# On Equivariant Fibrations

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## 1 Introduction

Fibrations are useful - even pervasive - in many areas of homotopy theory. The *Hurewicz fibration* concept, i.e. the condition that a map satisfies the *covering homotopy property* or *CHP*, is versatile and frequently used. It is probably the most familiar version of the fibration idea.

*Dold fibrations* are maps that satisfy the *weak covering homotopy property* or *WCHP* of [D], and have important technical advantages over Hurewicz fibrations. They arise naturally when pasting procedures are used, and are invariant under *fibre homotopy equivalence* or *FHE*.

Similar remarks apply to the role of equivariant fibrations in equivariant homotopy theory. Some of the deeper questions concerning the equivariant version of Hurewicz fibrations - maps that satisfy the *equivariant covering homotopy property* or *equivariant CHP* - are dealt with in [W1] and [W2].

With suitable base spaces, the non-equivariant Dold fibration concept is equivalent to three other possible definitions of fibration ([D, Thm.6.4] and [B, Thm.6.3]), including the appropriate homotopy version of local triviality. In this paper we consider the equivariant version of Dold fibrations, i.e. maps that satisfy the *equivariant WCHP*. We extend these results of [D] and [B] to the equivariant case. Thus **we determine conditions under which five potential or possible definitions of equivariant fibration are equivalent.** These are the equivariant WCHP, three versions of the condition of being *equivariantly local homotopy trivial* or *equivariant LHT*, and the *equivariant homotopy induced property* or *equivariant HIP*. The

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latter is the condition that the pullbacks of the given equivariant map, over any pair of equivariantly homotopic equivariant maps into its codomain, are necessarily *equivariantly fibre homotopy equivalent* or *equivariant FHE*.

Our argument incorporates a short and easy proof of the "local implies global" result [D, Thm.5.12] for the non-equivariant WCHP. The proof of that result in [D] utilizes the *Section Extension Theorem* [D, Thm.2.7], and constructs the required covering homotopy by an iterative process. Our method uses routine properties of fibrations, and the "local implies global" property of FHEs, i.e. [D, Thm.3.3].

One reason, for using this approach, is that it is easy to identify other situations where it applies. We clarify this point, in section 2, by listing the fibration properties required for such an argument.

In particular, the method of section 3 is extended to the equivariant case in section 4. Thus the "local implies global" result for the equivariant FHE property [W1, Prop.1.10], leads to our "local implies global" result for the equivariant WCHP (Theorem 4.1). This allows us to prove our main results, Theorems 6.2 and 6.3, which equate various equivariant fibration concepts.

The equivalence of alternative definitions of fibration result, as given in [B, Thm.6.3], also applies to fibrations with extra structure, assuming that structure preserving versions of the WCHP and FHE are used. Examples include sectioned fibrations and principal fibrations. Our present equivariant results can be extended, by similar techniques, to apply to equivariant fibrations with additional equivariant structures, such as equivariant sectioned fibrations. However, that argument uses equivariant fibred mapping spaces, and will be given in a paper that develops those techniques.

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## 2 Notations and Background Material

This non-equivariant section - as we explained previously - lists the notation and results that are assumed in our proof of the "local implies global" theorem for the WCHP, as well as being relevant to sections 5 and 6. In cases where no proofs are referred to, these either follow easily from the definitions involved or can be found in standard texts that discuss fibrations.

**(2.1)** If $Y$ is a given space, then $Y^I$ will denote the space of all maps from $I$ to $Y$, with the compact-open topology. Then $e_0 : Y^I \to Y$, the function that evaluates at 0, is continuous.

**(2.2)** Given maps $f : A \to B$ and $p : X \to B$, we will use $X \times_B A$ to denote the corresponding pullback space, and $f^*p : X \times_B A \to A$ and $p^*f : X \times_B A \to X$ to denote the induced projections.

**(2.3)** If $p : X \to B$ and $q : Y \to B$ are maps and the homeomorphism $k : X \to Y$

is fibrewise, i.e. such that $q \circ k = p$, then will write $p \equiv q$.

**(2.4)** Given maps $p : X \to B$ and $q : Y \to C$, and homeomorphisms $h : B \to C$ and $k : X \to Y$ such that $q \circ k = h \circ p$, then $p \equiv h^*q$.

**(2.5)** If $f : A \to B,\ g : B \to C$ and $q : Y \to C$ are maps, then $(g \circ f)^*q \equiv f^*(g^*q)$.

**(2.6)** If $p : X \to B$ is a Hurewicz fibration and $f : A \to B$ is a map, then $f^*p$ is a Hurewicz fibration.

**(2.7)** If $p$ is a Hurewicz fibration and $f : A \to B$ is a homeomorphism, then $p^*f$ is a homeomorphism.

**(2.8)** If $p$ is a Hurewicz fibration and $f : A \to B$ is a homotopy equivalence, then $p^*f$ is a homotopy equivalence [BH, Cor.1.4].

**(2.9)** Let $p : X \to B$ and $q : Y \to B$ be maps and $U$ be a subspace of $B$. Then $p|p^{-1}(U) : p^{-1}(U) \to U$ will be denoted, more simply, as $p|U : X|U \to U$.

If $f : X \to Y$ is a map over $B$, then the restriction of $f$ over $U$ will be denoted by $f|U : X|U \to Y|U$.

**(2.10)** If $p : X \to B$ is a map and $j$ denotes the inclusion $U \subseteq B$, then $j^*p \equiv p|U$, via the projection and homeomorphism $X \times_B U \to X|U$.

**(2.11)** If $p$ is a Hurewicz fibration and $U \subseteq B$, then $p|U$ is a Hurewicz fibration.

**(2.12)** Given $U \subseteq B$, then $B^I|U$ will denote the subspace of $B^I$ with underlying set $\{l \in B^I |\ l(1) \in U\}$. The map $\varepsilon_0 : B^I|U \to B$ evaluates at 0.

**(2.13)** Let $f : X \to Y$ be a map. We recall the standard fact that $f$ is a composite $\phi(f) \circ \eta(f)$, where $\eta(f)$ is a homotopy equivalence and $\phi(f)$ a Hurewicz fibration. Taking $E(f) = X \times_Y Y^I$ to be the pullback space obtained using $e_0 : Y^I \to Y$ and $f$, the maps $\eta(f)$ and $\phi(f)$ are defined by
$\eta(f) : X \to E(f),\ x \rightsquigarrow (x, c_{f(x)})$, where $x \in X$ and $c_{f(x)}$ denotes the constant map value $f(x)$, and $\phi(f) : E(f) \to Y, (x,l) \rightsquigarrow l(1)$, where $(x,l) \in E(f)$.

The last three results in our list assume a pair of maps $p : X \to B$ and $q : Y \to B$, and a map $f : X \to Y$ over $B$.

**(2.14)** If $f$ is an FHE and $q$ satisfies the WCHP, then $p$ satisfies the WCHP.

**(2.15)** If $p$ and $q$ satisfy the WCHP and $f$ is a homotopy equivalence, then $f$ is a fibre homotopy equivalence [D, Thm.6.1].

**(2.16)** If $\mathcal{U}$ is a numerable cover of $B$ and, for all $U \in \mathcal{U},\ f|U : X|U \to Y|U$ is an FHE, then $f$ is an FHE [D, Thm.3.3.].

## 3 On the "Local Implies Global" Property for the WCHP

We continue in the non-equivariant mode. Let $p : X \to B$ be a map and $j : U \to B$ be the inclusion of a subspace $U$ in $B$. Our argument starts with the definition of four Hurewicz fibrations from this data.

(i) *The Fibrations $q : Q \to U$ and $\hat{q} : Q \to U^I$.* We define $q$ to be the Hurewicz fibration $\phi(p|U) : E(p|U) \to U$. Hence $Q = E(p|U)$ is the pullback space that is obtained using $p|U$ and $e_0 : U^I \to U$ or, equivalently, $p$ and $j \circ e_0 : U^I \to B$ (see (2.5) and (2.10)). The projections and induced Hurewicz fibrations $(e_0)^*(p|U)$ and $(j \circ e_0)^* p$ coincide. We denote this fibration by $\hat{q} : Q \to U^I$.

(ii) *The Fibrations $r : R \to U$ and $\hat{r} : R \to B^I|U$.* Recalling that the map $\phi(p) : E(p) \to B$ is a Hurewicz fibration, we define $r : R \to U$ to be the restriction and Hurewicz fibration $\phi(p)|U : E(p)|U \to U$. Equivalently $R = E(p)|U$ is the pullback space obtained using $p$ and $\varepsilon_0 : B^I|U \to B$ (see (2.12)). We define $\hat{r} : R \to B^I|U$ to be the projection, so it is the induced Hurewicz fibration $(\varepsilon_0)^* p$.

We notice that $Q \subseteq R$, $U^I \subseteq B^I|U$ and $\hat{q} : Q \to U^I$ is the restriction $\hat{r}|U^I : R|U^I \to U^I$. So if $\alpha : U^I \to B^I|U$ and $\beta : Q \to R$ denote the inclusions, then there is commutative diagram as shown.

$$\begin{array}{ccccc} U & \xleftarrow{\;q\;} & Q & \xrightarrow{\;\hat{q}\;} & U^I \\ \big\downarrow 1_U & & \big\downarrow \beta & & \big\downarrow \alpha \\ U & \xleftarrow{\;r\;} & R & \xrightarrow{\;\hat{r}\;} & B^I|U \end{array}$$

**Lemma 3.1** *The inclusion $\beta$ is an FHE from $q$ to $r$.*

*Proof.* If $k \in B^I|U$, then we define $c_{k(1)} \in U^I$ to be the constant map value $k(1)$. The map $B^I|U \to U^I, k \rightsquigarrow c_{k(1)}$, is clearly a homotopy inverse to $\alpha$, so $\alpha$ is a homotopy equivalence.

The commutative square $\hat{r} \circ \beta = \alpha \circ \hat{q}$ is a pullback space (2.10), so it follows by (2.8) that $\beta$ is a homotopy equivalence. Now $r \circ \beta = q$, so we see via (2.15) that $\beta$ is an FHE over $U$.

**Theorem 3.2** *(see [D, Thm.5.12]). Let $p : X \to B$ be a map and $\mathcal{U}$ be a numerable cover of $B$. If, for each $U \in \mathcal{U}$, the map $p|U : X|U \to U$ satisfies the WCHP, then so also does $p$.*

*Proof.* Let $U$ be a given member of $\mathcal{U}$. We recall that there are homotopy equivalences $\eta(p) : X \to E(p)$ and $\eta(p|U) : X|U \to E(p|U)$, both defined by $x \rightsquigarrow (x, c_{p(x)})$, where $x$ is in $X$ or $X|U$, respectively.

Now $\beta : Q \to R$ is the inclusion, so $\eta(p)|U : X|U \to E(p)|U$ is the composite $\beta \circ \eta(p|U)$. Further, $\eta(p|U)$ is an FHE from $p|U$ to $q$ (2.15), and $\beta$ is an FHE from $q$ to $r$ (Lemma 3.1), so $\eta(p)|U$ is an FHE from $p|U$ to $r = \phi(p)|U$.

It follows, by (2.16), that $\eta(p) : p \to \phi(p)$ is an FHE. Now $\phi(p)$ is a Hurewicz fibration, so we see via (2.14) that $p$ satisfies the WCHP.

**Remarks** We notice that if the proper analogues of the results of section 2 are valid, for a theory of fibrations, then there will be a corresponding version of our Theorem 3.2 for that theory. This applies, for example, to structured fibrations in the sense of [B], including sectioned fibrations and principal fibrations. The appropriate analogue of (2.16) is then [B, Thm.5.3]. However, a version of Theorem 3.2 has already been proved for those theories [B, Thm.4.7], and our main concern here is the equivariant case.

## 4 On the "Local Implies Global" Property for the Equivariant WCHP

Dold proved "local implies global" theorems for FHE, the CHP and the WCHP. Waner did the same for equivariant versions of FHE [W1, Thm.1.10] and the CHP [W2, Lem.3.2.4]. We prove the remaining case here.

We assume, from this point on, that $G$ is a *compact Lie group* and work in the category of left $G$-spaces and $G$-maps.

The spaces, maps and homotopies of section 2 can now be replaced by the corresponding $G$-spaces, $G$-maps and $G$-homotopies. Thus $G$ acts trivially on $I$, $G$-subspaces have $G$-invariant underlying sets, $G$-pullbacks have the diagonal action and so on. If $Y$ is a $G$-space, then so too is $Y^I$, under the action $(g, l) \rightsquigarrow g.l$, where $(g.l)(t) = g.(l(t)), g \in G$ and $l \in Y^I$.

Let $A$ and $B$ be $G$ spaces and $F : A \times I \to B$ be a $G$-homotopy. Then $F$ will be said to be *stationary on* $A \times [0, 1/2]$ if, for all $a \in A$ and $t \in [0, 1/2]$, $F(a, t) = F(a, 0)$.

*Let $p : X \to B$ be a $G$-map. Then $p$ will be said to satisfy the $G$-WCHP if, for all choices of a $G$-space $A$, a $G$-map $f : A \times \{0\} \to X$, and a $G$-homotopy $F : A \times I \to B$ that is stationary on $A \times [0, \frac{1}{2}]$ and satisfies $p \circ f = F|A \times \{0\}$, there is a $G$-homotopy $H : A \times I \to X$ such that $p \circ H = f$ and $H|A \times \{0\} = f$.*

A *$G$-cover* is a cover by $G$-invariant sets. An open $G$-cover $\mathcal{U}$ will be said to be *$G$-numerable* if there is a locally finite partition of unity $\{\lambda_U\}_{U \in \mathcal{U}}$ such that each $\lambda_U$ is a $G$-map, with $U = \lambda_U^{-1}(0, 1]$.

The results of section 2 are valid equivariantly. For (2.1) ... (2.15) this is straightforward, the equivariant version of (2.16) is [W1, Prop.1.10].

**Theorem 4.1** *Let $p : X \to B$ be a $G$-map and $\mathcal{U}$ be a $G$-numerable $G$-cover of $B$. If, for all $U \in \mathcal{U}$, $p|U : X|U \to U$ satisfies the $G$-WCHP, then so also does $p$.*

*Proof.* The arguments of section 3 carry over directly to the equivariant case.

We remind the reader of [L, Lem.1.12], which asserts that:

**(4.2)** *every open $G$-cover of a paracompact $G$-space has a $G$-numerable refinement.*

It follows that, if $B$ is a paracompact space, the requirement that the open $G$-cover is $G$-numerable can be deleted from Theorem 4.1.

## 5 On Equivariant Local Homotopy Triviality

Let $H$ be a closed subgroup of $G$, and $F$ be a left $H$-space. Then the equivariant map

$$p_F : G \times_H F \to G/H, \quad [g, x] \rightsquigarrow gH,$$

where $g \in G$ and $x \in F$, is an equivariant fibration in the sense that it satisfies the $G$-CHP [tD, p.54, Ex.7]. Any such fibration will be referred to as a *$G$-model trivial fibration.*

Let $B$ be a $G$-space and $U$ be an open $G$-invariant subspace of $B$. Then $U$ will be said to be a *tube* in $B$ if it is the image $GV$ of a $G$-embedding (= $G$-homeomorphism into)

$$\mu : G \times_H V \to B, \quad [g, v] \rightsquigarrow gv,$$

where $V$ is an $H$-invariant subspace of $B$, $g \in G$ and $v \in V$.

Let $F$ and $W$ be left $H$-spaces. Then, using the diagonal action of $H$ on $W \times F$, there is a $G$-map

$$p_{W\times F} : G \times_H (W \times F) \to G \times_H W, \quad [g, (w, x)] \rightsquigarrow [g, w],$$

where $g \in G$, $w \in W$ and $x \in X$. We see by the following Lemma that it is a $G$-fibration, and will refer to it as a *$G$-trivial fibration*

**Lemma 5.1** *(i) The $G$-map $p_{W\times F}$ is the pullback of the model trivial $G$-fibration $p_F : G \times_H F \to G/H$ over $p_W : G \times_H W \to G/H$, i.e. we have $p_{W\times F} \equiv (p_W)^*(p_F)$ in the equivariant sense.*

*(ii) $p_{W\times F}$ satisfies the $G$-CHP.*

*Proof.* (i) The required $G$-homeomorphism

$$G \times_H (W \times F) \to (G \times_H W) \times_{G/H} (G \times_H F)$$

over $G \times_H W$, is determined by the rule $[g, (w, x)] \rightsquigarrow ([g, w], [g, x])$ where $g \in G$, $w \in W$ and $x \in F$.

(ii) We know that $p_F$ satisfies the $G$-CHP; the result follows by $G$-(2.6).

Let the $G$-space $T$ be a tube, i.e. relative to a closed subgroup $H$ of $G$, the $H$-invariant subspace $V$ of $T$ and the $G$-homeomorphism $\mu : G \times_H V \to GV = T$. If $q : Y \to T$ is a $G$-map that is $G$-FHE to the $G$-fibration $\mu \circ p_{V \times F} : G \times_H (V \times F) \to T$, then $q$ will be said to be a *$G$-homotopy trivial fibration.*

We notice, via $G$-(2.14), that such a $q$ necessarily satisfies the $G$-WCHP.

*If $p : X \to B$ is a $G$-map and $\mathcal{U}$ is an open $G$-cover of tubes for $B$ such that, for each $U \in \mathcal{U}$, $p|U$ is a $G$-homotopy trivial fibration, then $p$ will be said to be $G$-locally homotopy trivial or $G$-LHT.*

If the cover $\mathcal{U}$ is $G$-numerable, then $p$ will be said to be *$G$-numerably locally homotopy trivial or $G$-numerably LHT.*

Note that we are not insisting on a fixed fibre $F$; thus if the restriction of $p$ over one tube in the cover utilizes a fibre $F_1$, then the restriction of $p$ over another such tube may involve a quite different fibre $F_2$.

If $G$ acts on the space $B$ and $b \in B$, then we will use the standard notation that $G_b$ denotes the *isotropy subgroup of $G$ at $b$.* The following, alternative, definition of $G$-local homotopy triviality is similar to that used, by Bierstone, for differentiable $G$-fibre bundles [Bi, p.619-620].

*Let $p : X \to B$ be a $G$-map. We will write that $p$ is $G$-locally homotopy trivial in the sense of Bierstone or $G$-BLHT if, for all $b \in B$, there is a $G_b$-invariant neighbourhood $V_b$ of $b$ such that $p|V_b$ is $G_b$-FHE to a projection and $G_b$-map $V_b \times F_b \to V_b$, where $F_b$ is a $G_b$-space and $G_b$ acts diagonally on $V_b \times F_b$.*

**Proposition 5.2** *Let $p : X \to B$ be a $G$-map.*

(i) *If $p$ is $G$-LHT, then $p$ is $G$-BLHT.*

(ii) *If $B$ is completely regular, then the converse argument holds, and the two $G$-local homotopy triviality conditions on $B$ are equivalent.*

*Proof.* Similar results are given in [L, Lem.1.1 and Lem.1.3], but for $G$-$A$-fibre bundles. If we take those proofs, forget about $A$, allow the fixed fibre $F$ to be replaced by a variable fibre $F_b$, where $b \in B$, and remember that we are now dealing with $G$-FHEs rather than fibrewise $G$-homeomorphisms, the arguments of [L] extend to our case without difficulty.

# 6 On Alternative and Equivalent Definitions of Equivariant Fibration

We now give a further possible definition of equivariant fibration. *Let $p : X \to B$ be a $G$-map. If, for all choices of a $G$-space $A$ and a pair of $G$-homotopic $G$-maps $f$ and $g$ from $A$ to $B$, the induced $G$-maps $f^*p$ and $g^*p$ are necessarily $G$-FHE, then $p$ will be said to satisfy the $G$-homotopy induced property or $G$-HIP.*

**Proposition 6.1** *Let $p : X \to B$ be a $G$-map. The conditions that $p$ satisfies: (i) the $G$-numerably LHT property, (ii) the $G$-WCHP, and (iii) the $G$-HIP are related* *according to the scheme (i) ⇒ (ii) ⇒ (iii).*

*Proof.* (i) ⇒ (ii). We are assuming a $G$-numerable cover $\mathcal{U}$ and that, for each $U \in \mathcal{U}$, $p|U$ is a $G$-homotopy trivial fibration. Hence, as remarked in section 5, each $p|U$ satisfies the $G$-WCHP. The result follows by Theorem 4.1.

(ii) ⇒ (iii). This follows from the (ii) ⇒ (iii) argument in the proof of [B, Prop.6.2], and the proof of [B, Lem.4.1], on which that Proposition depends. We simply interpret spaces, maps and homotopies as $G$-spaces, $G$-maps and $G$-homotopies, and ignore references to base points and $\mathcal{E}$-structures.

Let $B$ be a $G$-space. *Then $B$ will be said to be $G$-numerably contractible if it has a $G$-numerable $G$-cover $\mathcal{U}$ of tubes such that, for all $U \in \mathcal{U}$, the inclusion $U \to B$ is $G$-homotopic to a $G$-map of $U$ into a single orbit $Q$ of $B$, where $Q$ may vary with $U$.*

**Theorem 6.2** *Let $G$ be a compact Lie group and $B$ be a $G$-numerably contractible space. If $p : X \to B$ is a $G$-map, then the $G$-numerably LHT, $G$-WCHP and $G$-HIP assumptions concerning $p$ are equivalent.*

*Proof.* It follows from Proposition 6.1 that we just have to show that, if $p$ satisfies the $G$-HIP, then it also satisfies the $G$-numerably LHT property.

Let $U$ be a member of a $G$-numerably contractible $G$-cover of $B$. We will use $j : U \to B$ to denote the inclusion, and $k : U \to Q \subseteq B$ to denote a $G$-map into an orbit $Q$ in $B$, such that $j$ and $k$ are $G$-homotopic.

Selecting $b \in Q$, we have $Q = Gb$, and will use $H$ to denote the isotropy subgroup $G_b$ of $G$ at $b$. Then, according to [Br, prop.4.1, p.40], there is a G-homeomorphism $\alpha : G/H \to Q$, $gH \rightsquigarrow g.b$. We have $G$-maps $p|Q : X|Q \to Q$ and $k : U \to Q$, and will define $F = (p|Q)^{-1}\alpha(eH)$ and $W = k^{-1}(\alpha(eH))$. Then, according to [Br, Prop.3.2, p.80], there are $G$-homeomorphisms $\phi_F : G \times_H F \to X|Q$ and $\phi_W : G \times_H W \to U$ over $Q$. In fact there is a commutative diagram as shown, where $p_F$ and $p_W$ are the obvious model trivial fibrations.

$$\begin{array}{ccccc}
G \times_H F & \xrightarrow{\;p_F\;} & G/H & \xleftarrow{\;p_W\;} & G \times_H W \\
\cong \Big\downarrow \phi_F & & \cong \Big\downarrow \alpha & & \cong \Big\downarrow \phi_W \\
X|Q & \xrightarrow{\;p|Q\;} & Q & \xleftarrow{\;k\;} & U
\end{array}$$

$$\begin{array}{llll}
\text{Then} & p|U \equiv j^*p & & \text{by } G\text{-(2.10)} \\
& \text{is } G\text{-FHE to } k^*p & & \text{by the } G\text{-HIP} \\
& = k^*(p|Q) & & \text{since } k(U) \subseteq Q \\
& \equiv k^*(\alpha^{-1})^*(p_F) & & \text{by } G\text{-(2.4)}
\end{array}$$

$$\equiv (\alpha^{-1} \circ k)^*(p_F) \qquad \text{by } G\text{-(2.5)}$$

$$\equiv (p_W \circ \phi_W^{-1})^*(p_F) \qquad \text{since } \alpha^{-1} \circ k = p_W \circ \phi_W^{-1}$$

$$\equiv (\phi_W^{-1})^*(p_W)^*(p_F) \qquad \text{by } G\text{-(2.5)}$$

$$\equiv (\phi_W^{-1})^*(p_{W\times F}) \qquad \text{by Lemma 5.1}$$

$$\equiv \phi_W \circ (p_{W\times F}) \qquad \text{by } G\text{-(2.7)}.$$

Hence $p|U$ is $G$-FHE to $\phi_W \circ (p_{W\times F})$, and $p|U$ is a $G$-homotopy trivial fibration. Hence $p$ is $G$-numerably LHT.

**Theorem 6.3** *Let $G$ be a compact Lie group and the $G$-space $B$ be paracompact. Further, let $B$ have an open $G$-cover $\mathcal{U}$ such that, for each $U \in \mathcal{U}$, the inclusion $U \to B$ is $G$-homotopic to a $G$-map $U \to Q \subseteq B$, where $Q$ is a single orbit in $B$ that may vary with $U$.*

*Then five conditions that a $G$-map $p : X \to B$ might satisfy, ie. the $G$-numerably LHT property, the $G$-LHT property, the $G$-BLHT property, the $G$-WCHP and the $G$-HIP, are equivalent.*

*Proof.* We know, via (4.2), that every open cover of $B$ has a $G$-numerable refinement. Hence the $G$-LHT and $G$-numerably LHT conditions are equivalent. Also, $B$ must be $G$-numerably contractible, so Theorem 6.2 applies and the $G$-numerably LHT condition, the $G$-WCHP and the $G$-HIP are equivalent.

A paracompact space is defined to be Hausdorff, and is known to be normal [W, Thm.20.10], so it is completely regular [W, Ex.15.3(a)]. It follows by Proposition 5.2(ii) that $p$ satisfies the $G$-LHT condition if and only if it satisfies the $G$-BLHT condition.

## References

[Bi.] E. Bierstone, The Equivariant Covering Homotopy Property for Differentiable $G$-Fibre Bundles, J. Diff. Geom. 8 (1973), 615-622.

[B.] P.I. Booth, Local to Global Properties in the Theory of Fibrations, Cahiers de Topologie et Géométrie Différentielle Catégoriques XXXIV-2 (1993), 127-151.

[BH.] R. Brown and P.R. Heath, Coglueing Homotopy Equivalences, Math. Z. 113 (1970), 313-325.

[Br.] G.E. Bredon, Introduction to Compact Transformation Groups, Academic Press, New York, 1972.

[D.] A. Dold, Partitions of Unity in the Theory of Fibrations, Ann. of Math. 78 (1963), 223-255.

[L.] R.K. Lashof, Equivariant Bundles, Ill. J. Math. 26 (1982), 257-270.

[tD.] T. tom Dieck, Transformation Groups, Walter de Gruyter, Berlin, 1987.

[W1.] S. Waner, Equivariant Fibrations and Transfer, Trans. Amer. Math. Soc. 258 (1980), 369-384.

[W2.] S. Waner, Equivariant Classifying Spaces and Fibrations, Trans. Amer. Math. Soc. 258 (1980), 385-405.

[W.] S. Willard, General Topology, Addison-Wesley Publishing Co. Inc., Reading Massachusetts, 1970.

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