High Singer invariant and equality of curvature

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Abstract

The author proves, by giving explicit examples, that the Singer invariant of a locally homogeneous Riemannian manifold can become arbitrarily high. In a second step it is shown that for each $k \in \mathbb{N}$ there exist pairs of nonisometric homogeneous Riemannian manifolds of Singer invariant k which have the same curvature up to order k.

1 Introduction

Let (M, g) be a locally homogeneous Riemannian manifold and let $V = T_p M$ be the tangent space at a point p which, by local homogeneity, can be chosen arbitrarily. Furthermore, let $\mathfrak{so}(V)$ be the Lie algebra of skew-symmetric endomorphisms of (V, g_p). For $k \in \mathbb{N}_0$ we define Lie subalgebras $\mathfrak{g}(k)$ of $\mathfrak{so}(V)$ by

$$\mathfrak{g}(k) := \{\mathcal{P} \in \mathfrak{so}(V) \mid \mathcal{P} \cdot R_p = \mathcal{P} \cdot (\nabla R)_p = \ldots = \mathcal{P} \cdot (\nabla^k R)_p = 0\}, \ k \geq 0.$$

Here $(\nabla^k R)_p$ is the value at p of the k-th covariant derivative of the curvature tensor R, and the endomorphism $\mathcal{P}$ acts on the tensor algebra of V as a derivation (see [1], p.25). By definition, $\nabla^0 R = R$. We speak of $\mathfrak{g}(k)$ as the stabiliser of the k-th covariant derivative of the Riemannian curvature tensor.

Definition 1.1. *The **Singer invariant** k_g of a locally homogeneous Riemannian manifold (M, g) is defined by*

$$k_g := min\{k \in \mathbb{N} \mid \mathfrak{g}(k) = \mathfrak{g}(k+1)\}.$$

Only for a few examples the Singer invariant has been calculated and it has been an open question whether the Singer invariant can be higher than 1 (see [4]). Presenting a class of Lie groups with left invariant metric with arbitrarily high Singer invariant will be the first result in our paper.

The second question concerns the equality of curvature, that is, whether there exist two nonisometric homogeneous manifolds (M, g), $(\widetilde{M}, \tilde{g})$ and a linear isometry $F \colon (T_p M, g_p) \longrightarrow (T_{\tilde{p}}\widetilde{M}, \tilde{g}_{\tilde{p}})$ which preserves the curvature tensor up to a certain order k. The two problems are related by the following theorem of L. Nicolodi and F. Tricerri (see [3]):

Theorem 1.2. *Let (M, g) and $(\tilde{M}, \tilde{g})$ be two Riemannian manifolds. Suppose that $(\tilde{M}, \tilde{g})$ is locally homogeneous. If for each point p of M there exists an isometry $F \colon T_p M \longrightarrow T_o \tilde{M}$ ($o \in \widetilde{M}$ is supposed fixed) such that*

$$F^* \nabla^s \tilde{R}_{|o} = \nabla^s R_{|p}$$

for $0 \leq s \leq k_{\tilde{g}} + 1$, then (M, g) is locally homogeneous and locally isometric to $(\tilde{M}, \tilde{g})$.

In [2] F. Lastaria discussed families of nonisometric homogeneous manifolds with Singer invariant 1 which have the same curvature. The author improves the result by constructing pairs of nonisometric homogeneous manifolds which have the same curvature up to their (arbitrarily high) Singer invariant.

In section 2, a class of two-step solvable Lie algebras is introduced and basic formulas are deduced. For a certain subclass the Singer invariant will be calculated in section 3. Section 4 discusses the question of curvature equality. Finally, in section 5, the isometry classes are determined.

2   A class of two-step solvable Lie algebras

Let n be the dimension of the real Lie algebra V, $\{a, X_2, \ldots, X_n\}$ an orthonormal basis of left invariant vector fields and $V' := span\{X_2, \ldots, X_n\}$. We define the Lie bracket by

$$[X_i, X_j] = 0 \qquad [a, X_i] = A(X_i) \qquad i, j = 2, \ldots n,$$

with A being a linear endomorphism of V'. Without changing the notation, we will sometimes consider A as an endomorphism of V by setting $A(a) = 0$. For the following discussion, it is helpful to set $A = D + S$ with D being the symmetric and S the skew-symmetric part of A. In the following, X, Y will always denote elements of V'.

A straightforward calculation gives the Levi-Civita connection

$$\nabla_a = S, \tag{2.1}$$
$$\nabla_X = DX \wedge a, \tag{2.2}$$

the Riemannian curvature tensor

$$
\begin{aligned}
R_{a,X} &= a \wedge (D^2 + [D,S])X, & (2.3) \\
R_{X,Y} &= DX \wedge DY & (2.4)
\end{aligned}
$$

and the Ricci tensor

$$
\begin{aligned}
Ric(a,a) &= -tr(D^2), & (2.5) \\
Ric(a,X) &= 0, & (2.6) \\
Ric(X,Y) &= -\langle [D,S]X, Y\rangle - tr(D)\langle DX, Y\rangle & (2.7)
\end{aligned}
$$

and hence, in particular:

$$
Ric(X,X) = -2\langle SX, DX\rangle - tr(D)\langle DX, X\rangle. \qquad (2.8)
$$

For the covariant derivative of the curvature we introduce the following notation: for $\ell \in \mathbb{N}_0$, let

$$
N_\ell := \underbrace{[S, [\ldots, [S, D]\ldots]]}_{\ell \text{ times}},
$$

and let $\mathcal{M}$ be the set of finite sums of finite products of the form $N_{\ell_1} \cdot \ldots \cdot N_{\ell_m}$. We define the weight of $M = N_{\ell_1} \cdot \ldots \cdot N_{\ell_m}$ as the greatest ℓ_i and the weight of the sum of such terms as the greatest weight of the factors.

Proposition 2.1. *(i) Let $Y_1, \ldots, Y_r \in V'$, $r \in \mathbb{N}_0$, $k \in \mathbb{N}_0$, c_0, $\ldots$, $c_r \in \mathbb{N}_0$ with $\sum_{i=0}^r c_i + r = k+2$. The k-th covariant derivative of the curvature $(\nabla^k R)(\underbrace{a, \ldots, a}_{c_0}, Y_1, \underbrace{a, \ldots, a}_{c_1}, Y_2, \ldots, Y_r, \underbrace{a, \ldots, a}_{c_r})$ is a finite sum of terms of the form:*

$$
\pm \langle M_1 Y_{\sigma(1)}, Y_{\sigma(2)}\rangle \cdot \ldots \cdot \langle M_{\frac{r-1}{2}} Y_{\sigma(r-2)}, Y_{\sigma(r-1)}\rangle a \wedge M_{\frac{r+1}{2}} Y_{\sigma(r)}
$$

$$
\text{if } r \text{ is odd and}
$$

$$
\pm \langle M_1 Y_{\sigma(1)}, Y_{\sigma(2)}\rangle \cdot \ldots \cdot \langle M_{\frac{r-2}{2}} Y_{\sigma(r-3)}, Y_{\sigma(r-2)}\rangle M_{\frac{r}{2}} Y_{\sigma(r-1)} \wedge M_{\frac{r+2}{2}} Y_{\sigma(r)}
$$

$$
\text{if } r \text{ is even,}
$$

with $M_i \in \mathcal{M}, \sigma \in \Sigma_r$.
The greatest weight of the M_i is less than or equal to $k+1$.
(ii) For each $k \geq 0$ and each $X \in V'$ we get:

$$
\begin{aligned}
(\nabla_a^k R)_{a,X} &= a \wedge \underbrace{[S, [\ldots, [S, D^2 - [S,D]]\ldots]]}_{k \text{ times}} X \\
&= a \wedge (-N_{k+1} + M)X,
\end{aligned}
$$

with $M \in \mathcal{M}$ of weight k.

Proof. Induction over k yields the first statement and also the first part of the second statement. The last part is immediate from the definition of N_{k+1} and $\mathcal{M}$. ∎

3   Manifolds with high Singer invariant

We now specialize for the rest of the article on a particular subclass of the Lie algebras from the previous section. We discuss Lie algebras where the symmetric part D on V' has two different eigenvalues a_1 and a_2, the first with multiplicity $(n-2)$ and the other with simple multiplicity. Furthermore, let $a_1 \neq a_2$, $a_1 \neq 0 \neq a_2$ and $a_1 \cdot a_2 > 0$. Let the dimension n of the Lie algebra be at least 4.

We will assume now, that there exists an orthonormal basis $\{a, X_2, \ldots, X_n\}$ as above so that on V' the symmetric part D and the skew-symmetric part S of the linear map A on the basis $\{X_2, \ldots, X_n\}$ of V' are of the form

$$
\mathbf{D} = \begin{pmatrix} a_1 & & & & \\ & a_1 & & 0 & \\ & & \ddots & & \\ & 0 & & a_1 & \\ & & & & a_2 \end{pmatrix}, \quad
\mathbf{S} = \begin{pmatrix} 0 & b_1 & & & \\ -b_1 & 0 & b_2 & & \\ & -b_2 & 0 & \ddots & \\ & & \ddots & \ddots & b_{n-2} \\ & & & -b_{n-2} & 0 \end{pmatrix}.
$$

Lie algebras of this type will be written as (V, D, S).

Theorem 3.1. *Let $\mathcal{G}$ be the simply connected Lie group with left invariant metric associated to (V, D, S). The Singer invariant k_g of $\mathcal{G}$ is as follows:*

$$
\begin{aligned}
k_g &= n - 4 \quad if \quad b_1, \ldots, b_{n-2} \neq 0, \\
k_g &= k \quad if \quad b_{n-k-3} = 0 \quad and \quad b_{n-k-2}, \ldots, b_{n-2} \neq 0, \ 0 \leq k \leq n - 4, \\
k_g &= 0 \quad if \quad b_{n-2} = 0.
\end{aligned}
$$

Corollary 3.2. *The Singer invariant is unbounded.*

For the proof, we will first consider the stabilizer of the Ricci tensor. Afterwards, we will show that - apart from a special case - the stabilizer of the Ricci tensor is the same as the stabilizer of the Riemannian curvature tensor. In the following we will calculate the stabilizers of the covariant derivatives of the curvature tensor.

Lemma 3.3. *The stabilizer of the Ricci tensor $stabRicci \subset \mathfrak{so}(V)$ is:*
if $b_{n-2} = 0$ and $(n-2)a_1^2 = (n-3)a_1a_2 + a_2^2$, then $stabRicci = \mathfrak{so}(\{a\}^\perp)$; otherwise $stabRicci = \mathfrak{so}(\{a, SX_n, X_n\}^\perp)$.

Proof. Since

$$
\begin{aligned}
Ric(X, Y) &= -\langle [D, S]X, Y \rangle - tr(D)\langle DX, Y \rangle \\
&= -((n-2)a_1 + a_2)a_1 \langle X, Y \rangle \quad X, Y \in \{X_2, \ldots, X_{n-2}\}, \\
Ric(a, X) &= 0 \quad X \in \{X_2, \ldots, X_n\}
\end{aligned}
$$

and $Ric(X_i, X_n) = Ric(X_i, X_{n-1}) = 0, \quad i \leq n - 2,$

the Ricci eigenvalues are easily calculated. Using the assumptions on a_1 and a_2, we get:

Case 1: If $b_{n-2} = 0$ and $(n-2)a_1^2 \neq (n-3)a_1a_2 + a_2^2$, then the Ricci tensor has three different eigenvalues and one easily attains $stabRicci = \mathfrak{so}(\{a, X_n\}^\perp)$.

Case 2: If $b_{n-2} = 0$ and $(n-2)a_1^2 = (n-3)a_1a_2 + a_2^2$, then only the eigenvalue of a is different from the others and we have $stabRicci = \mathfrak{so}(\{a\}^\perp)$.

Case 3: If $b_{n-2} \neq 0$, then the Ricci tensor has four different eigenvalues and simple calculations give $stabRicci = \mathfrak{so}(\{a, X_{n-1}, X_n\}^\perp)$. The condition $a_1 \cdot a_2 > 0$ is used to prove, that $-(n-2)a_1^2 - a_2^2$ is a simple eigenvalue.

Finally, $SX_n = b_{n-2}X_{n-1}$ yields the lemma. ∎

Proposition 3.4. *The stabilizer of the Riemannian curvature is*

$$\mathfrak{g}(0) = \mathfrak{so}(\{a, SX_n, X_n\}^\perp).$$

Proof. Obviously we have: $\mathfrak{g}(0) \subset stabRicci$. So we only have to show:
(i) if $\mathcal{P} \in \mathfrak{so}(\{a, SX_n, X_n\}^\perp)$, then $\mathcal{P} \in \mathfrak{g}(0)$, and
(ii) if $stabRicci = \mathfrak{so}(\{a\}^\perp)$ and $\mathcal{P} \in \mathfrak{so}(\{a\}^\perp) \setminus \mathfrak{so}(\{a, X_n\}^\perp)$, then $\mathcal{P} \notin \mathfrak{g}(0)$.
For $\mathcal{P} \in \mathfrak{so}(\{a, SX_n, X_n\}^\perp)$ we calculate $[\mathcal{P}, D] = 0$, $[\mathcal{P}, D^2] = 0$ and $[P, [D, S]] = 0$. For the first claim we compute:

$$
\begin{aligned}
(\mathcal{P}R)(a, X) &= [\mathcal{P}, R_{a,X}] - R_{\mathcal{P}a,X} - R_{a,\mathcal{P}X} \\
&= a \wedge \mathcal{P}[D^2 + [D, S]](X) - a \wedge [D^2, [D, S]]\mathcal{P}(X) \\
&= a \wedge [\mathcal{P}, D^2 + [D, S]](X) \\
&= 0;
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{P}R)(X, Y) &= [\mathcal{P}, R_{X,Y}] - R_{\mathcal{P}X,Y} - R_{X,\mathcal{P}Y} \\
&= \mathcal{P}DX \wedge DY + DX \wedge \mathcal{P}DY \\
&\quad - D\mathcal{P}X \wedge DY - DX \wedge D\mathcal{P}Y \\
&= [\mathcal{P}, D]X \wedge DY + DX \wedge [\mathcal{P}, D]Y \\
&= 0.
\end{aligned}
$$

For the second claim, let $\mathcal{P} \in \mathfrak{so}(\{a\}^\perp) \setminus \mathfrak{so}(\{a, X_n\}^\perp)$. Then there exists a $Y \in V'$ with $Y \perp X_n$ and $Y \perp \mathcal{P}X_n$ (recall that $n \geq 4$, so dim $V' \geq 3$). We have $\mathcal{P}Y \perp X_n$, thus $\mathcal{P}DY = a_1\mathcal{P}Y = D\mathcal{P}Y$ and also $[\mathcal{P}, D]Y = 0$. Finally, $0 \neq \mathcal{P}X_n \in V'$ and

$$
\begin{aligned}
(PR)(X_n, Y) &= [\mathcal{P}, D]X_n \wedge DY + DX_n \wedge [\mathcal{P}, D]Y \\
&= (a_2\mathcal{P}X_n - D\mathcal{P}X_n) \wedge DY \\
&= (a_2 - a_1)\mathcal{P}X_n \wedge a_1Y \neq 0.
\end{aligned}
$$

This is exactly what we wanted to show. ∎

The following proposition simplifies the calculation of the stabilizers of the k-th covariant derivatives of the curvature tensor.

Proposition 3.5. *Let $\mathcal{P} \in \mathfrak{so}(V)$. Then $\mathcal{P} \in \mathfrak{g}(k+1)$ if and only if $\mathcal{P} \in \mathfrak{g}(k)$ and $[\mathcal{P}, \nabla_X] - \nabla_{\mathcal{P}X} \in \mathfrak{g}(k)$ for all $X \in V$.*

Proof. Let $\mathcal{P} \in \mathfrak{g}(k)$ and $X \in V$. Then we have:

$$
\begin{aligned}
(\mathcal{P}(\nabla^{k+1}R))(X, .) &= \mathcal{P}((\nabla^{k+1}R)(X, .)) - (\nabla^{k+1}R)(\mathcal{P}X, .) - (\nabla^{k+1}R)(X, \mathcal{P}(.)) \\
&= \mathcal{P} \circ (\nabla_X(\nabla^k R)) - (\nabla_X(\nabla^k R)) \circ \mathcal{P} - \nabla_{\mathcal{P}X}(\nabla^k R) \\
&= \mathcal{P}(\nabla_X(\nabla^k R)) - \nabla_{\mathcal{P}X}(\nabla^k R) \\
&= ([\mathcal{P}, \nabla_X] - \nabla_{\mathcal{P}X})(\nabla^k R),
\end{aligned}
$$

where in the last equation we use $\mathcal{P} \in \mathfrak{g}(k)$.   ∎

Lemma 3.6. *Let $\mathcal{P} \in \mathfrak{g}(0)$. Then*

$$
\begin{aligned}
[\mathcal{P}, \nabla_X] - \nabla_{\mathcal{P}X} &= 0 \text{ for all } X \in V', \\
[\mathcal{P}, \nabla_a] - \nabla_{\mathcal{P}a} &= [\mathcal{P}, S].
\end{aligned}
$$

In particular, we have for $\mathcal{P} \in \mathfrak{g}(k)$ with proposition 3.5:

$$
\mathcal{P} \in \mathfrak{g}(k+1) \iff [\mathcal{P}, S] \in \mathfrak{g}(k).
$$

Proof. Let $\mathcal{P} \in \mathfrak{g}(0)$. Then:

$$
\begin{aligned}
[\mathcal{P}, \nabla_X] - \nabla_{\mathcal{P}X} &= [\mathcal{P}, DX \wedge a] - D\mathcal{P}X \wedge a \\
&= [\mathcal{P}, D]X \wedge a \\
&= 0, \\
[\mathcal{P}, \nabla_a] - \nabla_{\mathcal{P}a} &= [\mathcal{P}, S].
\end{aligned}
$$

∎

Straightforward inductions yield the following two lemmas.

Lemma 3.7. *For each $k \in \{1, 2, \ldots, n-2\}$ we have:*

$$
S^k X_n = b_{n-k-1} \ldots b_{n-2} X_{n-k} + \text{terms in } X_{n-k+1}, \ldots, X_n.
$$

Lemma 3.8. *If $b_{n-k-1}, \ldots b_{n-2} \neq 0$, then we have for $k \in \{1, 2, \ldots, n-2\}$:*

$$
span\{S^k X_n, \ldots, S X_n, X_n\} = span\{X_{n-k}, \ldots, X_{n-1}, X_n\}.
$$

Proposition 3.9. *The stabilizer of the k-th covariant derivative of the curvature tensor is*

$$
\mathfrak{g}(k) = \mathfrak{so}(\{a, S^{k+1} X_n, \ldots, S X_n, X_n\}^\perp).
$$

Proof. By induction.

For $k = 0$ the claim is proved by proposition 3.4.

Step of induction $k \longrightarrow k+1$: Using lemma 3.6, we have to show that the following statements for $\mathcal{P} \in \mathfrak{so}(\{S^{k+1} X_n, \ldots, X_n\}^\perp)$ are equivalent:

  (i) $\mathcal{P} S^{k+2} X_n = 0$,
  (ii) $[\mathcal{P}, S] \in \mathbf{so}(\{S^{k+1} X_n, \ldots, X_n\}^\perp)$.

But this is an immediate consequence of

$$
[\mathcal{P}, S] S^i X_n = \mathcal{P} S^{i+1} X_n - S \mathcal{P} S^i X_n \text{ for each } i \in \{0, 1, \ldots, k+1\}.
$$

∎

If we have $\mathcal{P}S^{k+2}X_n = 0$ for all $\mathcal{P} \in \mathfrak{g}(k)$, proposition 3.9 gives $\mathfrak{g}(k) = \mathfrak{g}(k+1)$.

Proof of theorem 3.1. Proposition 3.9 gives:

$$\begin{aligned}
\mathfrak{g}(0) &= \mathfrak{so}(\{a, SX_n, X_n\}^{\perp}), \\
\mathfrak{g}(1) &= \mathfrak{so}(\{a, S^2 X_n, SX_n, X_n\}^{\perp}), \\
&\vdots \\
\mathfrak{g}(n-4) &= \mathfrak{so}(\{a, S^{n-3}X_n, \ldots, X_n\}^{\perp}).
\end{aligned}$$

Case (1): There exists a k, with $b_{n-k-2}, \ldots, b_{n-2} \neq 0$, $b_{n-k-3} = 0$ and $0 \leq k \leq n-4$. Lemma 3.7 gives for $\mathcal{P} \in \mathfrak{g}(k)$:

$$\mathcal{P}S^{k+2}X_n = b_{n-k-3} \ldots b_{n-2}\mathcal{P}X_{n-k-2} + \mathcal{P}(\text{terms in}X_{n-k-1}, \ldots, X_{n-2}).$$

By lemma 3.8 and the assumption $\mathcal{P} \in \mathfrak{g}(k)$ the last term vanishes and the first also, since $b_{n-k-3} = 0$. This yields $\mathfrak{g}(k) = \mathfrak{g}(k+1)$. Since $b_{n-k-2} \neq 0$, we get with the same argument $\mathfrak{g}(k-1) \neq \mathfrak{g}(k)$, which shows that the Singer invariant is $k_g = k$.

Case (2): Let $b_1, \ldots, b_{n-2} \neq 0$. Then lemma 3.8 shows:

$$\begin{aligned}
\mathfrak{g}(n-4) &= \mathfrak{so}(\{a, S^{n-3}(X_n), \ldots, X_n\}^{\perp}) \\
&= \mathfrak{so}(\{a, X_3, \ldots, X_n\}^{\perp}) \\
&= \mathfrak{so}(span\{X_2\}) = 0.
\end{aligned}$$

As above, one shows $\mathfrak{g}(n-5) \neq \mathfrak{g}(n-4)$, so the Singer invariant is $k_g = n-4$.

Case (3): Let $b_{n-2} = 0$. We have $S^k X_n = 0$ for all $k \geq 0$, so $\mathfrak{g}(0) = \mathfrak{g}(1)$. Hence, the Singer invariant is $k_g = 0$. ∎

4 Equality of curvature

Let (V, D, S) and $(V, \tilde{D}, \tilde{S})$ be metric Lie algebras of our special class with coefficients $a_1, a_2, b_1, \ldots, b_{n-2}$ and $\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \ldots, \tilde{b}_{n-2}$.

Theorem 4.1. *If $a_1 = \tilde{a}_1$, $a_2 = \tilde{a}_2$ and $|b_{n-k-2}| = |\tilde{b}_{n-k-2}|, \ldots, |b_{n-2}| = |\tilde{b}_{n-2}|$, then we have $\nabla^s R = \widetilde{\nabla}^s \tilde{R}$ for all $0 \leq s \leq k$. In particular, (V, D, S) and $(V, \tilde{D}, \tilde{S})$ have the same curvature up to order k.*

Proposition 2.1 shows that the k-th covariant derivative of the curvature tensor depends only on some $M_i \in \mathcal{M}$ with weight at most $k+1$, i.e., some iterative commutators of S with $[S, D]$.

Let $Sym(V')$ be the space of symmetric endomorphisms of V'. We define for $r \in \{3, \ldots, n\}$:

$$U_r := \{Q \in Sym(V') \mid Q(X_2) = \ldots = Q(X_{r-1}) = 0\}.$$

Lemma 4.2. *Let $Q \in U_r$.*
   (i) *We have $[S, Q] \in U_{r-1}$.*
   (ii) *If the entries of the matrix representation of Q depend only on a_1, a_2,*
      $b_{r-1}, \ldots, b_{n-2}$, then those of $[S, Q]$ depend at most on a_1, a_2, $b_{r-2}, \ldots, b_{n-2}$.

Proof. The first claim follows from direct computation. For the second assertion, let $i \in \{r - 1, \ldots, n\}$. Then

$$
\begin{aligned}
[S, Q](X_i) &= SQ(X_i) - QS(X_i) \\
&= S(\text{terms in } a_1, a_2, b_{r-1}, \ldots, b_{n-2}, X_r, \ldots, X_n) \\
&\quad - b_{i-2}Q(X_{i-1}) + b_{i-1}Q(X_{i+1}).
\end{aligned}
$$

In all terms the smallest index i of the b_i is $r - 2$, which gives the claim.   ∎

Lemma 4.3.   (i) *The entries of the matrix representation of $N_0 = D$ depend only on a_1 and a_2.*
   (ii) *Let $k \in \{1, \ldots, n - 2\}$. Then $N_k \in U_{n-k}$, and the entries of the matrix representation depend at most on $a_1, a_2, b_{n-k-1}, \ldots, b_{n-2}$.*

Proof. The first claim is trivial and the second follows by induction using lemma 4.2.   ∎

Lemma 4.4. *Let (V, D, S) and $(V, \tilde{D}, \tilde{S})$ be two metric Lie algebras of our special class. If $a_1 = \tilde{a}_1, a_2 = \tilde{a}_2$ and $|b_i| = |\tilde{b}_i|$ for $i = 1, \ldots, n - 2$, then (V, D, S) and $(V, \tilde{D}, \tilde{S})$ are isomorphic as metric Lie algebras.*

Proof. We construct an orthogonal transformation φ of V with $\varphi^{-1} D \varphi = \widetilde{D}$ and $\varphi^{-1} S \varphi = \widetilde{S}$. Let $\bar{\sigma}_i$ be given by $\bar{\sigma}_i b_i = \tilde{b}_i$, for $i = 1, \ldots, n - 2$. We now define recursively:

$$
\begin{aligned}
\sigma_n &= 1 \quad \text{and} \\
\sigma_{i-1} &= \sigma_i \bar{\sigma}_{i-2}, \quad i = n, \ldots, 3.
\end{aligned}
$$

Let $\varphi \in O(V)$ be given by $\varphi(a) = a$ and $\varphi(X_i) = \sigma_i X_i$ for $i = 2, \ldots, n$. It is now a straightforward computation to show the claim.   ∎

Proof of theorem 4.1. Using lemma 4.4, we can assume that we have for the coefficients $b_{n-k-2} = \tilde{b}_{n-k-2}, \ldots, b_{n-2} = \tilde{b}_{n-2}$. Further, by lemma 4.3, the entries of N_r depend only on $a_1, a_2, b_{n-r-1}, \ldots b_{n-2}$. The assumptions of the theorem yield therefore $N_r = \widetilde{N}_r, 0 \leq r \leq k + 1$. Finally, the k-th covariant derivative of the curvature tensor is, by proposition, 2.1 completely determined by some $M_i \in \mathcal{M}$ with weight at most $k + 1$, i.e., by $N_0, \ldots, N_{k+1}$. Hence, the covariant derivatives of the curvature are the same up to order k.   ∎

5   Isometry classes

As above, let (V, D, S) and $(V, \tilde{D}, \tilde{S})$ be metric Lie algebras of our special class with coefficients a_1, a_2, $b_1, \ldots, b_{n-2}$ and $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{b}_1, \ldots, \tilde{b}_{n-2}$.

Theorem 5.1. *Let k be the Singer invariant of (V, D, S) and furthermore, let $a_1 = \tilde{a}_1$, $a_2 = \tilde{a}_2$ and $b_{n-2} \neq 0$. The simply connected Lie groups $\mathcal{G}, \tilde{\mathcal{G}}$ associated to (V, D, S) and $(V, \tilde{D}, \tilde{S})$ are isometric if and only if*

$$|b_{n-k-3}| = |\tilde{b}_{n-k-3}|, \ldots, |b_{n-2}| = |\tilde{b}_{n-2}|.$$

We will prove this theorem below.

Lemma 5.2. *Let k be the Singer invariant of (V, D, S) and furthermore, let $a_1 = \tilde{a}_1$, $a_2 = \tilde{a}_2$, $|b_{n-k-3}| = |\tilde{b}_{n-k-3}|, \ldots, |b_{n-2}| = |\tilde{b}_{n-2}|$. Then the simply connected Lie groups associated to the Lie algebras are isometric.*

Proof. By theorem 4.1, the Lie algebras have the same curvature up to order $k+1$ and hence, by theorem 1.2, they are locally isometric. Finally, they are both simply connected, thus isometric. ∎

Lemma 5.3. *Let (V, D, S) as above. Then:*

$$
\begin{aligned}
(i) \quad & \langle (\nabla_a^r R)_{a, X_{n-r-1}} a, X_n \rangle = -\langle N_{r+1} X_{n-r-1}, X_n \rangle \\
& = b_{n-r-2} \cdot \ldots \cdot b_{n-2}(a_1 - a_2), \quad \text{for all } r \in \{0, \ldots, n-3\}; \\
(ii) \quad & \langle (\nabla_a^r R)_{a, X_{n-r-1}} a, X_{n-1} \rangle = -\langle N_{r+1} X_{n-r-1}, X_{n-1} \rangle = 0, \\
& \text{for all } r \in \{1, \ldots, n-3\}; \\
(iii) \quad & (\nabla_a^r R)_{a, X_i} a = 0 \text{ for all } r \geq 1, i \leq n - r - 2.
\end{aligned}
$$

Proof.
(i) Using proposition 2.1 (ii) we have only to show that $\langle M X_{n-r-1}, X_n \rangle = 0$ for each $M \in \mathcal{M}$ with weight at most r. For $r = 0$ this is immediate and for $r \geq 1$ it can be concluded by using lemma 4.3 (ii). Considering the same lemma, the second equation is proved by induction on r.
(ii) and (iii) are proved with similar arguments.

∎

Let $r \in \mathbb{N}_0$ and define $\mathcal{F}^r := span\{a, S^{r+1} X_n, \ldots, S X_n, X_n\} \subset V$.

Lemma 5.4. *Let $a_1 = \tilde{a}_1$ and $a_2 = \tilde{a}_2$, $r \in \mathbb{N}_0$, and let further $\varphi \colon V \longrightarrow V$ be an orthogonal map with $\varphi^* \widetilde{\nabla^s R} = \nabla^s R$ for all $s \in \{0, \ldots, r\}$. Then we have:*
(i) $\varphi(a) = \pm a$, $\varphi(V') = V'$;
(ii) $\varphi(\mathcal{F}^s) = \tilde{\mathcal{F}}^s$ for all $s \in \{0, \ldots, r\}$ with $s \leq n - 4$.

Proof.
(i) The map φ preserves curvature, hence the Ricci tensor and also its eigenspaces. Since a is an eigenvector for a simple eigenvalue, this together with φ being a linear isometry gives the claim.

(ii) Let $s \in \{0, \ldots, r\}$. The assumption $\varphi^* \widetilde{\nabla}^s \widetilde{R} = \nabla^s R$ implies that $\varphi^{-1} \circ \widetilde{\mathfrak{g}}(s) \circ \varphi = \mathfrak{g}(s)$. By proposition 3.9 we get for $\mathcal{P} \in \mathfrak{so}(V)$:

$$\mathcal{P}(\widetilde{\mathcal{F}}^s) = 0 \iff \mathcal{P}(\varphi(\mathcal{F}^s)) = 0.$$

Since $s \leq n - 4$, we have $dim\mathcal{F}^s, dim\widetilde{\mathcal{F}}^s \leq s + 3 \leq n - 1$ and hence $dim\mathcal{F}^s = dim\widetilde{\mathcal{F}}^s$. If this dimension is at most $n - 2$, we immediately get $\varphi(\mathcal{F}^s) = \widetilde{\mathcal{F}}^s$. In the case $dim\mathcal{F}^{n-4} = dim\widetilde{\mathcal{F}}^{n-4} = n - 1 = s + 3$, one has to argue more carefully. Distinguishing the cases $n = 4$ and $n > 4$ and using the Ricci eigenvalues and lemma 5.3 yields the claim.

$\blacksquare$

Lemma 5.5. *Let $a_1 = \tilde{a}_1$, $a_2 = \tilde{a}_2$, $b_{n-2} \neq 0$. Furthermore, let $\varphi \in O(V)$ with $\varphi^* \widetilde{R} = R$. Then we have:*
(i) *$\varphi(span\{X_{n-1}, X_n\}) = span\{X_{n-1}, X_n\}$;*
(ii) *$|b_{n-2}| = |\tilde{b}_{n-2}|$.*

Proof. By lemma 5.4 (ii) and $b_{n-2} \neq 0$ we get $\mathcal{F}^0 = \widetilde{\mathcal{F}}^0 = span\{a, X_{n-1}, X_n\}$. Furthermore, by lemma 5.4 (i) we have $\varphi(a) = \pm a$, which yields the claim $\varphi(span\{X_{n-1}, X_n\}) = span\{X_{n-1}, X_n\}$.

The second statement follows by looking at the restriction of the determinant of Ric to the space $End(span\{X_{n-1}, X_n\})$.

$\blacksquare$

Lemma 5.6. *Let (V, D, S), $(V, \widetilde{D}, \widetilde{S})$ be two metric Lie algebras with the same Singer invariant $k \geq 1$. Let $a_1 = \tilde{a}_1$, $a_2 = \tilde{a}_2$ and $b_{n-2} \neq 0$. Furthermore, let $r \in \{1, \ldots, k\}$, and let $\varphi : V \longrightarrow V$ be an orthogonal map, which preserves the curvature up to order r. Then*
(i) *$\varphi(X_{n-s-1}) = \pm X_{n-s-1}$ for all $s \in \{1, \ldots, r\}$;*
(ii) *$\varphi(X_n) = \pm X_n$, $\varphi(X_{n-1}) = \pm X_{n-1}$, $\varphi(a) = \pm a$;*
(iii) *$|b_{n-s-2}| = |\tilde{b}_{n-s-2}|$ for all $s \in \{0, \ldots, r\}$.*

Proof.
(i) By $k \geq 1$ and theorem 3.1, we have $b_{n-k-2}, \ldots, b_{n-2} \neq 0$ and in particular, since $r \leq k$, also $b_{n-r-2}, \ldots, b_{n-2} \neq 0$. Similarly for $\tilde{b}_i$. By lemma 3.8 and the definition of $\mathcal{F}^s, \widetilde{\mathcal{F}}^s$, we get $\mathcal{F}^s = \widetilde{\mathcal{F}}^s = span\{a, X_{n-s-1}, \ldots, X_n\}$ for each $s \in \{0, \ldots, r\}$. By lemma 5.4, φ preserves the subspaces $span\{X_{n-s-1}, \ldots, X_n\}$ for each $s \in \{0, \ldots, r\}$. Together with lemma 5.5 this implies $\varphi(X_{n-s-1}) = \pm X_{n-s-1}$ for each $s \in \{1, \ldots, r\}$.
(ii) By lemma 5.4 (i) and lemma 5.5 (i), we know that $\varphi(a) = \pm a$ and $\varphi(span\{X_{n-1}, X_n\}) = span\{X_{n-1}, X_n\}$. By (i) we know $\varphi(X_{n-2}) = \pm X_{n-2}$. Using lemma 5.3 and comparing the terms of $\langle (\widetilde{\nabla}_a \widetilde{R})_{a, \varphi(X_{n-1})} a, X_{n-2} \rangle$ and $\langle (\nabla_a R)_{a, X_{n-1}} a, X_{n-2} \rangle$, we finally get $\varphi(X_{n-1}) \perp X_n$, which gives the claim.
(iii) Using lemma 5.3 and (i), (ii) above, the statement follows by induction over s.

$\blacksquare$

Corollary 5.7. *Let (V, D, S) and $(V, \widetilde{D}, \widetilde{S})$ be two metric Lie algebras with $a_1 = \tilde{a}_1$, $a_2 = \tilde{a}_2$, $b_{n-2} \neq 0$ and the same Singer invariant $k \in \{0, \ldots, n - 4\}$. Let further be $r \in \{0, \ldots, k + 1\}$. If both Lie algebras have the same curvature up to order r, then $|b_{n-r-2}| = |\tilde{b}_{n-r-2}|, \ldots, |b_{n-2}| = |\tilde{b}_{n-2}|$.*

Proof. Lemma 5.5 (ii) gives the claim in the case of $r = 0$ and also for $k = 0$ and $r = 1$ (since $\widetilde{b}_{n-2} \neq 0$, theorem 3.1 together with $k = 0$ gives $b_{n-3} = \widetilde{b}_{n-3} = 0$). In the case of $k \geq 1$ and $r \leq k$, the claim follows from lemma 5.6 (iii). In the case $k \geq 1$ and $r = k + 1$, it remains to show $|b_{n-k-3}| = |\widetilde{b}_{n-k-3}|$. For $k < n - 4$ we have by theorem 3.1 that $b_{n-k-3} = \widetilde{b}_{n-k-3} = 0$. This leaves us with $k = n - 4$ and $r = n - 3$. By lemma 5.6, we have $\varphi(X_i) = \pm X_i$ for all $i \leq n - (n-4) - 1 = 3$ and $\varphi(a) = \pm a$, and hence $\varphi(X_2) = \pm X_2$. The claim $|b_1| = |\widetilde{b}_1|$ can be proven by comparing $\langle (\widetilde{\nabla}_a^{n-3}\widetilde{R})_{a,X_n}a, X_2 \rangle$ and $\langle (\nabla_a^{n-3}R)_{a,X_n}a, X_2 \rangle$. ∎

Proof of theorem 5.1.

Lemma 5.2 shows one direction. For the other, observe that the Singer invariant is obviously invariant under isometries of manifolds, hence the Singer invariant of $(V, \widetilde{D}, \widetilde{S})$ is also k. Further, an isometry preserves the curvature tensor and all covariant derivatives of the curvature tensor. So we can apply corollary 5.7 in the particular case $r := k + 1$ and this yields the claim. ∎

It is straightforward to see that the simply connected Lie groups of two Lie algebras (V, D, S) and $(V, \widetilde{D}, \widetilde{S})$ with $a_1 = \widetilde{a}_1$, $a_2 = \widetilde{a}_2$ and $b_{n-2} = 0$ are isomorphic if and only if $\widetilde{b}_{n-2} = 0$.

Theorem 5.8. *Let $k \in \mathbb{N}_0$ be arbitrary. There exist pairs of locally non-isometric homogeneous manifolds with Singer invariant k, which have the same curvature up to order k.*

Proof. Let $n := k + 4$. Choose $a_1 = \widetilde{a}_1$, $a_2 = \widetilde{a}_2$ with $a_1 \neq a_2$, $a_1 a_2 > 0$ and $b_1, \ldots, b_{n-2} \neq 0$, $\widetilde{b}_1, \ldots, \widetilde{b}_{n-2} \neq 0$ with $|b_2| = |\widetilde{b}_2|, \ldots, |b_{n-2}| = |\widetilde{b}_{n-2}|$, but $|b_1| \neq |\widetilde{b}_1|$. This defines Lie algebras (V, D, S) and $(V, \widetilde{D}, \widetilde{S})$. By theorem 3.1, they have Singer invariant k and the associated simply connected Lie groups are by theorem 5.1 non-isometric. Since both Lie groups are simply connected, they are also not locally isometric. Finally, theorem 4.1 shows that they have the same curvature up to order $k = n - 2 - 2 = n - 4$. ∎

Remark 5.9. *Let (V, D, S), $(V, \widetilde{D}, \widetilde{S})$ be two metric Lie algebras with Singer invariant $0 < k < n - 4$ and $a_1 = \widetilde{a}_1$, $a_2 = \widetilde{a}_2$. If the associated simply connected Lie groups are non-isometric, they have at most the same curvature up to order $k - 1$.*

Proof. Since $0 < k < n - 4$, theorem 3.1 implies $b_{n-2}, \widetilde{b}_{n-2} \neq 0$ and $b_{n-k-3} = 0$, $\widetilde{b}_{n-k-3} = 0$, that is $b_{n-k-3} = \widetilde{b}_{n-k-3}$. If the two Lie algebras had the same curvature up to order k, corollary 5.6 would imply $|b_{n-k-2}| = |\widetilde{b}_{n-k-2}|, \ldots, |, b_{n-2}| = |b_{n-2}|$; hence by theorem 5.1, the simply connected Lie groups would be isomorphic, contradicting the assumption. ∎

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