Synthetic Differential Geometry of Jet Bundles

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Abstract

The theory of infinite jet bundles provides the very foundation for the geometric theory of nonlinear partial differential equations, but it is hard to say that orthodox differential geometry is an appropriate vehicle of the former theory, as has been exhorted by Vinogradov and his Russian school. We contend that synthetic differential geometry initiated by Lawvere is the veriest framework for the theory of infinite jet bundles. Kock (1980) gave a synthetic treatment of the theory of jet bundles, but his approach was restricted to formal manifolds and inherited clumsiness from the standard theory. This paper gives an alternative synthetic treatment of infinite jet bundles, in which the pinpointed notion of (nonlinear) connection will play a predominant role and no remainders of coordinates can be seen. Contact vector fields of finite type are completely determined in this new context.

0 Introduction

The notion of jet is a far-reaching generalization of that of tangent vector. The theory of jet bundles provides a good framework for the general theory of nonlinear partial different equations as well as the calculus of variations, for which the reader is referred to Bocharov et al. (1999), Gamkrelidze (1991, Chapter 5) and Krasil'shchik, Lychagin and Vinogradov (1986) as well as Saunders (1989). All the same the standard theory of jet bundles appears clumsy mainly because of its heavy use of coordinate manipulations. Although the coordinate representation of jets bears

Received by the editors April 2000.
Communicated by Y. Félix.
1991 *Mathematics Subject Classification :* 58A03, 53C05, 58A20, 35A30, 35G20, 51K10.
Key words and phrases : synthetic differential geometry, jet bundle, connection, contact transformation, contact vector field, prolongation, preconnection, geometric theory of nonlinear partial differential equations, Cartan connection.

some resemblance to the traditional coordinate representation in tensor calculus, the transformation rules are no longer linear in the former case while they are in the latter.

Synthetic differential geometry is an avant-garde branch of differential geometry as logical anathema, in which nilpotent infinitesimals, once ostracized from orthodox differential geometry, are abundantly and coherently available. We believe firmly that synthetic differential geometry can contribute much towards making the theory more neat and noble. In one of our previous papers [Nishimura (1999)] we have outdated Bunge and Heggie's (1984) synthetic treatment of the calculus of variations by making use of the synthetic notion of (pointwise) connection. Our approach there is completely coordinate-free and applicable to a wider class of microlinear spaces possibly without a stitch of coordinates. Kock (1980) gave a synthetic treatment of jet bundles, but his approach is restricted to formal manifolds, remaining on the periphery of its standard counterpart just as Bunge and Heggie (1984) did. The principal objective of this paper is to give an alternative synthetic approach to the theory of jet bundles, which is applicable to a wider class of microlinear spaces than that of formal manifolds. As in our synthetic treatment of the calculus of variations, our synthetic theory of jet bundles is completely coordinate-free and makes essential use of a synthetic sort of connection. The technical end of the paper is Theorem 3.5, which is a synthetic version of Theorem 2.5 of Bocharov et al. (1999, Chapter 4).

The basic idea of this paper is simple enough. We are much more interested in how jets function than how they are constructed, just as category theorists are interested in how the direct product of two sets functions in the category of sets and mappings while set theorists are more interested in how to construct the direct product of two sets. We prefer to approach the matter from a functional viewpoint. Kock (1980) began his synthetic treatment of jets by transcribing the standard construction of jets into his favorite machinery of monads with a synthetic flavor, which has limited the applicability of his theory to a considerable extent. We begin our consideration by remarking that a first-order jet at a point of a space gives a decomposition of the tangent space to the space at the point, for which the reader is referred to Theorem 4.3.2 of Saunders (1989) as well as Theorem 6.3.2 and Theorem 7.3.2 there. Since we know well that such a decomposition can be identified with a sort of (nonlinear) connection [cf. Nishimura (1998, Theorem 2.1)], we are naturally inclined to identify a first-order jet with something like a pinpointed connection. The idea can be repeated to higher-order jets for free, finally resulting in our synthetic theory of infinite jet bundles. The exact relationship between Kock's (1980) synthetic approach to jet bundles and ours will be discussed elsewhere.

First of all, we must pinpoint the notion of connection, which results in our notion of *preconnection*. Our notion of preconnection is essentially nonlinear, and the notion of (nonlinear) connection in our previous paper [Nishimura (1998)] is simply an assignment of a preconnection to each point of the space at issue. In Section 1 we will present our synthetic theory of infinite jet bundles based on the notion of preconnection. In Section 2 an important class of contact transformations of infinite jet bundles is determined completely, while in Section 3 contact vector fields of finite type of infinite jet bundles are determined. Our standard reference on synthetic differential geometry is Lavendhomme (1996) and those on jet bundles are Saunders (1989) and Bocharov et al. (1999).

Now we fix our notation and terminology. We denote by $\mathbb{R}$ the extended set of real numbers with cornucopia of infinitesimals, which is expected to satisfy the so-called general Kock axiom [cf. Lavendhomme (1996, §2.1)]. We denote by D the totality of elements of $\mathbb{R}$ whose squares vanish. Given a microlinear space M, its tangent space at $x \in M$ is denoted by $T_x(M)$ and its tangent bundle is denoted by $\tau_M : T(M) \to M$. A mapping $\pi : E \to M$ of microlinear spaces is called a *bundle over* M, in which E is called the *total space* of π, M is called the *base space* of π, and $E_x = \pi^{-1}(x)$ is called the *fiber* over $x \in M$. The totality of sections of π is denoted by $\Gamma(\pi)$. Given $x \in E$, we denote by $V_x(\pi)$ the $\mathbb{R}$-submodule of the $\mathbb{R}$-module $T_x(E)$ consisting of all vertical tangent vectors to E at x. The vertical bundle of π is denoted by $\upsilon_\pi : V(\pi) \to E$. If all the fibers of the bundle $\pi : E \to M$ are Euclidean $\mathbb{R}$-modules, then the bundle is called a *vector bundle over* M. Given two bundles $\pi : E \to M$ and $\pi' : P \to M$ over the same microlinear space M, a mapping $\eta : E \to P$ is said to be *over* M if it induces the identity mapping of M. In this case η is called a *morphism of bundles over* M *from* π *to* π'. In synthetic differential geometry a vector field X on a microlinear space M can be viewed equally in three different ways, namely, as mappings $M \to M^D$, $M \times D \to M$ and $D \to M^M$, for which the reader is referred to Lavendhomme (1996, §3.2). Given a mapping $\theta : M \to N$ of microlinear spaces, a *vector field along* θ is a mapping X from M to $T(N)$ with $\theta = \tau_N \circ X$.

A bundle $\pi : E \to M$ shall be chosen and fixed once and for all. It will be assumed to be a vector bundle in Section 3.

1 Cartan Connection

Let us begin by pinpointing the notion of connection. A *preconnection over* the bundle $\pi : E \to M$ *at* $x \in E$ is a mapping $\nabla_x : T_{\pi(x)}(M) \to T_x(E)$ such that for any $t \in T_{\pi(x)}(M)$ and any $\alpha \in \mathbb{R}$, we have the following:

$$\nabla_x(\alpha t) = \alpha \nabla_x(t) \tag{1.1}$$

$$\pi \circ \nabla_x(t) = t. \tag{1.2}$$

A *connection* ∇ *over* π is simply an assignment of a preconnection ∇_x over π at x to each point x of E, in which we will often write $\nabla(t, x)$ in place of $\nabla_x(t)$. Our present definition of connection is essentially that of nonlinear connection in our previous paper (Nishimura, 1998, §2) simply tailored to our present bundle formalism. More specifically, our notion of nonlinear connection in that paper is no other than a connection over the vector bundle $\tau_M : T(M) \to M$ in our present nomenclature.

As in Theorem 2.1 of Nishimura (1998) we have

Proposition 1.1. *Given a preconnection* ∇_x *over the bundle* $\pi : E \to M$ *at a point* x *of* E*, the set* $H_x(\pi, \nabla_x) = \{\nabla_x(t) \mid t \in T_{\pi(x)}(M)\}$ *is an* $\mathbb{R}$*-submodule of the* $\mathbb{R}$*-module* $T_x(E)$*, and the* $\mathbb{R}$*-module* $T_x(E)$ *is the direct sum of* $\mathbb{R}$*-submodules* $H_x(\pi, \nabla_x)$ *and* $V_x(\pi)$ *of* $T_x(E)$*. Conversely, given an* $\mathbb{R}$*-submodule* H *of* $T_x(E)$ *with*

$T_x(E) = H \oplus V_x(\pi)$, there exists a unique preconnection ∇_x with $H_x(\pi, \nabla_x) = H$ provided that there exists a preconnection over π at x at all. In short, providing that there exists a preconnection over π at x at all, the assignment $\nabla_x \mapsto H_x(\pi, \nabla_x)$ gives a bijective correspondence between preconnections over π at x and $\mathbb{R}$-submodules of $T_x(E)$ complementary to $V_x(\pi)$.

Proof: Essentially the same as in the proof of Theorem 2.1 of Nishimura (1998). ■

Vectors in $H_x(\pi, \nabla_x)$ in the above proposition are called *horizontal*. The canonical projections of $T_x(E)$ into $H_x(\pi, \nabla_x)$ and $V_x(\pi)$ with respect to the decomposition $T_x(E) = H_x(\pi, \nabla_x) \oplus V_x(\pi)$ in the above proposition are denoted respectively by h_{∇_x} and v_{∇_x}. Note that $h_{\nabla_x}(t) = \nabla_x(\pi \circ t)$ for any $t \in T_x(E)$.

Let $\eta : E \to P$ be a morphism of bundles over M from the bundle $\pi : E \to M$ to anotherbundle $\pi' : P \to M$ over the same base space M. We say that a preconnection ∇_x over π at a point x of E *is η-related to* a preconnection ∇_y over π' at the point $y = \eta(x)$ of P provided that

$$\eta \circ \nabla_x(t) = \nabla_y(t) \tag{1.3}$$

for any $t \in T_a(M)$ with $a = \pi(x) = \pi'(y)$. Then we have

Proposition 1.2. *The preconnection ∇_x over π at x is η-related to the preconnection ∇_y over π' at $y = \eta(x)$ iff $h_{\nabla_y}(\eta \circ t) = \eta \circ h_{\nabla_x}(t)$ for any $t \in T_x(E)$, or equivalently, iff $v_{\nabla_y}(\eta \circ t) = \eta \circ v_{\nabla_x}(t)$ for any $t \in T_x(E)$.*

Proof: It is easy to see that for any $t \in T_x(E)$,

$$v_{\nabla_y}(\eta \circ t) = \eta \circ t - h_{\nabla_y}(\eta \circ t) \text{ and} \tag{1.4}$$

$$\eta \circ (v_{\nabla_x}(t)) = \eta \circ t - \eta \circ h_{\nabla_x}(t), \tag{1.5}$$

from which it follows directly that $h_{\nabla_y}(\eta \circ t) = \eta \circ h_{\nabla_x}(t)$ iff $v_{\nabla_y}(\eta \circ t) = \eta \circ v_{\nabla_x}(t)$. Now suppose that ∇_x is η-related to ∇_y. Then it is easy to see that for any $t \in T_x(E)$,

$$\begin{aligned} \eta \circ (h_{\nabla_x}(t)) &= \eta \circ (\nabla_x(\pi \circ t)) \\ &= \nabla_y(\pi' \circ t) \\ &= \nabla_y(\pi \circ \eta \circ t) \\ &= h_{\nabla_y}(\eta \circ t). \end{aligned} \tag{1.6}$$

Conversely suppose that $h_{\nabla_y}(\eta \circ t) = \eta \circ h_{\nabla_x}(t)$ for any $t \in T_x(E)$. Then it is easy to see that for any $t \in T_{\pi(x)}(M)$,

$$\begin{aligned} \eta \circ \nabla_x(t) &= \eta \circ h_{\nabla_x}(\nabla_x(t)) \\ &= h_{\nabla_y}(\eta \circ \nabla_x(t)) \\ &= \nabla_y(\pi' \circ \eta \circ \nabla_x(t)) \\ &= \nabla_y(\pi \circ \nabla_x(t)) \\ &= \nabla_y(t). \end{aligned} \tag{1.7}$$

Therefore we are now certain that ∇_x is η-related to ∇_y iff $h_{\nabla_y}(\eta \circ t) = \eta \circ h_{\nabla_x}(t)$ for any $t \in T_x(E)$, which completes the proof. ■

We are now going to define *jet bundles* $\pi_i : J^i(\pi) \to M$ of the bundle $\pi : E \to M$ inductively ($i \geq 0$). The zeroth-order jet bundle $\pi_0 : J^0(\pi) \to M$ of $\pi : E \to M$ shall be $\pi : E \to M$ itself with $J^0(\pi) = E$. For each $x \in E$, let $J^1(\pi)_x$ be the totality of preconnections over π at x. The total space $J^1(\pi)$ of the first-order jet bundle $\pi_1 : J^1(\pi) \to M$ of the bundle $\pi : E \to M$ shall be the set-theoretic union of $J^1(\pi)_x$ for all $x \in E$. We define a mapping $\pi_{1,0} : J^1(\pi) \to E$ to be

$$\pi_{1,0}(\nabla^1) = x \tag{1.8}$$

for $\nabla^1 \in J^1(\pi)_x$ together with $\pi_1 = \pi \circ \pi_{1,0}$. Now we proceed inductively. Given the i-th order jet bundle $\pi_i : J^i(\pi) \to M$ of the bundle $\pi : E \to M$ with $\nabla^i \in J^i(\pi)$, let $J^{i+1}(\pi)_{\nabla^i}$ be the totality of preconnections over π_1 at ∇^i that are $\pi_{i,i-1}$-related to ∇^i ($i \geq 2$). The total space $J^{i+1}(\pi)$ of the $(i+1)$-th-order jet bundle $\pi_{i+1} : J^{i+1}(\pi) \to M$ of $\pi : E \to M$ shall be the set-theoretic union of $J^{i+1}(\pi)_{\nabla^i}$ for all $\nabla^i \in J^i(\pi)$. We define a mapping $\pi_{i+1,i} : J^{i+1}(\pi) \to J^i(\pi)$ to be

$$\pi_{i+1,i}(\nabla^{i+1}) = \nabla^i \tag{1.9}$$

for $\nabla^{i+1} \in J^{i+1}(\pi)_{\nabla^i}$ together with $\pi_{i+1} = \pi_i \circ \pi_{i+1,i}$. Given $i \leq j$, we define $\pi_{j,i}$ to be the composition $\pi_{i+1,i} \circ ... \circ \pi_{j,j-1}$.

Now we present one of our two basic assumptions.

Assumption $[J^\infty\mathrm{I}]$**:** The mapping $\pi_{i+1,i}$ is surjective for all $i \geq 0$.

Now we define the infinite jet bundle $J^\infty(\pi)$ of the bundle $\pi : E \to M$ as the inverse limit of the sequence

$$J^0(\pi) \xleftarrow{\pi_{1,0}} J^1(\pi) \xleftarrow{\pi_{2,1}} J^2(\pi) \xleftarrow{\pi_{3,2}} J^3(\pi)... \tag{1.10}$$

or equivalently as the inverse limit of the sequence

$$J^1(\pi) \xleftarrow{\pi_{2,1}} J^2(\pi) \xleftarrow{\pi_{3,2}} J^3(\pi)... \tag{1.11}$$

Therefore a point $\mathbf{x}$ of $J^\infty(\pi)$ is represented by a sequence $\{x_i\}_{i\geq 0}$ with $\pi_{i+1,i}(x_{i+1}) = x_i$ or by a sequence $\{\nabla^i\}_{i\geq 1}$ with $\pi_{i+1,i}(\nabla^{i+1}) = \nabla^i$. We define a mapping $\pi_{\infty,k} : J^\infty(\pi) \to J^k(\pi)$ to be $\pi_{\infty,k}(\{x_i\}_{i\geq 0}) = x_k$. We define $\pi_\infty : J^\infty(\pi) \to M$ to be $\pi_\infty(\{x_i\}_{i\geq 0}) = \pi(x_0)$.

Now we are going to define a connection on the bundle $\pi_\infty : J^\infty(\pi) \to M$ to be called the *Cartan connection* and to be denoted by ∇^∞:

$$\nabla^\infty(t, \mathbf{x})(d) = \{\nabla^{i+1}(t, x_i)(d)\}_{i\geq 0} \tag{1.12}$$

for any tangent vector t to M, any $d \in D$ and any $\mathbf{x} = \{x_i\}_{i\geq 0} = \{\nabla^i\}_{i\geq 1} \in J^\infty(\pi)$. The assignment $\mathbf{x} \in J^\infty(\pi) \mapsto H_{\mathbf{x}}(\pi, \nabla^\infty_{\mathbf{x}})$ is called the *Cartan distribution*. The existence of the Cartan connection ∇^∞ on $J^\infty(\pi)$ makes the infinite jet bundle $J^\infty(\pi)$ absolutely predominant over higher-order jet bundles $J^i(\pi)$'s in theory and applications.

Now we present the remaining one of our two basic assumptions, which claims flatness of the Cartan connection ∇^∞:

Assumption $[J^\infty\mathrm{II}]$**:** For any microsquare γ on M, any $d_1, d_2 \in D$ and any $\mathbf{x} \in J^\infty(\pi)$ with $\pi_\infty(\mathbf{x}) = \gamma(0,0)$, we have

$$\begin{aligned}(1.13)\qquad & \nabla^\infty(\gamma(\cdot, d_2), \nabla^\infty(\gamma(0,\cdot), \mathbf{x})(d_2))(d_1) \\ &= \nabla^\infty(\gamma(d_1,\cdot), \nabla^\infty(\gamma(\cdot,0), \mathbf{x})(d_1))(d_2).\end{aligned}$$

Given a vector field X along $\pi_\infty : J^\infty(\pi) \to M$, we define its *horizontal lift* $\hat{X}$ to be the following vector field on $J^\infty(\pi)$:

$$(1.14)\qquad \hat{X}(\mathbf{x}) = \nabla^\infty(X(\pi_\infty(\mathbf{x})), \mathbf{x})$$

for any $\mathbf{x} \in J^\infty(\pi)$.

A mapping $F : J^\infty(\pi) \to J^\infty(\pi)$ is called a *contact transformation* of $J^\infty(\pi)$ provided that the differential of F maps horizontal vectors to horizontal ones with respect to the Cartan connection ∇^∞. A vector field X on $J^\infty(\pi)$ is called *contact* provided that $X_d : J^\infty(\pi) \to J^\infty(\pi)$ is a contact transformation of $J^\infty(\pi)$ for any $d \in D$.

We denote by $\mathcal{F}^f_M(\pi_\infty, \pi)$ the totality of morphisms of bundles over M from π_∞ to π that decompose into the projection $\pi_{\infty,i} : J^\infty(\pi) \to J^i(\pi)$ and a morphism of bundles over M from π_i to π for some $i \geq 0$. We denote by $\mathcal{F}^f_E(\pi_{\infty,0}, \upsilon_\pi)$ the totality of morphisms of bundles over E from $\pi_{\infty,0}$ to υ_π that decompose into the projection $\pi_{\infty,i} : J^\infty(\pi) \to J^i(\pi)$ and a morphism of bundles over E from $\pi_{i,0}$ to υ_π for some $i \geq 0$. Note that if the bundle $\pi : E \to M$ happens to be a vector bundle over M, then we can naturally identify $\pi^f_E(\pi_{\infty,0}, \upsilon_\pi)$ and $\pi^f_M(\pi_\infty, \pi)$, since for any $\varphi \in \pi^f_E(\pi_{\infty,0}, \upsilon_\pi)$ there exists a unique $\hat{\varphi} \in \pi^f_M(\pi_\infty, \pi)$ such that

$$(1.15)\qquad \varphi(\mathbf{x})(d) = \pi_{\infty,0}(\mathbf{x}) + d\hat{\varphi}(\mathbf{x})$$

for any $\mathbf{x} \in J^\infty(\pi)$ and any $d \in D$.

A transformation F of $J^\infty(\pi)$ is said to be of *finite type* if there exists a natural number k such that $\pi_{\infty,i} \circ F$ decomposes into the projection $\pi_{\infty,k+i} : J^\infty(\pi) \to J^{k+i}(\pi)$ and a mapping from $J^{k+i}(\pi)$ to $J^i(\pi)$ for any $i \geq 0$. In this case we say that F is of *degree* k *at most*. A vector field X on $J^\infty(\pi)$ is said to be of *finite type* provided that there exists a natural number k such that the transformation X_d is of degree k at most for any $d \in D$.

Any $\psi \in \Gamma(\pi)$ determines a series $\psi^{(i)} \in \Gamma(\pi_i)$ of its *i-th prolongations* $(i \geq 0)$ together with its *infinite prolongation* $\psi^{(\infty)} \in \Gamma(\pi_\infty)$. We proceed inductively. Let $\psi^{(0)} = \psi$. For any $x \in M$, $\psi^{(i+1)}(x)$ shall be the preconnection over π_i at $\psi^{(i)}(x)$ as follows:

$$(1.16)\qquad \psi^{(i+1)}(x)(t, \psi^{(i)}(x))(d) = \psi^{(i)}(t(d))$$

for any $t \in T_x(M)$ and any $d \in D$.

Lemma 1.3. *For any $x \in M$, the preconnection $\psi^{(i+2)}(x)$ over π_{i+1} at $\psi^{(i+1)}(x)$ is $\pi_{i+1,i}$-related to the preconnection over $\psi^{(i+1)}(x)$ over π_i at $\psi^{(i)}(x)$.*

Proof: For any $t \in T_x(M)$ and any $d \in D$, we have

$$(1.17)\qquad \pi_{i+1,i} \circ \psi^{(i+2)}(x)(t, \psi^{(i+1)}(x))(d)$$

$$\begin{aligned}&= \pi_{i+1,i} \circ \psi^{(i+1)}(t(d)) \\ &= \psi^{(i)}(t(d)) \\ &= \psi^{(i+1)}(x)(t, \psi^{(i)}(x))(d).\end{aligned}$$ ■

The above lemma enables us to define $\psi^{(\infty)} \in \Gamma(\pi_\infty)$ as follows:

(1.18) $$\psi^{(\infty)}(x) = \{\psi^{(i)}(x)\}_{i \geq 0} \text{ for any } x \in M.$$

2 Contact Transformations

The principal objective of this section is to determine contact transformations of finite type of $J^\infty(\pi)$ over M. Let us begin with

Proposition 2.1. *For any morphism $f : J^k(\pi) \to E$ of bundles over M from π_k to π, there exists a unique morphism $f^{(1)} : J^{k+1}(\pi) \to J^1(\pi)$ of bundles over M from π_{k+1} to π_1 satisfying the following two conditions:*

(2.1) *The square*
$$\begin{tikzcd} J^{k+1}(\pi) \arrow[r, "f^{(1)}"] \arrow[d, "\pi_{k+1,k}"'] & J^{(1)}(\pi) \arrow[d, "\pi_{1,0}"] \\ J^k(\pi) \arrow[r, "f"'] & E \end{tikzcd}$$
is commutative.

(2.2) *For any $x \in J^k(\pi)$ and any $\nabla_x \in J^{k+1}(\pi)_x$, the preconnection ∇_x over π_k at x is f-related to the preconnection $f^{(1)}(\nabla_x)$ over π at $f(x)$.*

Proof: For any $x \in J^k(\pi)$ and any $\nabla_x \in J^{k+1}(\pi)_x$, it is easy to see that the assignment $\nabla_{f(x)} : t \in T_{\pi_k(x)}(M) \mapsto f \circ \nabla_x(t)$ is a preconnection over π at $f(x)$, so that $\nabla_{f(x)} \in J^1(\pi)_{f(x)}$. Therefore the desired unique morphism $f^{(1)} : J^{k+1}(\pi) \to J^1(\pi)$ of bundles over M from π_{k+1} to π_1 can and should be determined by the requirement that $f^{(1)}(\nabla_x) = \nabla_{f(x)}$ for any $x \in J^k(\pi)$ and any $\nabla_x \in J^{k+1}(\pi)_x$. ■

The above proposition can be generalized to higher orders with a bit more effort.

Proposition 2.2. *Let $i \geq 0$. Let $f^{(i)} : J^{k+1}(\pi) \to J^i(\pi)$ be a morphism of bundles over M from π_{k+i} to π_i and $f^{(i+1)} : J^{k+i+1}(\pi) \to J^{i+1}(\pi)$ a morphism of bundles over M from π_{k+i+1} to π_{i+1} subject to the following two conditions:*

(2.3) *The square*
$$\begin{tikzcd} J^{k+i+1}(\pi) \arrow[r, "f^{(i+1)}"] \arrow[d, "\pi_{k+i+1,k+i}"'] & J^{i+1}(\pi) \arrow[d, "\pi_{i+1,i}"] \\ J^{k+i}(\pi) \arrow[r, "f^{(i)}"'] & J^i(\pi) \end{tikzcd}$$
is commutative.

(2.4) *For any $x \in J^{k+i}(\pi)$ and any $\nabla_x \in J^{k+i+1}(\pi)_x$, the preconnection ∇_x over π_{k+i} at x is $f^{(i)}$-related to the preconnection $f^{(i+1)}(\nabla_x)$ over π_i at $f^{(i)}(x)$.*

Then there exists a unique morphism $f^{(i+2)} : J^{k+i+2}(\pi) \to J^{i+2}(\pi)$ of bundles over M from π_{k+i+2} to π_{i+2} satisfying the following two conditions:

(2.5) *The square*
$$\begin{tikzcd} J^{k+i+2}(\pi) \arrow[r, "f^{(i+2)}"] \arrow[d, "\pi_{k+i+2,k+i+1}"'] & J^{i+2}(\pi) \arrow[d, "\pi_{i+2,i+1}"] \\ J^{k+i+1}(\pi) \arrow[r, "f^{(i+1)}"'] & J^{i+1}(\pi) \end{tikzcd}$$
is commutative.

(2.6) *For any $x \in J^{k+i+1}(\pi)$ and any $\nabla_x \in J^{k+i+2}(\pi)_x$, the preconnection ∇_x over π_{k+i+1} at x is $f^{(i+1)}$-related to the preconnection $f^{(i+2)}(\nabla_x)$ over π_{i+1} at $f^{(i+1)}(x)$.*

Proof: For any $x \in J^{k+i+1}(\pi)$ and any $\nabla_x \in J^{k+i+2}(\pi)_x$, it is easy to see that the assignment $\nabla_f(i+1)_{(x)} : t \in T_{\pi_{k+1}(x)}(M) \mapsto f^{(i+1)} \circ \nabla_x(t)$ is a preconnection over π_{i+1} at $f^{(i+1)}(x)$. Let $\nabla_{\pi_{k+i+1,k+i}(x)} = \pi_{k+i+2,k+i+1}(\nabla_x) = x$ and $\nabla_{f^{(i)} \circ\pi_{k+i+1,k+i}(x)} = \nabla_{\pi_{i+1,i} \circ f^{(i+1)}(x)} = f^{(i+1)}(x)$. Since ∇_x is $\pi_{k+i+1,k+i}$-related to $\nabla_{\pi_{k+i+1,k+i}(x)}$ and $\nabla_{\pi_{k+i+1,k+i}(x)}$ is in turn $f^{(i)}$-related to $\nabla_{f(i) \circ\pi_{k+i+1,k+i}(x)}$, we have that for any $t \in T_{\pi_{k+i+1}(x)}(M)$,

$$\begin{aligned} \pi_{i+1,i} \circ \nabla_f(i+1)_{(x)}(t) &= \pi_{i+1,i} \circ f^{(i+1)} \circ \nabla_x(t) \\ &= f^{(i)} \circ \pi_{k+i+1,k+i} \circ \nabla_x(t) \\ &= f^{(i)} \circ \nabla_{\pi_{k+i+1,k+i}(x)}(t) \\ &= \nabla_{f^{(i)} \circ\pi_{k+i+1,k+i}(x)}(t) \\ &= \nabla_{\pi_{i+1,i}\circ f^{(i+1)}(x)}(t), \end{aligned} \tag{2.7}$$

so that $\nabla_{f^{(i+1)}(x)} \in J^{i+2}(\pi)_{f^{(i+1)}(x)}$. Therefore the desired unique morphism $f^{(i+2)} : J^{k+i+2}(\pi) \to J^{i+2}(\pi)$ of bundles over M from π_{k+i+2} to π_{i+2} can and should be determined by the requirement that $f^{(i+2)}(\nabla_x) = \nabla_f(i+1)_{(x)}$ for any $x \in J^{k+i+1}(\pi)$ and any $\nabla_x \in J^{k+i+2}(\pi)_x$. ■

A *c-family of degree k* ("*c*" for "contact") is a family $\{f^{(i)} : J^{k+i}(\pi) \to J^i(\pi)\}_{i\geq 0}$ of morphisms of bundles over M from π_{k+i} to π_i subject to the following conditions ($i \geq 0$):

(2.8) The square
$$\begin{tikzcd} J^{k+i+1}(\pi) \arrow[r, "f^{(i+1)}"] \arrow[d, "\pi_{k+i+1,k+i}"'] & J^{i+1}(\pi) \arrow[d, "\pi_{i+1,i}"] \\ J^{k+i}(\pi) \arrow[r, "f^{(i)}"'] & J^{i}(\pi) \end{tikzcd}$$
is commutative.

(2.9) For any $x \in J^{k+i}(\pi)$ and any $\nabla_x \in J^{k+i+1}(\pi)_x$, the preconnection ∇_x over π_{k+i} at x is $f^{(i)}$-related to the preconnection $f^{(i+1)}(\nabla_x)$ over π_i at $f^{(i)}(x)$.

Now we introduce an equivalence relation among c-families. Two c-families $\mathbf{f} = \{f^{(i)} : J^{k+i}(\pi) \to J^i(\pi)\}_{i\geq 0}$ and $\mathbf{g} = \{g^{(i)} : J^{m+i}(\pi) \to J^i(\pi)\}_{i\geq 0}$ of degrees k and m are said to be *c-equivalent* if, assuming without loss of generality that $k \leq m$, we have $g^{(i)} = f^{(i)} \circ \pi_{m+i,k+i}$ for any $i \geq 0$.

Each c-family $\mathbf{f} = \{f^{(i)} : J^{k+i}(\pi) \to J^i(\pi)\}_{i \geq 0}$ determines its associated contact transformation $F_{\mathbf{f}} : J^\infty(\pi) \to J^\infty(\pi)$ of finite type over M by the requirement that for any $\mathbf{x} = \{x^{(i)}\}_{i\geq 0} \in J^\infty(\pi)$,

$$F_{\mathbf{f}}(\mathbf{x}) = \{f^{(i)}(x^{(k+i)})\}_{i\geq 0}, \tag{2.10}$$

which gives, under the assumption $[J^\infty \mathrm{I}]$, a bijective correspondence between c-families up to c-equivalence and contact transformations of finite type of $J^\infty(\pi)$ over M. Given two c-families $\mathbf{f} = \{f^{(i)} : J^{k+i}(\pi) \to J^i(\pi)\}_{i\geq 0}$ and $\mathbf{g} = \{g^{(i)} : J^{m+i}(\pi) \to J^i(\pi)\}_{i\geq 0}$, the composition $F_{\mathbf{g}} \circ F_{\mathbf{f}}$ of their associated contact transformations of finite type of $J^\infty(\pi)$ over M is represented by the c-family $\{g^{(i)} \circ f^{(m+i)} : J^{k+m+i}(\pi) \to J^i(\pi)\}_{i\geq 0}$.

A c_0*-function of degree* k is a morphism $f : J^k(\pi) \to E$ of bundles over M from π_k to π, which, owing to Propositions 2.1 and 2.2, determines its associated c-family $\mathbf{f}_f = \{f^{(i)} : J^{k+i}(\pi) \to J^i(\pi)\}_{i\geq 0}$ by the requirement that $f^{(0)} = f$.

Now we introduce an equivalence relation among c_0-functions. Two c_0-functions $f : J^k(\pi) \to E$ and $g : J^m(\pi) \to E$ are said to be c_0*-equivalent* if, assuming without loss of generality that $k \leq m$, we have $g = f \circ \pi_{m,k}$. Evidently there exists a bijective correspondence between $\mathcal{F}^f_M(\pi_\infty, \pi)$ and c_0-functions up to c_0-equivalence under the assumption $[J^\infty \mathrm{I}]$.

Proposition 2.3. *The above assignment $f \mapsto \mathbf{f}_f$ gives a bijective correspondence between c_0-functions up to c_0-equivalence and c-families up to c-equivalence.*

Proof: It suffices to show that, given two c-families $\mathbf{f} = \{f^{(i)} : J^{k+i}(\pi) \to J^i(\pi)\}_{i\geq 0}$ and $\mathbf{g} = \{g^{(i)} : J^{m+i}(\pi) \to J^i(\pi)\}_{i\geq 0}$, if c_0-functions $f^{(0)}$ and $g^{(0)}$ are c_0-equivalent, then c-families $\mathbf{f}$ and $\mathbf{g}$ are c-equivalent. For simplicity we assume that $m = k+1$. We will show that the family $\{f^{(i)} \circ \pi_{k+i+1,k+i} : J^{k+i+1}(\pi) \to J^i(\pi)\}_{i\geq 0}$ is indeed a c-family. We note that

$$\pi_{i+1,i} \circ f^{(i+1)} \circ \pi_{k+i+2,k+i+1} = f^{(i)} \circ \pi_{k+i+1,k+i} \circ \pi_{k+i+2,k+i+1}, \tag{2.11}$$

so that the family satisfies condition (2.5). Let $x \in J^{k+i+1}(\pi)$ and $\nabla_x \in J^{k+i+2}(\pi)_x$. Let $\nabla_{\pi_{k+i+1,k+i}(x)} = \pi_{k+i+2,k+i+1}(\nabla_x)$ and $\nabla_{f^{(i)}\circ\pi_{k+i+1,k+i}(x)} = f^{(i+1)}(\nabla_{\pi_{k+i+1,k+i}(x)})$. We note that for any $t \in T_{\pi_{k+i+1}(x)}(M)$,

$$\begin{aligned} f^{(i)} \circ \pi_{k+i+1,k+i} \circ \nabla_x(t) &= f^{(i)} \circ \nabla_{\pi_{k+i+1,k+i}(x)}(t) \\ &= \nabla_{f^{(i)}\circ\pi_{k+i+1,k+i}(x)}(t), \end{aligned} \tag{2.12}$$

so that the family satisfies the condition (2.6). Therefore the family $\{f^{(i)} \circ \pi_{k+i+1,k+i} : J^{k+i+1}(\pi) \to J^i(\pi)\}_{i\geq 0}$ is indeed a c-family. Since $\mathbf{g} = \{g^{(i)} : J^{m+i}(\pi) \to J^i(\pi)\}_{i\geq 0}$ is also a c-family with $f^{(0)} \circ \pi_{k+1,k} = g^{(0)}$ by assumption, we have $g^{(i)} = f^{(i)} \circ \pi_{k+i+1,k+i}$ for any $i \geq 0$, which completes the proof. ■

The above proposition finally yields the following characterization of contact transformations of finite type of $J^\infty(\pi)$ over M as its direct consequence.

Theorem 2.4. *There exists a bijective correspondence between $\mathcal{F}^f_M(\pi_\infty, \pi)$ and contact transformations of finite type of $J^\infty(\pi)$ over M.*

3 Contact Vector Fields

The principal objective of this section is to determine contact vector fields of finite type on $J^\infty(\pi)$. We assume that the bundle $\pi : E \to M$ is a vector bundle, i.e., that the fiber E_x over each $x \in M$ is a Euclidean $\mathbb{R}$-module. Let us begin with complete integrability of the Cartan distribution.

Proposition 3.1. *For any horizontal vector field X on $J^\infty(\pi)$ and any $d \in D$, X_d is a contact transformation of $J^\infty(\pi)$. In other words, X is a contact vector field on $J^\infty(\pi)$.*

Proof: Let t be a tangent vector to $J^\infty(\pi)$ with $\mathbf{x} = t(0)$. We define a microsquare γ on M as follows:

$$\gamma(d, d') = \pi_\infty \circ X_d \circ t(d') \tag{3.1}$$

for any $d, d' \in D$. By assumption $[J^\infty \mathrm{II}]$, we have

$$\begin{aligned} X_d \circ t(d') &= \nabla^\infty(\gamma(\cdot, d'), \nabla^\infty(\gamma(0, \cdot), \mathbf{x})(d'))(d) \\ &= \nabla^\infty(\gamma(d, \cdot), \nabla^\infty(\gamma(\cdot, 0), \mathbf{x})(d))(d'), \end{aligned} \tag{3.2}$$

so that the tangent vector $X_d \circ t$ to $J^\infty(\pi)$ is horizontal. This completes the proof.■

The following proposition shows that the class of contact vector fields on $J^\infty(\pi)$ is an $\mathbb{R}$-submodule of the $\mathbb{R}$-module of vector fields on $J^\infty(\pi)$.

Proposition 3.2. *The class of contact vector fields is closed under addition and scalar multiplication.*

Proof: The class is clearly closed under scalar multiplication, for we have

$$(\alpha X)_d = X_{\alpha d} \tag{3.3}$$

for any vector field X on $J^\infty(\pi)$, any $\alpha \in \mathbb{R}$ and any $d \in D$. The closedness of the class under addition follows from the formula

$$(X + Y)_d = X_d \circ Y_d \tag{3.4}$$

for any vector fields X, Y on $J^\infty(\pi)$ and any $d \in D$, for which the reader is referred to Lavendhomme [1996, §3.2, Proposition 6]. ■

Now we will determine vertical contact vector fields of finite type on $J^\infty(\pi)$. It is easy to see that

Proposition 3.3. *A vector field X on $J^\infty(\pi)$ is vertical iff the transformation $X_d : J^\infty(\pi) \to J^\infty(\pi)$ is over M for any $d \in D$.*

Now we are ready to present our fundamental theorem on vertical contact vector fields of finite type on $J^\infty(\pi)$.

Theorem 3.4. *There is a bijective correspondence between vertical contact vector fields of finite type on $J^\infty(\pi)$ and $\mathcal{F}^f_M(\pi_\infty, \pi)$.*

Proof: By Theorem 2.4 and Proposition 3.3 there is a bijective correspondence between vertical contact vector fields of finite type on $J^\infty(\pi)$ and $\mathcal{F}^f_E(\pi_{\infty,0}, \upsilon_\pi)$. Since we can naturally identify $\mathcal{F}^f_E(\pi_{\infty,0}, \upsilon_\pi)$ and $\mathcal{F}^f_M(\pi_\infty, \pi)$, the proof is complete. ■

Given $\varphi \in \mathcal{F}^f_M(\pi_\infty, \pi)$, we will denote by $\ni_\varphi$ the vertical contact vector field of finite type on $J^\infty(\pi)$ corresponding to φ under Theorem 3.4.

Now we are ready to present the fundamental theorem on contact vector fields of finite type on $J^\infty(\pi)$.

Theorem 3.5. *A contact vector field X of finite type on $J^\infty(\pi)$ can be written uniquely as*

$$X = \hat{Y} + \ni_\varphi \tag{3.5}$$

for a vector field Y along $\pi_\infty : J^\infty(\pi) \to M$ and $\varphi \in \mathcal{F}^f_M(\pi_\infty, \pi)$, in which $Y_d = \pi_\infty \circ X_d$ for any $d \in D$ and $X - \hat{Y} = \ni_\varphi$.

Proof: This follows from Propositions 3.1 and 3.2 and Theorem 3.4. ∎

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