A Groebner Basis Algorithm for Computing the Rational L.-S. Category of Elliptic Pure Spaces

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Abstract

The rational Lusternik-Schirelmann category, $\mathrm{cat}_0(S)$, of an elliptic space S has been characterized in terms of its Sullivan minimal model $(\Lambda V, d)$ as $\mathrm{cat}_0(S) = \sup\{k \mid \exists w \in \Lambda^{\geq k}V, [w] \text{ is a top class}\}$. We combine a method for computing a representative of the fundamental class of any elliptic space with a Groebner basis approach so that, for S a pure elliptic space, reduction of this representative provides one that achieves the $\mathrm{cat}_0(\Lambda V, d)$ upper bound.

Introduction

The *Lusternik-Schirelmann category* [9, 13], cat S, of a topological space S is the least integer m such that S is the union of $m+1$ open sets, each contractible in S. For S a simply connected CW complex, the *rational L.-S. category*, $\mathrm{cat}_0(S)$, introduced by Felix and Halperin in [5] is given by $\mathrm{cat}_0(S) = \mathrm{cat}(S_{\mathbb{Q}}) \leq \mathrm{cat}\,(S)$.

Recently [6], the rational L.-S. category of an elliptic space S has been characterized in terms of its minimal model $(\Lambda V, d)$ as $\mathrm{cat}_0(S) = \sup\{k \mid \exists w \in \Lambda^{\geq k}V, [w] \text{ is a top class}\}$. We combine a method [12] for computing a representative of the fundamental class of any elliptic space with a Groebner basis approach so that, for a pure elliptic space, reduction of this representative yields one that achieves the $\mathrm{cat}_0(\Lambda V, d)$ upper bound.

In this paper, all spaces are CW-complexes that are simply connected and whose rational homology is finite dimensional in each degree.

Received by the editors June 2000 - In revised form September 2000.
Communicated by R. Delanghe.
1991 *Mathematics Subject Classification :* Primary 58E05,55P62,secondary 13P10.
Key words and phrases : Rational homotopy, L-S category, Groebner Basis.

This paper is organized as follows, first in §1 we recall some notions and basic facts in Sullivan's theory of minimal models, then in §2 we give an algorithm for the rational category of spaces with vanishing Euler homotopy characteristic, finally in §3 we generalizes our algorithm to pure elliptic spaces.

1 Basic facts

We recall here some basic facts and notation we shall need from Sullivan's theory of minimal models, for which [8, 16] are standard references.

A simply connected space S such that $\dim H^*(S) < \infty$ is called *rationally elliptic* if $\dim \pi_*(S) \otimes \mathbb{Q} < \infty$, otherwise S is called *rationally hyperbolic*. A commutative graded algebra H is said to have *formal dimension* N if $H^p = 0$, $p > N$, and $H^N \neq 0$. An element $0 \neq w \in H^N$ is called a *fundamental* or *top class*.

For S a simply connected elliptic space, by [6, Lemma 10.1] the rational Toomer's invariant $\mathrm{e}_0(\Lambda V, d)$ is the largest integer k such that the top class can be represented by a cocycle in $\Lambda^{\geq k} V$.

In [6, Th.3] is proven that for (rationally) elliptic spaces $\mathrm{cat}_0(S) = \mathrm{e}_0(S)$. Hence $\mathrm{cat}_0(S) = \sup\{k \mid \exists w \in \Lambda^{\geq k} V, [w] \text{ is a top class}\}$.

1.1 Pure spaces

Henceforth, if S is a space with minimal model $(\Lambda V, d)$ we shall denote $X = V^{\mathrm{even}}$, $Y = V^{\mathrm{odd}}$, $n = \dim V^{\mathrm{even}}$, $m = \dim V^{\mathrm{odd}}$. The integer $\chi_\pi = n - m$ is called the *Euler homotopy characteristic* of S, and $\sum_i (-1)^i \dim(H^i(S; Q))$ is called the *Euler characteristic* of S.

A pure space S is a space whose minimal model $(\Lambda V, d) = \Lambda X \otimes \Lambda Y$ satisfies $d\,X = 0$ and $d\,Y \subset \Lambda X$. Spheres and compact homogeneous spaces are examples of pure spaces. If $\dim V < \infty$ then it is called a finite pure space.

A bigradation on $(\Lambda V, d)$ is given by $\Lambda V = \sum_{n,j \geq 0} (\Lambda_j V)^n$ where $(\Lambda_j V)^n = (\Lambda X \otimes \Lambda^j Y)^n$. Since $d(\Lambda_j V)^n \subset (\Lambda_{j-1} V)^{n+1}$, the differential d has bidegree $(1, -1)$ and this induces a bigradation in cohomology.

If $(\Lambda V, d)$ is pure and elliptic then $H(\Lambda V, d)$ is a Poincare duality algebra and for $k = -\chi_\pi$ it is verified that $H_k(\Lambda V) \neq 0$ and $H_{k+p}(\Lambda V) = 0$, $p \geq 1$. Hence, if $n = m$ then $H(\Lambda V, d) = H_0(\Lambda V, d)$. As an immediate consequence of theses properties we obtain:

Lemma 1. *Let $(\Lambda V, d)$ be a pure elliptic space. Then there is a cocycle w_1 in $\Lambda_{m-n} V$ that represents the top class and lives in $\Lambda^{\geq k} V$ with $k = cat_0(\Lambda V, d)$.*

In [14] is given a formula for computing a cocycle representing the fundamental class of a pure elliptic space $(\Lambda X \otimes \Lambda Y, d)$. A slight modification [12] of this formula gives the following algorithm.

Let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ be homogeneous basis of X and Y respectively and $\bar{X} = sX$ the suspension of X with $d\bar{x}_i = x_i$. Choose elements $\Psi_j \in \Lambda X \otimes \Lambda^1 \bar{X}$ for which $d\Psi_j = dy_j, j = 1, \ldots, m$. Then

Proposition 2.

$$w = \textit{coefficient of } \prod_{i=1}^{n} \bar{x}_i \textit{ in the development of } \prod_{j=1}^{m} (y_j - \Psi_j)$$

is a cocycle in $\Lambda_{m-n}V$ that represents the fundamental class of $(\Lambda V, d)$.

Observe that to construct Ψ_j it suffices to replace in each term of dy_j one $x_k \in \{x_1, \ldots, x_n\}$ by its suspension $\bar{x}_k$.

Proposition 2 provides a lower bound for $\mathrm{cat}_0\,(S)$. We shall apply a modification of standard Groebner basis (as in [1]). We begin by giving the necessary definitions and the modifications required in the proofs. Then we will show that reduction of w with respect to a certain Groebner basis provides a presentative w_0 that achieves the upper bound $\mathrm{cat}_0(\Lambda V, d)$. We start with a special case, spaces with positive Euler characteristic and Groebner basis for ideals. Then we proceed to the general case, that is, pure elliptic spaces and Groebner basis for modules.

2 $\mathrm{cat}_0\, S$ for elliptic spaces with positive Euler characteristic

We will denote by $\mathbb{K}$ a field of characteristic 0. In [7] is proven that, for an elliptic space S, the three conditions: positive Euler characteristic, $\chi_\pi = 0$ and $H^{\mathrm{odd}} = 0$ are equivalent. Furthermore, for an elliptic space, $\chi_\pi = 0$ implies that, up to isomorphism, the model of S is a pure model. Hence, if the pure model associated to S is $(\Lambda V, d)$ and I is the ideal $< dy_1, \ldots, dy_n >$ in $\mathbb{K}[x_1, \ldots, x_n]$ then $H(\Lambda V, d) = \mathbb{K}[x_1, \ldots, x_n]/I$.

2.1 Groebner basis for ideals and the $\leq_{\mathrm{opglex}}$ order

We have in mind an audience more expert in rational homotopy than in Groebner basis theory, so we will recall some standard facts and definitions on Groebner basis for which [1, 4] are standard references.

First, we recall that the set of power products in $\mathbb{K}[x_1, \ldots, x_n]$ is denoted by $\mathbb{T}^n = \{x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} \mid \beta_i \in \mathbb{N}, i = 1, \ldots, n\}$.

Definition 3. *By a term order on $\mathbb{T}^n$ we mean a total order $\leq$ on $\mathbb{T}^n$ satisfying the following conditions:*

1. $1 \leq x^\alpha$ *for all* $\alpha \in \mathbb{N}^n$.
2. *if* $x^\alpha \leq x^\beta$ *then* $x^\alpha \cdot x^\gamma \leq x^\beta \cdot x^\gamma$, *for all* $\gamma \in \mathbb{N}^n$

Theorem 4. *Every term order on $\mathbb{T}^n$ is a well-ordering, that is, for every subset $A \subset \mathbb{T}^n$, there exists $x^\alpha \in A$ such that for all $x^\beta \in A$, $x^\alpha \leq x^\beta$.*

Definition 5. *Let $f = \sum_{\alpha \in A} a_\alpha x^\alpha \in \mathbb{K}[x_1, \ldots, x_n]$, with $\forall \alpha \in A, a_\alpha \neq 0$, and let $\leq$ be a total order on $\mathbb{N}^n$. Then*

1. *The* total degree *of $x^\beta \in \mathbb{T}^n$ is $||\beta|| = \sum_{i=1}^{n} \beta_i$.*
2. *The* total degree *of f, denoted* $\mathrm{tdeg}(f)$, *is* $\max(||\alpha||,\, \alpha \in A)$.

3. *The* multidegree *of* f *is* $\text{multideg}(f) = \max(\alpha,\, \alpha \in A)$.

4. *The* leading coefficient *of* f *is* $\text{lc}(f) = a_{\text{multideg}(f)} \in \mathbb{K}$.

5. *The* leading power product *of* f *is* $\text{lp}(f) = x^{\text{multideg}(f)}$.

6. *The* leading term *of* f *is* $\text{lt}(f) = \text{lc}(f) \cdot \text{lp}(f)$.

We will write $\text{hdeg}(f)$ for the homological degree of a homogeneous element $f \in (\Lambda V, d)$.

Definition 6. *The lexicographical order* $\leq_{lex}$ *on* $\mathbb{T}^n$ *with* $x_1 > x_2 > \cdots > x_n$ *is defined by*

$$x^\alpha \leq_{lex} x^\beta \Leftrightarrow \alpha = \beta \text{ or } \alpha_1 < \beta_1 \text{ or } \exists k \,|\, \alpha_i = \beta_i \text{ for } 1 \leq i < k \text{ and } \alpha_k < \beta_k.$$

Definition 7. *The opposite of the graded lexicographic order on* $\mathbb{T}^n$*, that we denote by* $\leq_{opglex}$*, is given by*

$$x^\alpha \leq_{opglex} x^\beta \Longleftrightarrow \alpha = \beta \text{ or } ||\alpha|| > ||\beta|| \text{ or } ||\alpha|| = ||\beta|| \text{ and } x^\beta \leq_{lex} x^\alpha.$$

Clearly, the $\leq_{\text{opglex}}$ order is not a well-ordering. But it is a total order and verifies that if $x^\alpha \leq_{\text{opglex}} x^\beta$ then $x^\alpha \cdot x^\gamma \leq_{\text{opglex}} x^\beta \cdot x^\gamma$ for all $\gamma \in \mathbb{N}^n$. This is a key property in Groebner basis theory. Since $\leq_{\text{opglex}}$ is not a well-ordering, we will have to modify the proof of some of the following theorems. The key idea is that since I is homogeneous (w.r.t homological degree) and finite dimensional in each degree, we will prove that the the division algorithm and Buchberger's algorithm terminate in a finite number of steps without applying the well-order property.

Definition 8. *Given* f, g, h *in* $\mathbb{K}[x_1, \ldots, x_n]$ *with* $g \neq 0$*, we say that* f *reduces to* h *modulo* g *in one step, written* $f \xrightarrow{g} h$*, if and only if* $\text{lp}(g)$ *divides a non zero term* X *that appears in* f *and* $h = f - \dfrac{X}{\text{lt}(g)} g$.

Let $f, h,$ and $f_1, \ldots, f_s$ be polynomials in $\mathbb{K}[x_1, \ldots, x_n]$, with $f_i \neq 0$, and let $F = \{f_1, \ldots, f_s\}$. We say that f reduces to h modulo F, denoted $f \xrightarrow{F}{}_+ h$, if and only if there exist a sequence of indices $i_1, i_2, \ldots, i_t \in \{1, \ldots, s\}$ and a sequence of polynomials $h_1, \ldots, h_{t-1}$ such that

$$f \xrightarrow{f_{i_1}} h_1 \xrightarrow{f_{i_2}} h_2 \xrightarrow{f_{i_3}} \cdots \xrightarrow{f_{i_{t-1}}} h_{t-1} \xrightarrow{f_{i_t}} h.$$

A polynomial r is called reduced with respect to a set of non zero polynomials $F = \{f_1, \ldots, f_s\}$ if $r = 0$ or no power product that appears in r is divisible by any one of the $\text{lp}(f_i)$, $i = 1, \ldots, s$.

If $f \xrightarrow{F}_+ r$ and r is reduced with respect to F, then we call r a remainder for f with respect to F. Note that r is not unique in general. The reduction process allow us to define a division algorithm that mimics the Division Algorithm in one variable. Given f and a family of non zero polynomials $\{g_i \in \mathbb{K}[x_1, \ldots, x_n] \,:\, g_i \neq 0\}_{i=1}^s$, this algorithm returns quotients $u_1, \ldots, u_s \in \mathbb{K}[x_1, \ldots, x_n]$ and a remainder $r \in \mathbb{K}[x_1, \ldots, x_n]$, such that $f = u_1 g_1 + \cdots + u_s g_s + r$.

The division algorithm

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INPUT:An order on T^n and f, g_1, ..., g_s ∈ K[x_1, ..., x_n] with g_i ≠ 0(1 ≤ i ≤ s)
OUTPUT: u_1, ..., u_s, r such that f = u_1 g_1 + ··· + u_s g_s + r and
r is reduced with respect to {g_1, ..., g_s} and
max(lp(u_1) lp(g_1), ..., lp(u_s) lp(g_s), lp(r)) = lp(f).
INITIALIZATION: u_1 = 0, u_2 := 0, ..., u_s := 0, r := 0, h := f
WHILE h ≠ 0 DO
IF there exists i such that lp(g_i) divides lp(h) THEN
choose i least such that lp(g_i) divides lp(h)
u_i := u_i + lt(h)/lt(g_i)
h := h - (lt(h)/lt(g_i)) g_i
ELSE
r := r + lt(h)
h := h - lt(h)
ENDIF
ENDWHILE
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Theorem 9 (Finiteness of the division algorithm). *If f and g_i are homogeneous element of I (with respect to the homological degree) then the division algorithm with respect to $\leq_{opglex}$ terminates in a finite number of steps.*

Proof. Write $f_0 = f$, at each step the reduction $f_k \xrightarrow{g_j} f_{k+1}$ with f_k and g_j homogeneous (w.r.t. homological degree) gives f_{k+1} homogeneous with $\mathrm{hdeg}(f_{k+1}) = \mathrm{hdeg}(f_k)$ and $\mathrm{lp}(f_{k+1}) \leq_{\mathrm{opglex}} \mathrm{lp}(f_k)$. Then $||\,\mathrm{lp}(f_k)|| \leq ||\,\mathrm{lp}(f_{k+1})||$. Obviously, there are only a finite number of power products X in the same homological degree as f. Hence the algorithm terminates in a finite number of steps. With the same proof as in [1, Theorem 1.5.9] we conclude that $f = u_1g_1+\cdots+u_sg_s+r$, where r is reduced with respect to $\{g_1, \ldots, g_s\}$ and that $\mathrm{lp}(f) = \max\left(\mathrm{lp}(u_1)\,\mathrm{lp}(g_1), \ldots, \mathrm{lp}(u_s)\,\mathrm{lp}(g_s), \mathrm{lp}(r)\right)$. ■

Definition 10. *Let $f, g \in \mathbb{K}[x_1, \ldots, x_n]$ be non zero polynomials. Let $L = \mathrm{lcm}(\mathrm{lp}(f), \mathrm{lp}(g))$. The polynomial $S(f, g) = \dfrac{L}{\mathrm{lt}(f)}f - \dfrac{L}{\mathrm{lt}(g)}g$ is called the S-polynomial of f and g.*

Definition 11. *For a subset S of $\mathbb{K}[x_1, \ldots, x_n]$, we define the* leading term ideal *of S to be the ideal $\mathrm{lt}(S) =< \mathrm{lt}(s) \mid s \in S >$.*

Definition 12. *A set of non zero polynomials $G = \{g_1, \ldots, g_t\}$ contained in an ideal I, is called a Groebner (or standard) basis for I if and only if $\mathrm{lt}(G) = \mathrm{lt}(I)$.*

In other words, for all $f \in I$ such that $f \neq 0$, there exists $i \in \{1, \ldots, t\}$ such that $lp(g_i)$ divides $\mathrm{lp}(f)$. A set G of non zero polynomials it is called a Groebner basis if it is a Groebner basis of $< G >$.

Theorem 13 (Buchberger's Criterion). *Let $G = \{g_i\}_{i=1}^{s}$ be a set of non zero polynomials in $\mathbb{K}[x_1, \ldots, x_n]$. Then G is a Groebner basis with respect to $\leq_{opglex}$ if and only if for all $i \neq j$, $S(g_i, g_j) \xrightarrow{G}_+ 0$.*

Proof. The classical proof [1, Theorem 1.7.4] use the well-ordering property to establish that we can choose an expression $f = \sum_{i=1}^{t} h_i g_i$ with $X = \max_{1 \le i \le s}(\mathrm{lp}(h_i)\,\mathrm{lp}(g_i))$ least. Observe that $\mathrm{lp}(f) \le_{\mathrm{opglex}} X \Rightarrow ||X|| \le ||\,\mathrm{lp}(f)||$. And clearly, $\{Z \in \mathbb{T}^n$ such that $||Z|| \le ||\,\mathrm{lp}(f)||\}$ is a finite set, hence it has a least element. So we have established the existence of such an expression. The rest of the proof is the same as in the classical proof. This proves the theorem. ∎

2.2 Buchberger's algorithm for computing Groebner basis

Let $F = \{f_1, \dots, f_r\} \subset \mathbb{K}[x_1, \dots, x_n]$ with $f_i \neq 0$. The following algorithm will produce a Groebner basis $G = \{g_1, \dots, g_s\}$ for the ideal $< f_1, \dots, f_r >$.

Buchberger's Algorithm

Initialization: $G := F$, $\mathcal{G} := \{\{f_i, f_j\} \mid f_i \neq f_j \text{ and } (f_i, f_j) \in G \times G\}$
While $\mathcal{G} \neq \emptyset$ **Do**
Choose any $\{f, g\} \in \mathcal{G}$
$\mathcal{G} := \mathcal{G} \setminus \{\{f, g\}\}$
$S(f, g) \xrightarrow{G}_+ h$, where h is reduced with respect to G.
If $h \neq 0$ **then**
$\mathcal{G} := \mathcal{G} \cup \{\{u, h\} \mid \text{ for all } u \in G\}$
$G := G \cup \{h\}$

Theorem 14 (Finiteness of Buchberger Algorithm for the $\le_{\mathbf{opglex}}$ order). *Buchberger's Algorithm with the $\le_{opglex}$ order terminates in a finite number of steps and provides a Groebner basis of I with respect to the $\le_{opglex}$ order.*

Proof. At each stage, if there is $h \neq 0$, the ideal $\mathrm{lt}(G \cup h)$ is strictly larger that $\mathrm{lt}(G)$, so by the noetherian property of $\mathbb{K}[x_1, \dots, x_n]$ after finitely many steps it has to be all $h = 0$, and at this stage, by Buchberger's criterion G is a Groebner Basis. ∎

The following theorem provides a usefull characterization of a Groebner basis.

Theorem 15. *Let $G = \{g_i\}_{i=1}^s$ be a set of non zero polynomials in $\mathbb{K}[x_1, \dots, x_n]$. Then G is a Groebner basis with respect to $\le_{opglex}$ if and only if for all $f \in \mathbb{K}[x_1, \dots, x_n]$, the remainder r of the division of f by G is unique.*

Proof. The proof is the same as in the standard case [1, Theorem 1.6.7] ∎

If G is a Groebner basis then the remainder r is called the normal form of f with respect to G, and we write $r = \mathrm{NF}_G(f)$. As an immediate consequence of Theorem 15, if $G \subset \mathbb{K}[x_1, \dots, x_n]$ is a Groebner basis and $f_1, f_2 \in \mathbb{K}[x_1, \dots, x_n]$ then $f_1 - f_2 \in < G > \Leftrightarrow \mathrm{NF}_G(f_1) = \mathrm{NF}_G(f_2)$.

Theorem 16. *Let $(\Lambda V, d)$ be a pure elliptic model with $\chi_\pi = 0$, $w \in \Lambda X$ be a cocycle representing the top class of $(\Lambda V, d)$ and G a Groebner basis of $< dy_1, \dots, dy_n >$ with respect to $\le_{opglex}$. Then the reduction $w \xrightarrow{G}_+ w_0$ gives $w_0 \in \Lambda^k X$, where $k = cat_0(\Lambda V, d)$.*

Proof. Observe that

i) $(w \xrightarrow{G}_+ w_0) \Longrightarrow ||\,\mathrm{lt}(w)|| \le ||\,\mathrm{lt}(w_0)||$

ii) $||\,\mathrm{lt}(w)|| = l \Longleftrightarrow w \in \Lambda^{\ge l}X$ and $w \notin \Lambda^{\ge l+1}X$.

By Lemma 1 there is a top class $w_1 \in \Lambda^{\ge k}X$ with $k = \mathrm{cat}_0(\Lambda V, d)$. Let w_0 be then the normal form of w_1 with respect to G then by (i) $w_0 \in \Lambda^{\ge k}X$. Now, let w be any cocycle that represents the top class, then by Theorem 15 $w - w_1 \in G \Rightarrow \mathrm{NF}_G(w) = \mathrm{NF}_G(w_1) = w_0$. Finally, observe that w_0 is a monomial because if $w_0 = t_1 + t_2$ with t_1 and t_2 in ΛX and reduced with respect to G then both t_1 and t_2 are cocycles representing the top class so that there are $\lambda \in \mathbb{K}\backslash\{0\}$ and $f \in < dy_1, \ldots, dy_n >$ such that $t_1 = \lambda t_2 + f$, hence $t_1 = \mathrm{NF}_G(t_1) = \mathrm{NF}_G(\lambda t_2 + f) = \mathrm{NF}_G(\lambda t_2) = \lambda \mathrm{NF}_G(t_2) = \lambda t_2$ and this implies $w_0 \in \Lambda^k X$. ■

As an immediate consequence we obtain the following algorithm for computing $\mathrm{cat}_0(\Lambda V, d)$ for S a pure elliptic spaces with positive Euler characteristic.

Algorithm for computing the rational category

Input: $n = \dim X$ and $\{dy_1, \ldots, dy_n\}$
Output: The rational category of the pure elliptic model $(\Lambda V, d)$ with $\chi_\pi = 0$.

Initialization: Let $I = < dy_1, \ldots, dy_n >$, this is the coboundary ideal.
Apply Proposition 2 to compute a cocycle $w \in \Lambda X$ that represents the fundamental class.
Apply Buchberger's Algorithm to compute a Groebner basis G for I with respect to $\le_{\mathrm{opglex}}$.
Apply the division algorithm to obtain the normal form $w_0 = \mathrm{NF}_G(w)$.
Compute $k = ||w_0||$.
Then k is the rational category of $(\Lambda V, d)$.

Example 17. *Let S be the rational space whose minimal model $(\Lambda V, d)$ is given by $V^{even} = \{x_1, x_2, x_3\}$, $V^{odd} = \{y_1, y_2, y_3\}$ with $|x_i| = 2i$, $y_i = 12$ for $i = 1, 2, 3$, and the differential given by*

$$\begin{aligned} dy_1 &= x_1^6 + 36x_2^3 + 28x_1x_2x_3 + 35x_3^2 \\ dy_2 &= 26x_2^3 + 11x_1x_2x_3 + 44x_3^2 \\ dy_3 &= 10x_2^3 + 14x_1x_2x_3 + 39x_3^2 \end{aligned}$$

First, apply Proposition 2 (and construct ψ_i by taking the suspension of the x_i with greatest degree) and obtain a representative of the top class

$$w = 254x_1^6x_2^3 + 574x_1^5x_2^2x_3$$

Then we apply Buchberger's algorithm to obtain a Groebner basis of I given by $G = x_1x_2x_3 + 287/6957x_1^6$, $x_1^6x_3 + 127/287x_1^7x_2$, $x_1^6x_2^2 + 23749/164738x_1^8x_2$, $x_1^9x_2 + \frac{13569304322}{20983167711}x_1^{11}$, x_1^{13}, $x_3^2 - \frac{127}{6957}x_1^6$, $x_2^3 + \frac{187}{13914}x_1^6$.

Finally, we reduce w with respect to G and the normal form is given by $\dfrac{1737050805429097}{145979897765427}x_1^{12}$, hence $cat_0(S) = 12$.

Remark 18. *Standard software for computing Groebner Basis are CoCoA [3] and Macaulay2. The above Groebner Basis was calculated with version 4.1 of the computer algebra system CoCoA. This is the input:*

```
Use R::=Q[xyz],DegLex;
Opposite:= - Ord(R);
Use R::=Q[xyz], Ord(Opposite);
I:=Ideal(x^6 + 36y^3 + 28x y z + 35z^2,
         26 y^3 + 11x y z + 44z^2 ,
         10 y^3 + 14x y z + 39z^2);
 ReducedGBasis(I);
 NF(254 x^6 y^3+ 574 x^5 y^2 z,I);
```

Remark 19. *If* $cat_0(\Lambda V, d) = \infty$ *then Proposition 2 provides* w *such that* $[w] = 0$*, hence* w *reduces to 0. This is because by [14]* w *is in the image of the evaluation map,* $ev_{(\Lambda V,d)}$*, and by [15, Theorem 3.2] image of* $ev_{(\Lambda V,d)}$ *is non zero if and only if* $(\Lambda V, d)$ *is elliptic. See [12] for applications of this result.*

3 $\mathbf{cat}_0(\Lambda V, d)$ for pure elliptic spaces

We generalize the above to any pure elliptic space $(\Lambda V, d)$. We shall apply reduction and computation of Groebner bases for R-modules.

Write $A = \mathbb{K}[x_1, \ldots, x_n]$, $M = \Lambda X \otimes \Lambda^{m-n} Y$, $p = m - n$. Let $B \subset M$ be the A-module of coboundaries in M then B is an A-submodule of M, and B is generated by $\{d(y_{i_1} \cdot y_{i_2} \ldots \cdot y_{i_{p+1}}) \ : \ 1 \leq i_1 < i_2 \cdots < i_{p+1} \leq m\}$.

3.1 A Groebner basis for the submodule of coboundaries

We consider $\{e_k\}_{k=1}^q = \{y_{i_1} \cdot y_{i_2} \ldots \cdot y_{i_p} \ : \ 1 \leq i_1 < i_2 < \ldots < i_p \leq m\}$ a basis of the A-module M. By Hilbert's basis theorem M is a noetherian A-module.

Then by a *monomial* in M we mean a vector of the type Xe_1 $(1 \leq i \leq q)$ where X is a power product in A. If Xe_i and Ye_j are monomials in M, we say that Xe_i divides Ye_j provided that $i = j$ and X divides Y. In this case we define $\dfrac{Xe_i}{Ye_i} = \dfrac{X}{Y}$. Similarly, by a *term*, we mean a vector of the type $c\boldsymbol{X}$ where $c \in k$, and $\boldsymbol{X}$ is a monomial.

Definition 20. *By a* term order *on the monomials of* M *we mean a total order* $<$ *on these monomials satisfying the following two conditions:*

(i) $\boldsymbol{X} < Z\boldsymbol{X}$*, for every monomial* $\boldsymbol{X}$ *of* M *and power product* $Z \neq 1$ *of* A*.*

(ii) *If* $\boldsymbol{X} < \boldsymbol{Y}$*, then* $Z\boldsymbol{X} < Z\boldsymbol{Y}$ *for all monomials* $\boldsymbol{X}, \boldsymbol{Y} \in M$ *and every power product* $Z \in A$*.*

Definition 21. *For monomials* $\boldsymbol{X} = Xe_i$ *and* $\boldsymbol{Y} = Ye_j$ *of* M*, we define* $\boldsymbol{X} \leq_{top} \boldsymbol{Y} \iff (X \leq_{opglex} Y)$ *or* $(X = Y$ *and* $i < j)$

As for the $\leq_{\text{opglex}}$ order, the $\leq_{\text{top}}$ order is neither a term order nor a well-order, but again it is a total order and verifies that $x^\alpha \leq_{\text{top}} x^\beta$ then for all $\gamma \in \mathbb{N}^n$, $x^\alpha \cdot x^\gamma \leq_{\text{top}} x^\beta \cdot x^\gamma$. Since $\leq_{\text{top}}$ is not a well-ordering, we will have to modify the proof of some of the following theorems. The key fact is that B is homogeneous (w.r.t homological degree) and finite dimensional in each degree. We will prove that the the division algorithm and Buchberger's algorithm terminate in a finite number of steps.

Definition 22. *For all $f \in M$, with $f \neq 0$, me may write $f = a_1\boldsymbol{X}_1 + a_2\boldsymbol{X}_2 + \cdots + a_r\boldsymbol{X}_r$, where $a_i \in \mathbb{K}\setminus\{0\}$ for $1 \leq i \leq r$ and $\boldsymbol{X}_i$ are monomials in M satisfying $\boldsymbol{X}_r \leq_{top} \boldsymbol{X}_{r-1} \leq_{top} \cdots \leq_{top} \boldsymbol{X}_1$. We define*

- $\mathrm{lm}(\boldsymbol{f}) = \boldsymbol{X}_1$*, the leading monomial of* $\boldsymbol{f}$*;*
- $\mathrm{lc}(\boldsymbol{f}) = a_1$*, the leading coefficient of* $\boldsymbol{f}$*;*
- $\mathrm{lt}(\boldsymbol{f}) = a_1\boldsymbol{X}_1$*, the leading term of* $\boldsymbol{f}$*.*

We define $\mathrm{lt}(\mathbf{0}) = 0$, $\mathrm{lm}(\mathbf{0}) = 0$, and $\mathrm{lc}(\mathbf{0}) = 0$.

Now that we have defined monomials (in place of power products), divisibility and quotients of monomials, and term orders, the definitions of reduction, reduced vector, remainder, and the division algorithm can be lifted word for word as above. The basic idea behind the algorithm is the same as for polynomials: when dividing $\boldsymbol{f}$ by $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s$, we want to cancel monomials of $\boldsymbol{f}$ using the leading terms of the $\{\boldsymbol{f}_i\}$, and continue this process until it cannot be done anymore.

Theorem 23. *If $\boldsymbol{f}$ and $\boldsymbol{f}_i$, $i = 1, \ldots, s$ are homogeneous elements of M with respect to the homological degree. Then the division of $\boldsymbol{f}$ by $\{\boldsymbol{f}_i\}_{i=1}^s$ with respect to $\leq_{top}$ terminates in a finite number of steps.*

Proof. The proof is the same as that for Theorem 9. ∎

For a subset V of M, the leading term module of V is the submodule of M given by $\mathrm{lt}(W) =< \mathrm{lt}(\boldsymbol{w}) | \boldsymbol{w} \in W >$.

Definition 24. *A set on non zero vectors $G = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_t\}$ contained in the submodule $C \subset M$ is called a Groebner basis for C if and only if $\mathrm{lt}(G) = \mathrm{lt}(C)$. We say that the set G is a Groebner basis provided G is a Groebner basis for the submodule, $< G >$, it generates.*

Theorem 25. *Let $G = \{\boldsymbol{g}_i\}_{i=1}^s$ be a set of non zero vectors in M. Then G is a Groebner basis with respect to $\leq_{top}$ if and only if for all $f \in M$, the remainder $\boldsymbol{r}$ of the division of $\boldsymbol{f}$ by G is unique.*

Proof. The proof is the same as in the classical proof [1, Theorem 3.5.14]. ∎

We introduce the analog of S-polynomial. Let $\boldsymbol{X} = Xe_i$ and $\boldsymbol{Y} = Ye_j$ be two monomials in M. Then by the *least common multiple* of $\boldsymbol{X}$ and $\boldsymbol{Y}$ (denoted $\mathrm{lcm}(\boldsymbol{X}, \boldsymbol{Y})$, we mean $\mathrm{lcm}(X, Y)e_i$, if $i = j$ or $\mathbf{0}$ otherwise.

Definition 26. *Let* $\mathbf{0} \neq \boldsymbol{f}, \boldsymbol{g} \in M$. *Let* $\boldsymbol{L} = \mathrm{lcm}(\mathrm{lm}(\boldsymbol{f}), \mathrm{lm}(\boldsymbol{g}))$. *Then the vector* $S(\boldsymbol{f}, \boldsymbol{g}) = \dfrac{\boldsymbol{L}}{\mathrm{lt}(\boldsymbol{f})}\boldsymbol{f} - \dfrac{\boldsymbol{L}}{\mathrm{lt}(\boldsymbol{g})}\boldsymbol{g}$ *is called the S-polynomial of* $\boldsymbol{f}$ *and* $\boldsymbol{g}$.

Remark 27. *Observe that the reduction and the* S*-polynomial of two homogeneous element (with respect to the homological degree) is a homogeneous element.*

Theorem 28 (Buchberger's criterion). *Let* $G = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_t\}$ *be a set of non zero vector in* M. *Then* G *is a Groebner basis with respect to* $\leq_{top}$ *if and only if for all* $i \neq j$, $S(\boldsymbol{g}_i, \boldsymbol{g}_j) \xrightarrow{G}_+ \mathbf{0}$.

Proof. The proof is analogous to that of Theorem 13. ∎

This last Theorem allow us to give the analog of Buchberger's Algorithm for computing Groebner basis. This algorithm is exactly the same as above (with the new definitions).

Theorem 29. $G = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_t\}$ *be a set of homogeneous (with respect to the homological degree) non zero vector in* M. *Then Buchberger's Algorithm for* G *with the* $\leq_{top}$ *order terminates in a finite number of steps and gives a Groebner basis of the module generated by* G.

Proof. Observe that, if at stage i we denote by G_i the partial Groebner basis computed so far, then $\mathrm{lt}(G_i)$ is an ascending chain of submodules. Hence, by the noetherian property of M, the algorithm finitely terminates. Now, Buchberger's criterion implies that the algorithm terminates giving G a Groebner basis M. ∎

Theorem 30. *Let* G *be a Groebner basis for the coboundary module* B *with respect to top,* $w \in \Lambda_{m-n}V$ *be a cocycle that represents the top class and* $w_0 = NF_G(w)$. *Then* $cat_0(\Lambda V, d) = m - n + ||\,\mathrm{lt}(w_0)||$.

Proof. Let $p = \max\{l \,|\, \exists w \in \Lambda^{\geq l}X \otimes \Lambda^{m-n}Y$, and $[w]$ is a top class$\}$, then $\mathrm{cat}_0(\Lambda V, d) = m - n + p$. Observe that

i) $(w \xrightarrow{G}_+ w_0) \Longrightarrow ||\,\mathrm{lt}(w)|| \leq ||\,\mathrm{lt}(w_0)||$

ii) $||\,\mathrm{lt}(w)|| = l \Longleftrightarrow w \in \Lambda^{\geq l}X \otimes \Lambda^{m-n}Y$ and $w \notin \Lambda^{\geq l+1}X \otimes \Lambda^{m-n}Y$.

By Lemma 1 there is a cocycle $w_1 \in \Lambda^{\geq l}X \otimes \Lambda^{m-n}Y$ such that $[w_1]$ is a top class and $l + m - n = \mathrm{cat}_0(\Lambda V, d)$. Then $w_0 = \mathrm{NF}_G(w_1) \in \Lambda^{\geq l}X \otimes \Lambda^{m-n}Y$ and $\mathrm{cat}_0(\Lambda V, d) = m - n + ||\,\mathrm{lt}(w_0)||$. Finally, let $w \in \Lambda_{m-n}V$ be any cocycle that represents the top class. Then $w - w_1 \in G \Rightarrow \mathrm{NF}_G(w) = \mathrm{NF}(w_1) = w_0$ hence $\mathrm{cat}_0(\Lambda V, d) = m - n + ||\,\mathrm{lt}(w_0)||$. ∎

As an immediate consequence we obtain the following algorithm for computing $\mathrm{cat}_0(\Lambda V, d)$ for pure elliptic spaces.

Algorithm for computing the rational category
Apply Proposition 2 to compute $w \in \Lambda_{m-n}V$ that represents the top class. Compute the generators S (as above) of the coboundary module B. Apply Buchberger's algorithm to obtain a Groebner basis G for S with respect to $\leq_{\text{top}}$. Obtain the normal form w_o of w with respect to G by the division algorithm. Compute $l = \|\operatorname{lt}(w_0)\|$. Then $k = m - n + l$ is the rational category of the space.

Example 31. *Let $(\Lambda V, d)$ be the minimal model given by $X = X^2 =< x_1, \ldots, x_4 >$, $Y =< y_1, \ldots, y_6 >$ and the differential*

$$\begin{array}{lll} dy_1 = x_1x_4 & dy_2 = x_2^2x_3 & dy_3 = x_2^4 \\ dy_4 = x_1x_4^4 & dy_5 = x_1^7 + x_3^7 & dy_6 = x_4^7 \end{array}$$

Then a representative of the top class, w, is given by

$$w = x_2x_3^8x_4^6y_3y_4 - x_2^3x_3^7x_4^6y_2y_4 + x_2x_3^{11}x_4^6y_1y_3 - x_2^3x_3^{10}x_4^6y_1y_2.$$

The module M has 20 generators. The Groebner Basis of M has 34 elements. The reduction of w with respect to M gives again $w_0 = w$.
Now $p = \|\operatorname{lt}(w_0)\| = 15$, so that $cat_0(\Lambda V, d) = p + m - n = 17$.

Remark 32. *For the computation of the rational category we only need the reduction of w representing the top class with respect to a Groebner basis of coboundaries. Thus, it suffices to compute a truncate Groebner basis, that is, during the computation of Buchberger's algorithm the element with homological degree greater than the formal dimension are discarded.*

3.2 Final remark and open problems

The computation of a Groebner basis can be computationally very expensive (there are double exponential worst case bounds)[2]. But, in practice, the algorithm performs quite well for many problems. In [10] is given an algorithm that is not a polynomial time algorithm, but performs much better (in computation time) than this one, the tradeoff is that it only provides a lower bound for the rational category. On the other hand, in [11] is proven that the computation of the rational category is an NP-hard problem even for the restricted class of formal pure spaces. Since this result cannot be applied to determine the algorithmic complexity of computing the rational category of a pure elliptic space, this remains an open problem.

For non pure space, in [12] is given an algorithm for computing a cocycle that represents the top class. A straightforward modification of our algorithm with a stage by stage optimization yields only a lower bound for $\operatorname{cat}_0(\Lambda V, d)$. So, as above, we ask: what is the algorithmic complexity of computing the rational category of any elliptic space?. Is there a "good" approximate algorithm for this problem?.

I thank the referee for very useful suggestions which improved the presentation of this paper.

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