An operator representation for weighted inductive limits of spaces of vector valued holomorphic functions

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Dedicated to Professor Jean Schmets on the occasion of his 60th birthday

Abstract

We discuss the problem when an isomorphism $\mathcal{V}H(G, E'_b) = \mathcal{L}_b(E, \mathcal{V}H(G))$ holds for weighted (LB)-spaces $\mathcal{V}H(G)$ and Fréchet spaces E. There is a close connection to a special case of Grothendieck's "problème des topologies", which raises the (open) question under which conditions $\mathcal{V}H(G)$ has (QNo)' in the terminology of Peris [28]. We also point out an interesting relation to vector valued projective description. In the last section, a result on weighted inductive limits of spaces of holomorphic functions on product sets is given.

1 Introduction; preliminaries

We consider the vector valued generalization $\mathcal{V}H(G, E)$ of weighted (LB)-spaces $\mathcal{V}H(G)$ of holomorphic functions with O-growth conditions and discuss the formal analog of [8], Theorem 2 in this context : When do we have an operator representation

$$\mathcal{V}H(G, E'_b) = \mathcal{L}_b(E, \mathcal{V}H(G))$$

for a Fréchet space E ? Thus, this article is a sequel to [8], and only the results in that paper enabled us to start the present research. But the main problem considered here is of a completely different character and corresponds to a question

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which did not arise in the operator representation $HV(G, E_b') = \mathcal{L}_b(E, HV(G))$ of weighted spaces of holomorphic functions with values in the strong dual of a quasibarrelled locally convex space E; viz., whether the corresponding space of operators has "good" locally convex properties.

In order to explain the difference, we note that $H^\infty(G, E_b') = \mathcal{L}_b(E, H^\infty(G))$ is valid for any quasibarrelled space E and any open set $G \subset \mathbb{C}^N$ while there exists a reflexive Fréchet space E for which $\mathcal{L}_b(E, H^\infty(D))$ is not (DF), where D denotes the open unit disc (see [13]). Thus, the operator representation of weighted spaces of vector valued holomorphic functions does not necessarily imply that the corresponding space of operators is (DF) or bornological. On the other hand, it is easy to see that for a normed space E, the topological equality $\mathcal{V}H(G, E_b') = \mathcal{L}_b(E, \mathcal{V}H(G))$ holds if and only if this space is (ultra-) bornological.

In the case of o-growth conditions; i.e., of the spaces $\mathcal{V}_0H(G, E)$, one can use the ε-(tensor) product and discuss when the equality $\mathcal{V}_0H(G, E) = E\varepsilon\mathcal{V}_0H(G)$ holds. This amounts to interchanging the inductive limit with the ε-product and was first discussed in [9]. Even if the inductive limit is a (DFS)-space; i.e., has compact linking maps, this is not always possible, as Peris [27] showed. Peris [28] went on to solve the problem of interchanging the inductive limit of a (DFS)-space with the ε-product of arbitrary Banach spaces and introduced "locally convex properties by operators" like (QNo), quasinormable by operators, and (QNo)', the strict Mackey condition by operators. They are related to Grothendieck's famous "problème des topologies", which had been solved in the negative by Taskinen [31]. But only the positive results on special cases of this problem, which arose from the work of Taskinen, Peris and other authors after the negative solution of the general "problème des topologies", made our present research possible, as well as applicable.

The organization of the article is as follows : In the rest of this section, we give preliminaries on weighted inductive limits, their preduals, and projective description, as well as on Fréchet spaces and the properties (QNo) and (QNo)' of Peris. Section 2 contains the main results on $\mathcal{V}H(G, E_b') = \mathcal{L}_b(E, \mathcal{V}H(G))$. Let the bounded subsets of $\mathcal{V}H(G)$ be metrizable (which holds, in particular, if $\mathcal{V}$ is regularly decreasing), and let E be a Fréchet space for which the bounded subsets of E_b' are metrizable. Then the desired topological equality holds if and only if (E, F) has property (BB) (i.e., if the "problème des topologies" has a positive solution for $E\hat{\otimes}_\pi F$), where F denotes the predual of $\mathcal{V}H(G)$ (Theorem 1). For a decreasing sequence $\mathcal{V}$ of radial weights v_n on a balanced set $G \subset \mathbb{C}^N$ such that $H(v_1)_0(G)$ contains the polynomials, (E, F) does have property (BB) for any Fréchet space E if $\mathcal{V}$ satisfies condition (S). This reproves a result of [12] (Corollary 2). In terms of the conditions of Peris, the desired topological equality holds if $\mathcal{V}H(G)$ is (QNo)' and E is (QNo), which is true, in particular, if E is a Banach space (Proposition 3). Many examples in which Proposition 3 can be applied follow from work of Shields-Williams [30], Lusky [22] and [23], Galbis [18], and from a recent proposition of Peris [29]. At the end of Section 2, an easy application of the main theorem of [8] shows that whenever projective description holds in the scalar case, then the desired topological equality $\mathcal{V}H(G, E_b') = \mathcal{L}_b(E, \mathcal{V}H(G))$ is true if and only if E_b'-valued projective description holds (Proposition 6). Finally, Section 3 contains our results (Proposition 7, Corollary 8) on $(\mathcal{V} \otimes \mathcal{W})H(G_1 \times G_2)$.

1.1 Preliminaries on weighted inductive limits and projective description

Our notation on weighted spaces $HV(G,E)$ of vector valued holomorphic functions is standard; e.g. see [8]. Let $\mathcal{V} = (v_n)_n$ be a decreasing sequence of (strictly) positive continuous functions on an open set $G \subset \mathbb{C}^N$. For a quasicomplete locally convex space E, the *weighted inductive limits of spaces of vector valued holomorphic functions* are defined by

$$\mathcal{V}H(G,E) := \operatorname{ind}_n Hv_n(G,E), \; \mathcal{V}_0H(G,E) := \operatorname{ind}_n H(v_n)_0(G,E),$$

while the corresponding spaces of scalar functions will be denoted by $\mathcal{V}H(G)$ and $\mathcal{V}_0H(G)$. We also consider the *"projective hulls"* $H\overline{V}(G,E)$ and $H\overline{V}_0(G,E)$ of the spaces $\mathcal{V}H(G,E)$ and $\mathcal{V}_0H(G,E)$, respectively, where $\overline{V}$ denotes the Nachbin family

$$\overline{V} := \left\{ \overline{v} > 0 \text{ continuous on } G; \; \sup_G \frac{\overline{v}}{v_n} < \infty \; \forall n \in \mathbb{N} \right\}.$$

We remark that, under our conditions, the weighted topology of $H\overline{V}(G,E)$ is stronger than uniform convergence on the compact subsets of G. $\mathcal{V}H(G,E)$ is continuously injected in $H\overline{V}(G,E)$, and $\mathcal{V}_0H(G,E)$ is continuously injected in $H\overline{V}_0(G,E)$. We say that *E-valued projective description holds* if the first two spaces are equal algebraically and topologically. Then the inductive limit topology of $\mathcal{V}H(G,E)$ is given by a system of weighted sup-seminorms. In the scalar case, in which we again write $H\overline{V}(G)$ instead of $H\overline{V}(G,\mathbb{C})$, $\mathcal{V}H(G) = H\overline{V}(G)$ holds algebraically, the inductive limit is always complete (and hence regular), and the two spaces have the same bounded sets. If $\mathcal{V}H(G)$ is a semi-Montel space, then $\mathcal{V}H(G)$ and $H\overline{V}(G)$ are equal algebraically and topologically. But even under this assumption it remains an open problem for which, say, Banach or (DF)-spaces E the topological equality $\mathcal{V}H(G,E) = H\overline{V}(G,E)$ holds.

The sequence $\mathcal{V}$ satisfies *condition (S)* if for each $n \in \mathbb{N}$ there is $m > n$ such that v_m/v_n vanishes at infinity. Then $\mathcal{V}H(G)$ is a (DFS)-space; if n, m are as in condition (S), then the injection of $Hv_n(G)$ into $Hv_m(G)$ is compact. $\mathcal{V}$ is said to be *regularly decreasing* if

for each $n \in \mathbb{N}$ there is $m \geq n$ such that for each subset Y of X :

$$\inf_{y\in Y} \frac{v_m(y)}{v_n(y)} > 0 \implies \inf_{y\in Y} \frac{v_k(y)}{v_n(y)} > 0 \quad \text{for} \quad k = m+1, m+2, \ldots$$

Then $\operatorname{ind}_n Hv_n(G)$ is boundedly retractive (see [15], Corollary 2.7); in particular, the bounded subsets of $\mathcal{V}H(G)$ are metrizable. Clearly, (S) implies that $\mathcal{V}$ is regularly decreasing. See [15], [12], [14] for more information on these conditions and on the projective description problem. In particular, [14] contains a characterization of the boundedly retractive spaces $\mathcal{V}H(G)$.

As explained in [7], Section 3.B, letting B_n denote the closed unit ball of $Hv_n(G)$ for $n \in \mathbb{N}$, $\mathcal{V}H(G)$ is the inductive dual F_i' of the Fréchet space

$$F = \{u \in (\mathcal{V}H(G))^*; \; u_{|B_n} \text{ is continuous w.r.t. the compact-open topology } \forall n \in \mathbb{N}\},$$

endowed with the topology of uniform convergence on all B_n (or, equivalently, uniform convergence on all bounded subsets of $\mathcal{V}H(G)$). A different, quite interesting description of the predual F can be found in [12], Proposition 1.8.(a). Of course, if F is distinguished, we get $\mathcal{V}H(G) = F_b'$. In general, the topology of F_b' is stronger than the weighted topology of $H\overline{V}(G)$. If $\mathcal{V}$ satisfies *condition (D)* (see [10] and [5], II), then the bounded subsets of $H\overline{V}(G)$ are metrizable, and $F_b' = H\overline{V}(G)$ is equivalent to $\mathcal{V}H(G) = H\overline{V}(G)$ topologically by [12], Proposition 1.8.(b), (c). If $\mathcal{V}H(G)$ is boundedly retractive, then F is quasinormable, a fortiori distinguished, so that $\mathcal{V}H(G) = F_b'$ must hold in this case. If $\mathcal{V}$ satisfies condition (S), then F is even a Fréchet-Schwartz space. If G is balanced, the weights v_n are radial and if each $H(v_n)_0(G)$ contains the polynomials, the predual F of $\mathcal{V}H(G)$ is canonically topologically isomorphic to $(\mathcal{V}_0H(G))_b'$ by [12], 1.9.

1.2 Preliminaries on locally convex spaces, Grothendieck's "problème des topologies" and the "locally convex properties by operators" by Peris

For background information on locally convex spaces we refer to [25]; notation and facts on locally convex inductive limits can be found in the survey [2]. The symbol $\simeq$ will sometimes be used to denote topological isomorphisms between Banach spaces.

A locally convex space E has the *countable neighborhood property (cnp)* if, given any sequence $(p_n)_n$ of continuous seminorms on E, there are a continuous seminorm p on E and a sequence $(a_n)_n$ of positive numbers such that $p_n \leq a_n p$ for each $n \in \mathbb{N}$. All (DF)-spaces have cnp. A Fréchet space E satisfies the *density condition* if and only if the bounded subsets of its strong dual E_b' are metrizable, cf. [4]. If E is quasinormable or Montel, then it has the density condition, while, on the other hand, any Fréchet space with the density condition is distinguished.

A pair (E, F) of Fréchet spaces is said to have *property (BB)* if each bounded subset of the (complete) projective tensor product $E\hat{\otimes}_\pi F$ is contained in the closed absolutely convex hull of a set $B_1 \otimes B_2$ with B_1 bounded in E and B_2 bounded in F. Then

$$(E\hat{\otimes}_\pi F)_b' = \mathcal{L}_b(E, F_b')$$

holds topologically by [19]. Grothendieck asked if each pair of Fréchet spaces satisfies (BB), and this was his famous "problème des topologies". The "problème des topologies" was solved in the negative by J. Taskinen [31]. Since that time, also many positive results on property (BB) have been proved, by a number of authors.

If G is balanced, the weights v_n are radial, if each $H(v_n)_0(G)$ contains the polynomials, and if $\mathcal{V}$ satisfies condition (S), then $\mathcal{V}H(G) = \mathcal{V}_0H(G)$ is a (DFS)-space with the bounded approximation property (and the polynomials are dense in this space) by [12], Theorem 1.6.(a). Then its predual F equals its strong dual and is a Fréchet-Schwartz space with the bounded approximation property. Hence (E, F) satisfies property (BB) for each Banach space E by [12], Propositions 2.1 and 2.2.(b). (Note that it turned out later on that the existence of a total bounded set is not needed in 2.2.(b), cf. [3].)

The problem of interchanging the inductive limit of a (DFS)-space with the ε-product with arbitrary Banach spaces was well-known since the 1970's. It was solved only recently by A. Peris [28] (also see [3] for a survey). He defined two locally convex properties by operators, (QNo) (quasinormable by operators) and (QNo)' (strict Mackey condition by operators), which are dual to each other : If a quasibarrelled space is (QNo), then its strong dual satisfies (QNo)', and the strong dual of a locally convex space with (QNo)' is (QNo). Here are the main results relevant for the present discussion : Complemented subspaces of spaces with (QNo) are (QNo), too. If the Fréchet spaces E and F are (QNo), then (E, F) has property (BB). For a quasinormable Fréchet space F, (E, F) has property (BB) for each Banach space E if and only if F_b' satisfies (QNo)'. If the Fréchet space F is (QNo) and the (DF)-space H satisfies (QNo)', then $\mathcal{L}_b(F, H)$ is a bornological (DF)-space.

2 Operator representation for weighted inductive limits : vector valued functions

Throughout this section, $\mathcal{V} = (v_n)_n$ is a decreasing sequence of positive continuous functions on an open set $G \subset \mathbb{C}^N$, and E is a quasibarrelled locally convex space. From [8], Theorem 2 we obtain that

$$\mathcal{V}H(G, E_b') = \operatorname*{ind}_n Hv_n(G, E_b') = \operatorname*{ind}_n \mathcal{L}_b(E, Hv_n(G))$$

holds topologically, and this raises the *problem* when

$$\operatorname*{ind}_n \mathcal{L}_b(E, Hv_n(G)) = \mathcal{L}_b(E, \operatorname*{ind}_n Hv_n(G)) = \mathcal{L}_b(E, \mathcal{V}H(G))$$

holds algebraically and topologically; i.e., when the inductive limit $\operatorname{ind}_n Hv_n(G))$ interchanges with the space of continuous linear operators defined on E.

We start our discussion of this problem along the lines of [9], Section 3 : The algebraic equality is valid under very general conditions. Note first that, by identifying each $\mathcal{L}_b(E, Hv_n(G))$ with a subspace of $\mathcal{L}_b(E, \mathcal{V}H(G))$, one obtains a canonical continuous linear injection i of $\operatorname{ind}_n \mathcal{L}_b(E, Hv_n(G))$ into $\mathcal{L}_b(E, \mathcal{V}H(G))$. The surjectivity of i amounts to showing that all continuous linear mappings from E into $\operatorname{ind}_n Hv_n(G)$ factorize continuously through one of the step spaces $Hv_n(G)$. Since the step spaces are Banach, it is a standard consequence of Grothendieck's factorization theorem that i is surjective whenever E is a Fréchet space. (Note that it would suffice here that the step spaces in the inductive limit are Fréchet, and a modification of Grothendieck's factorization theorem, due to De Wilde and Floret, would allow the same conclusion whenever the inductive limit is regular – which, as we have already pointed out, always holds in the present framework.) But the case of a Fréchet space E is the most important one here anyway, and we will restrict our attention to this case as we discuss the topological equality, from this point on.

Then S. Dierolf [17], 4.2.(b) implies that $\operatorname{ind}_n \mathcal{L}_b(E, Hv_n(G))$ is regular and that this space has the same bounded sets as $\mathcal{L}_b(E, \mathcal{V}H(G))$. By [17], 4.4.(a), the two spaces are equal topologically whenever $\mathcal{L}_b(E, \mathcal{V}H(G))$ is bornological. If E is even a Banach space, then the (LB)-space $\operatorname{ind}_n \mathcal{L}_b(E, Hv_n(G))$ is nothing but the bornological space associated with $\mathcal{L}_b(E, \mathcal{V}H(G))$ (cf. the method of proof of [9], 3.12; in this context also see the method of proof of [9], 4.6, which allows to show a similar result for a larger class of Fréchet spaces E).

Moreover, whenever $\operatorname{ind}_n Hv_n(G)$ is boundedly retractive, it follows from [17], 4.2.(c) that also $\operatorname{ind}_n \mathcal{L}_b(E, Hv_n(G))$ is boundedly retractive and that this space induces the same topology as $\mathcal{L}_b(E, \mathcal{V}H(G))$ on the bounded subsets. In this case, the two spaces are equal topologically whenever $\mathcal{L}_b(E, \mathcal{V}H(G))$ is "only" a (DF)-space (cf. [17], 4.4.(b)).

At this point, we turn to the connection of our problem with (a special case of) Grothendieck's "problème des topologies", and here we need the Fréchet predual F of $\mathcal{V}H(G)$. We will always suppose that F is distinguished so that $\mathcal{V}H(G) = F'_b$ holds. Generally, $\mathcal{L}(E, F'_b)$ is the strong dual of the space $E\hat{\otimes}_\pi F$, when equipped with the topology of uniform convergence on the saturated system $\mathcal{A}$ of all subsets of bounded sets of the form $\overline{\Gamma(B_1 \otimes B_2)}$, where B_1 is bounded in E, B_2 is bounded in F and Γ denotes the absolutely convex hull. But if the pair (E, F) of Fréchet spaces has the property (BB), then

$$(E\hat{\otimes}_\pi F)'_b = \mathcal{L}_b(E, F'_b) = \mathcal{L}_b(E, \mathcal{V}H(G))$$

holds topologically. From our considerations so far, we get :

Let $\mathcal{V} = (v_n)_n$ be a decreasing sequence of positive continuous functions on an open subset G of $\mathbb{C}^N$ such that the predual F of $\mathcal{V}H(G)$ is distinguished. (This holds, in particular, if $\mathcal{V}H(G)$ is boundedly retractive, in which case F is even quasinormable, or if $\mathcal{V}$ satisfies condition (D) and $\mathcal{V}H(G)$ and $H\overline{V}(G)$ induce the same topology on their bounded sets.) Assume that E is a Fréchet space such that (E, F) has property (BB). Then $E\hat{\otimes}_\pi F$ distinguished implies $\mathcal{L}_b(E, \mathcal{V}H(G))$ bornological, and hence we obtain $\mathcal{V}H(G, E'_b) = \mathcal{L}_b(E, \mathcal{V}H(G))$ topologically.

Now Theorem 1.(a) below is a direct consequence of [6], Corollary 7 and [26], Corollary 1.2.

1. Theorem. *(a) Let the bounded subsets of $\mathcal{V}H(G)$ be metrizable, which holds whenever $\mathcal{V}H(G)$ is boundedly retractive, and let the Fréchet space E have the density condition. If we assume that, for the predual F of $\mathcal{V}H(G)$, the pair (E, F) has (BB), then we obtain*

$$(*)\ \mathcal{V}H(G, E'_b) = \mathcal{L}_b(E, \mathcal{V}H(G)) = \mathcal{L}_b(E, F'_b) = (E\hat{\otimes}_\pi F)'_b$$

topologically, and this space has a fundamental sequence of bounded subsets which are metrizable.

(b) Conversely, if (the last equality of) $(*)$ holds, then the pair (E, F) must have property (BB).

Proof of (b). By our previous discussion, $\mathcal{L}(E, F'_b)$ is the strong dual of the Fréchet space $G := E\hat{\otimes}_\pi F$, equipped with the topology of uniform convergence on the saturated family $\mathcal{A}$ of bounded subsets of G covering G. But if this strong dual equals $\mathcal{L}_b(E, F'_b)$, it is a (DF)-space with metrizable bounded subsets by [6], Proposition 6, and then [16], Lemma 4.1 (p. 216/7) yields that every bounded subset of G belongs to $\mathcal{A}$; i.e., the pair (E, F) satisfies (BB). (We would like to thank J. Bonet and A. Peris for indicating the relevance of [16], Lemma 4.1 in this context.) ■

Thus, in the case that E is a Fréchet space with the density condition and, say, $\mathcal{V}$ is regularly decreasing, the topological equality $\mathcal{V}H(G, E'_b) = \mathcal{L}_b(E, \mathcal{V}H(G))$ is reduced to a positive solution of Grothendieck's "problème des topologies" for the pair (E, F). Also note that for $\mathcal{V}H(G)$ boundedly retractive $E\hat{\otimes}_\pi F$ must even be quasinormable for quasinormable E by [21], 15.6.5.

The following known (cf. [12]) result now follows directly from Theorem 1 and from the preliminaries if E is a Banach space.

2. Corollary. *Let $\mathcal{V}$ be a decreasing sequence of radial weights on a balanced set $G \subset \mathbb{C}^N$ which satisfies (S) and such that $H(v_1)_0(G)$ contains the polynomials. Then for any Fréchet space E we have topologically*

$$\mathcal{V}H(G, E'_b) = H\overline{V}(G, E'_b) = \mathcal{L}_b(E, \mathcal{V}H(G)) = \mathcal{L}_b(E, H\overline{V}(G)).$$

Proof . By [12], Corollary 2.5, $\mathcal{V}H(G, E) = H\overline{V}(G, E)$ holds topologically even for all quasicomplete locally convex spaces E with the countable neighborhood property (but it was already noted there that this fact need not imply that the space is a bornological (DF)-space whenever E is). On the other hand, at the very end of [12], it was pointed out that, under the conditions of Corollary 2 on $\mathcal{V}$, one has $\mathcal{L}_b((\mathcal{V}_0H(G))'_b, E) = \mathcal{V}H(G, E)$ topologically for quasicomplete locally convex spaces E with the countable neighborhood property. Now let E be any Fréchet space and apply the above equality with E'_b instead of E (as a (DF)-space, E'_b has cnp) as well as, once more, [17], 5.10 to obtain

$$\mathcal{V}H(G, E'_b) = \mathcal{L}_b((\mathcal{V}_0H(G))'_b, E'_b) = \mathcal{L}_b(E, ((\mathcal{V}_0H(G))'_b)'_b) = \mathcal{L}_b(E, \mathcal{V}H(G)). \quad ■$$

By a result of Peris [28] (see the preliminaries), $\mathcal{L}_b(F, H)$ is a bornological (DF)-space if F is a Fréchet space with (QNo) and H is a (DF)-space with (QNo)'. Thus, in order to generalize Corollary 2 from sequences $\mathcal{V}$ with (S), say, to regularly decreasing sequences, one has to prove that $\mathcal{V}H(G)$ is (QNo)'.

3. Proposition. *If $\mathcal{V}H(G)$ is (QNo)' and E is Banach or (QNo), then one gets the desired topological equality*

$$\mathcal{V}H(G, E'_b) = \mathcal{L}_b(E, \mathcal{V}H(G)).$$

We note that for balanced G and radial functions v_n such that each $H(v_n)_0(G)$ contains the polynomials, $n = 1, 2, \ldots$, the predual F of $\mathcal{V}H(G)$ is the strong dual (and even the $'_c$-dual) of a space with the bounded approximation property by [12], 1.6.(b) and 1.8.(a). But this does not allow to conclude that, in this case (and under the conditions of Theorem 1), $\mathcal{V}H(G)$ must be (QNo)', as [28], Counterexample (2) after Theorem 4.7 shows. (We thank A. Peris for pointing this out to us.) At this point it remains an important *open problem* for which (e.g., regularly) decreasing sequences $\mathcal{V}$ the space $\mathcal{V}H(G)$ is (QNo)'.

However, due to the following result of Peris [29], there are many examples of spaces $\mathcal{V}H(G)$ with (QNo)' :

Proposition (Peris). *Let $X = \mathrm{ind}_n X_n$ be a boundedly retractive (LB)-space such that each of the step spaces X_n is isomorphic to l_∞. Then X has (QNo)'.*

In fact, Shields-Williams [30] and Lusky [22] have proved that for certain radial weights v on the unit disk D the space $Hv_0(D)$ is isomorphic to c_0, and Galbis [18] gave a condition for radial weights v on the complex plane such that $Hv_0(\mathbb{C})$ is isomorphic to c_0. From these results and known ε-product representations (see [1]), examples of radial weights v on $G =$ a polydisk or $\mathbb{C}^N$ follow such that $Hv_0(G)$ is again isomorphic to c_0. Let us say that a radial weight on a balanced domain $G \subset \mathbb{C}^N$ is *fine* if $Hv_0(G)$ contains the polynomials and if $Hv_0(G) \simeq c_0$. Then we know from [11] and [12] that $Hv(G)$ is the bidual of $Hv_0(G)$ and hence isomorphic to l_∞. Thus, if all the radial weights v_n in a decreasing sequence $\mathcal{V}$ on a balanced domain G are fine and if $\mathcal{V}H(G)$ is boundedly retractive, then the space $\mathcal{V}H(G)$ has (QNo)', and Proposition 3 applies. – Note that in the recent article [23] of Lusky, it is even shown that for radial weights v on the unit disk "with moderate decay" $Hv(D)$ can only be isomorphic to either l_∞ or $H^\infty(D)$.

Even if there are many cases in which all the step spaces of $\mathcal{V}H(G)$ are isomorphic to l_∞, it is still not trivial to get concrete examples in which Proposition 3 applies, but not already (the known) Corollary 2. This is mainly due to the fact that it is virtually impossible to construct regularly decreasing sequences $\mathcal{V} = (v_n)_n$ of *radial* weights v_n on the open unit disk D which do not satisfy condition (S).

But concrete examples of such sequences for which the weights v_n are not radial can easily be found. We thank W. Lusky for reminding us of the following simple, but useful fact : Let v denote a strictly positive continuous function on the unit disk, and let f be a holomorphic function on D with $f(z) \neq 0$ for all $z \in D$. Then $Hv(D)$ is (isometrically) isomorphic to $H(v|f|)(D)$ by the mapping $g \to g/f$.

4. Example. Let v be any radial weight on D such that $Hv(D) \simeq l_\infty$ and put for each $n \in \mathbb{N}$,

$$v_n(z) \ := 2^{-n}|1 - z|^n v(z).$$

Then $\mathcal{V} = (v_n)_n$ is a regularly decreasing sequence of weights on D which clearly does not satisfy (S). To sequences of this type Proposition 3 applies (since all step spaces are isomorphic to l_∞), but not Corollary 2.

This method also serves to show that there are many examples of spaces $\mathcal{V}H(G)$ for which the weights v_n are not radial, and yet all the step spaces $Hv_n(G)$ are isomorphic to l_∞.

Now we would like to demonstrate that our problem is connected with the question when vector valued projective description holds. In fact, [8], Theorem 2 directly applies to yield part (a) of the following corollary.

5. Corollary. *(a) For any quasibarrelled locally convex space E, we have*

$$H\overline{V}(G, E_b') = \mathcal{L}_b(E, H\overline{V}(G)).$$

(b) $H\overline{V}(G, E_b')$ has a fundamental sequence of bounded sets and the countable neighborhood property for any Fréchet space E.

PROOF OF (b). Since $\mathcal{V}H(G)$ and $H\overline{V}(G)$ have the same bounded sets, $H\overline{V}(G)$ has a fundamental sequence of bounded sets. Moreover, it is easy to see that $H\overline{V}(G)$ always has cnp : $H\overline{V}(G)$ is a topological subspace of the co-echelon space $K_\infty(\overline{V})$, which, as the strong dual of the corresponding echelon space, is a (DF)-space and hence has cnp. But the cnp is inherited by topological subspaces. Now (b) follows from (a) and [17], Propositions 2.7 and 2.8 (together with 2.2). ∎

Of course, [9], 4.3 implies in an easy way that, whenever $\mathcal{V}$ satisfies (S) and E is a Fréchet-Schwartz space, then $H\overline{V}(G, E_b')$ is a (DFS)-space. For general Fréchet spaces E, however, the question when $H\overline{V}(G, E_b')$ enjoys the (DF)-property is quite delicate, and we have preferred to discuss this problem for the more important spaces $\mathcal{V}H(G, E)$.

Since $H\overline{V}(G, E_b') = \mathcal{L}_b(E, H\overline{V}(G))$ holds for any quasibarrelled locally convex space E, it is obvious that our problem is closely connected with the *E_b'-valued projective description problem*; i.e., when do we have $\mathcal{V}H(G, E_b') = H\overline{V}(G, E_b')$ topologically? In fact :

6. Proposition. *Let E be a quasibarrelled space. If projective description holds in the scalar case; i.e., $\mathcal{V}H(G) = H\overline{V}(G)$ topologically, then*

$$H\overline{V}(G, E_b') = \mathcal{L}_b(E, \mathcal{V}H(G)),$$

and hence $\mathcal{V}H(G, E_b') = \mathcal{L}_b(E, \mathcal{V}H(G))$ is equivalent to $\mathcal{V}H(G, E_b') = H\overline{V}(G, E_b')$.

3 Operator representation for weighted inductive limits : scalar functions on product sets

Turning to the operator representation for weighted inductive limits of spaces of holomorphic functions in several variables, we start with the observation that [15], Corollary 5.12 implies the following result : If $\mathcal{V} = (v_n)_n$ and $\mathcal{W} = (w_n)_n$ are decreasing sequences of positive continuous functions on open sets $G_1 \subset \mathbb{C}^N$ and $G_2 \subset \mathbb{C}^M$ such that $\mathcal{W}H(G_2)$ is a boundedly retractive inductive limit, then

$$\operatorname*{ind}_n Hv_n(G_1, Hw_n(G_2)) = \mathcal{V}H(G_1, \mathcal{W}H(G_2)) = H\overline{V}(G_1, \mathcal{W}H(G_2))$$

holds algebraically and $\operatorname{ind}_n Hv_n(G_1, Hw_n(G_2))$ is the bornological space associated with $\mathcal{V}H(G_1, \mathcal{W}H(G_2))$ (or with $H\overline{V}(G_1, \mathcal{W}H(G_2))$). From [8], Corollary 5 and Theorem 7 it follows that, for each $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{L}_b(P_n, Q_n') &= Hw_n(G_2, P_n') = Hw_n(G_2, Hv_n(G_1)) \\ &= H(v_n \otimes w_n)(G_1 \times G_2) \\ &= \mathcal{L}_b(Q_n, P_n') = Hv_n(G_1, Q_n') = Hv_n(G_1, Hw_n(G_2)) \end{aligned}$$

algebraically and topologically, where P_n and Q_n denotes the predual of $Hv_n(G_1)$ and $Hw_n(G_2)$, respectively. As a consequence, if $\mathcal{V} \otimes \mathcal{W}$ denotes the sequence $(v_n \otimes w_n)_n$ on $G_1 \times G_2$, we have

$$(\mathcal{V} \otimes \mathcal{W})H(G_1 \times G_2) = \operatorname*{ind}_n Hv_n(G_1, Hw_n(G_2)).$$

On the other hand, let P and Q be the preduals of $\mathcal{V}H(G_1)$ and $\mathcal{W}H(G_2)$, respectively; we assume that P is distinguished so that $\mathcal{V}H(G_1) = P'_b$ and note that, by the preliminaries, $\mathcal{W}H(G_2) = Q'_b$ also holds since $\mathcal{W}H(G_2)$ is boundedly retractive (and hence Q satisfies the density condition). Then [8], Corollary 5 yields

$$Hv_n(G_1, \mathcal{W}H(G_2)) = Hv_n(G_1, Q'_b) = \mathcal{L}_b(P_n, Q'_b) = \mathcal{L}_b(P_n, \mathcal{W}H(G_2))$$

for $n = 1, 2, ...$ From this point on, we assume that all the spaces $Hv_n(G_1)$ are isomorphic to l_∞ so that for each $n \in \mathbb{N}$, P_n is a $\mathcal{L}_1$-space. Fix $n \in \mathbb{N}$. Then by a result of Taskinen [32], (P_n, Q) has property (BB). If P even satisfies the density condition (or, equivalently, if the bounded subsets of $\mathcal{V}H(G_1)$ are metrizable), then [6], Corollary 1.7 implies that $\mathcal{L}_b(P_n, Q'_b)$ is bornological, whence

$$\mathcal{V}H(G_1, \mathcal{W}H(G_2)) = \operatorname*{ind}_n Hv_n(G_1, \mathcal{W}H(G_2))$$

must also be bornological. Summing up, we have proved the following result.

7. Proposition. *Let $\mathcal{V} = (v_n)_n$ and $\mathcal{W} = (w_n)_n$ denote decreasing sequences of positive continuous functions on open sets $G_1 \subset \mathbb{C}^N$ and $G_2 \subset \mathbb{C}^M$. We suppose that*

(i) each step space $Hv_n(G_1)$ of $\mathcal{V}H(G_1)$ is isomorphic to l_∞,

(ii) the bounded subsets of $\mathcal{V}H(G_1)$ are metrizable,

(iii) $\mathcal{W}H(G_2)$ is a boundedly retractive inductive limit.

Then the space

$$(\mathcal{V} \otimes \mathcal{W})H(G_1 \times G_2) = \operatorname*{ind}_n Hv_n(G_1, Hw_n(G_2)) = \operatorname*{ind}_n Hw_n(G_2, Hv_n(G_1))$$

equals $\mathcal{V}H(G_1, \mathcal{W}H(G_2))$ algebraically and topologically.

8. Corollary. *If, in addition to the assumptions of Proposition 7, G_2 is balanced, all the weights w_n are radial, all the spaces $H(w_n)_0(G_2)$ contain the polynomials, and if each step space $Hw_n(G_2)$ of $\mathcal{W}H(G_2)$ is isomorphic to l_∞, then we obtain*

$$\mathcal{V}H(G_1, \mathcal{W}H(G_2)) = \mathcal{L}_b(Q, \mathcal{V}H(G_1)),$$

where Q denotes the predual of $\mathcal{W}H(G_2)$.

Proof. The last of our additional conditions implies that $\mathcal{W}H(G_2)$ has (QNo)' by the proposition of Peris in Section 2. We claim that Q must then be (QNo) :

In fact, given the other additional conditions, Q is the strong dual of $\mathcal{W}_0H(G_2)$. But then Q is complemented in its strong bidual $Q'' = \mathcal{W}_0H(G_2)''' = (\mathcal{W}H(G_2))'_b$: Let $T : \mathcal{W}_0H(G_2) \to Q'_b = \mathcal{W}_0H(G_2)'' = \mathcal{W}H(G_2)$ be the canonical inclusion. Then it is easy to see that its transpose $T^t : Q'' = \mathcal{W}_0H(G_2)''' \to Q = (\mathcal{W}_0H(G_2))'_b$, the natural restriction map, yields the desired projection. Now it follows from [28], Corollary 3.6.(1) and Proposition 3.3.(1) (cf. the preliminaries at the end of our Section 1) that both $Q'' = (\mathcal{W}H(G_2))'_b$ and Q are (QNo). (We thank J. Bonet for this argument.)

After our claim is established, Corollary 8 directly follows from Proposition 3. ∎

The results in this article are part of the second-named author's dissertation [20].

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