda^2)$ with $\det(\Lambda,\Lambda') = 1$, the surface*

$$f(u^1,u^2) = \frac{1}{\alpha\big|\Lambda^1(z)\big|^2 + \beta\big|\Lambda^2(z)\big|^2} f_{Q_2}(u^1,u^2), \quad z = u^1 + \sqrt{-1}u^2$$

is a definite self-dual centroaffine surface with affine mean curvature $H = \alpha\beta$.

5 Affine Mean Curvature of Centroaffine Surfaces of Non-parametric Type

In this section, we describe the affine mean curvature of a centroaffine surface of non-parametric type.

Theorem 5.1. *Let $f : M \to \mathbb{R}^4\setminus\{0\}$ be a centroaffine immersion given as $f(u) := {}^t(u^1,u^2,\varphi(u),\psi(u))$. The affine mean curvature of f is given by*

$$H = -\frac{1}{4}|\det{}_{\theta^0} h^0|^{1/4}\Big\{|\det{}_{\theta^0} h^0|^{1/4}\triangle_{h^0}|\det{}_{\theta^0} h^0|^{-1/4} + \mathrm{tr}_{h^0} T^0 - \mathrm{tr}\, S^0\Big\} \tag{5.1}$$

with

$$\begin{aligned}
\theta^0 &:= (\psi - \sum_{l=1}^{2} u^l \partial_l \psi) du^1 \wedge du^2, \\
h^0_{ij} &:= \frac{(\psi - \sum_{l=1}^{2} u^l \partial_l \psi)\partial_i\partial_j\varphi - (\varphi - \sum_{l=1}^{2} u^l \partial_l \varphi)\partial_i\partial_j\psi}{\psi - \sum_{l=1}^{2} u^l \partial_l \psi}, \\
T^0_{ij} &:= (\psi - \sum_{l=1}^{2} u^l \partial_l \psi)^{-1} \partial_i\partial_j\psi, \\
S^0 &:= 0,
\end{aligned} \tag{5.2}$$

where $\triangle_{h^0}$ denotes the Laplacian for h^0, and $\det_{\theta^0} h^0 = \det\big(h^0(e_i, e_j)\big)$ for a basis (e_1, e_2) of T_uM with $\theta^0(e_1, e_2) = 1$.

Proof. *Step 1.* We put $\xi_0 = {}^t(0, 0, 1, 0)$ and may assume that $M \subset \{u \in \mathbb{R}^2 | \det \Omega(u) > 0\}$ where

$$\begin{aligned}
\Omega(u) &:= [f_*\partial_1, f_*\partial_2, \xi_0, f]\,(u) \\
&= \begin{bmatrix} 1 & 0 & 0 & u^1 \\ 0 & 1 & 0 & u^2 \\ \partial_1\varphi & \partial_2\varphi & 1 & \varphi \\ \partial_1\psi & \partial_2\psi & 0 & \psi \end{bmatrix} (u) \in GL(4; \mathbb{R}).
\end{aligned}$$

Let ∇^0, h^0, T^0, S^0, τ^0, P^0 and θ^0 be the geometric quantities defined as in (2.3) and (2.2) with respect to ξ_0. Because ξ_0 is constant, S^0, τ^0 and P^0 vanish identically. We calculate (5.2) as

$$\begin{aligned}
&\theta^0\,(\partial_1, \partial_2) = \det \Omega, \\
&\begin{bmatrix} {\Gamma^0}^1_{ij} \\ {\Gamma^0}^2_{ij} \\ h^0_{ij} \\ T^0_{ij} \end{bmatrix} = \Omega^{-1} \begin{bmatrix} 0 \\ 0 \\ \partial_i\partial_j\varphi \\ \partial_i\partial_j\psi \end{bmatrix},
\end{aligned}$$

where $\nabla^0_{\partial_i}\partial_j = \sum_{k=1}^{2} {\Gamma^0}^k_{ij} \partial_k$.

Step 2. We choose a positive function ρ, a function a and a vector field U on M so that

$$\xi = \rho^{-1}(\xi_0 + af + f_*U)$$

is the Blaschke normal vector field of f. Then we have

$$\begin{aligned}
\rho &= |\det_{\theta^0} h^0|^{-1/4}, \\
U &= \mathrm{grad}_{h^0} \log \rho, \\
a &= \frac{1}{4}(\mathrm{tr}_{h^0} T^0 + \mathrm{tr}\, S^0 - \rho^{-1} \triangle_{h^0} \rho).
\end{aligned} \tag{5.3}$$

To prove it, we remark that (4.6) and (4.9) hold in this case with $\omega = 1$ (see also [3]). We get the first equation of (5.3) from $\theta = \rho^{-1}\theta^0$, the second equation of (4.6) and the third condition of (2.4), and the second equation of (5.3) from the second equation of (4.9). The third equation of (5.3) is obtained as follows. From (4.6) and (4.9), we have

$$\begin{aligned} &T(X,Y) + h(SX,Y) \\ &\qquad\qquad = T^0(X,Y) + h^0(S^0X,Y) - 2ah^0(X,Y) - h^0(\nabla_X U, Y), \end{aligned}$$

which implies

$$\begin{aligned} 0 &= \operatorname{tr}_h\{(X,Y) \mapsto T(X,Y) + h(SX,Y)\} \\ &= \rho^{-1}\left(\operatorname{tr}_{h^0} T^0 + \operatorname{tr} S^0 - 4a - \operatorname{div}^{\nabla} U\right). \end{aligned}$$

Noting $\nabla \operatorname{Vol}_h = 0$, we calculate

$$\begin{aligned} a &= \frac{1}{4}\left(\operatorname{tr}_{h^0} T^0 + \operatorname{tr} S^0 - \operatorname{div}^{\nabla} \operatorname{grad}_{h^0} \log\rho\right) \\ &= \frac{1}{4}\left(\operatorname{tr}_{h^0} T^0 + \operatorname{tr} S^0 - \operatorname{div}^{\nabla} \operatorname{grad}_h \rho\right) \\ &= \frac{1}{4}\left(\operatorname{tr}_{h^0} T^0 + \operatorname{tr} S^0 - \triangle_h \rho\right). \end{aligned}$$

Since $\dim M = 2$, we obtain the third equation of (5.3).

By (5.3), we obtain that

$$\begin{aligned} \operatorname{tr} S &= -\operatorname{tr}_h T \\ &= -\frac{1}{\rho}\left(\operatorname{tr}_{h^0} T^0 - 2a\right) \\ &= -\frac{1}{2\rho}\left(\operatorname{tr}_{h^0} T^0 - \operatorname{tr} S^0 + \rho^{-1}\triangle_{h^0}\rho\right), \end{aligned}$$

which implies (5.1). ∎

References

[1] Blaschke, W., *Vorlesungen über Differentialgeometrie II*, Springer, Berlin, 1923.

[2] Furuhata, H., Minimal centroaffine immersions of codimension two, Bull. Belgian Math. Soc. **7**(2000), 125–134.

[3] Nomizu, K. and Sasaki, T., Centroaffine immersions of codimension two and projective hypersurface theory, Nagoya Math. J. **132**(1993), 63–90.

[4] Rogers, C. and Shadwick, W. F., *Bäcklund Transformations and Their Applications*, Math. Sci. Engrg. 161, Academic Press, New York, 1982.

[5] Walter, R., Centroaffine differential geometry: submanifolds of codimension 2, Results in Math. **13**(1988), 386–402.

Department of Mathematics,
Hokkaido University,
Sapporo 060-0810, Japan
email : furuhata@math.sci.hokudai.ac.jp

Department of Applied Mathematics, Faculty of Sciences,
Fukuoka University,
Fukuoka 814-0180, Japan
email: sm036447@fukuoka-u.ac.jp