Bases for existence varieties of strict regular semigroups

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Abstract

Existence varieties (or e-varieties) were introduced by Hall as classes of regular semigroups closed for direct products, homomorphic images and regular subsemigroups. They can be characterized by the identities satisfied by all regular unary semigroups $(S,')$, that is $a \to a'$ is an inverse unary operation on S, where S is in the given e-variety. In this way, we may speak of a basis of (identities of) an e-variety.

We provide several bases for a number of sub-e-varieties of the e-variety $\mathcal{SR}$ of strict semigroups. The latter are best characterized as subdirect products of completely (0-) simple semigroups. These sub-e-varieties of $\mathcal{SR}$ include those of all of whose members are: completely regular, E-solid, orthodox, inverse, overabelian, combinatorial and semigroups whose core is either overabelian or combinatorial. These e-varieties are depicted in two diagrams.

1 Introduction and summary

For basic concepts, we freely follow the development of Hall [3] to which we refer for a complete discussion of the subject. We consider only regular semigroups the class of which we denote by $\mathcal{RS}$. A subclass $\mathcal{V}$ of $\mathcal{RS}$ is an *existence variety* (for short *e-variety*) if it is closed under direct products, homomorphic images and taking of regular subsemigroups. A *unary semigroup* S is a semigroup on which is given a unary operation. If S is a regular semigroup and the unary operation $a \to a'$ is

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inverse in the sense that for every $a \in S$, we have $a = aa'a$ and $a' = a'aa'$, then S is a *regular unary semigroup*. The class of these is denoted by $\mathcal{RUS}$. Members of $\mathcal{RS}$ ought to be denoted by $(S, \cdot)$ and members of $\mathcal{RUS}$ by $(S, \cdot, ')$; however, we will generally abbreviate both by writing simply S.

A *unary identity* is of the form $u = v$ where u and v are *unary words*, that is words in which figure both multiplication and unary operation. Then $S \in \mathcal{RS}$ is said to *satisfy* a unary identity $u = v$ if for all choices of inverse unary operations on S, the regular unary semigroup $(S, ')$ satisfies $u = v$. For a set I of unary identities, the class

$$[I] = \{S \in \mathcal{RS} \mid S \text{ satisfies all identities in } I\}.$$

is an e-variety. Conversely, for every e-variety $\mathcal{V}$, the set

$$[\mathcal{V}] = \{u = v \mid \text{ for every } S \in \mathcal{V},\ S \text{ satisfies } u = v\}$$

is the set of all unary identities satisfied by all $S \in \mathcal{V}$. If B is a basis for $[\mathcal{V}]$, we say that B is a *basis* for (the identities of) $\mathcal{V}$ and write $\mathcal{V} = [B]$.

This dichotomy should cause no confusion: there are regular semigroups and some of their classes are called e-varieties, and there are unary semigroups and the identities satisfied by them in the usual sense. The principal novelty here is the concept of a regular (that is non-unary) semigroup *satisfying* certain unary identities, and the notion of an e-variety essentially closed under the usual Birkhoff operations (relative to semigroup homomirphisms). We will not dwell here upon lucky and unlucky incidences of these, quite original, new concepts and constructions but proceed instead with concrete definitions needed for our development.

A regular semigroup S satisfies *$\mathcal{D}$-majorization* if for any idempotents e, f, g of S, $e \geq f$, $e \geq g$ and $f \ \mathcal{D}\ g$ imply that $f = g$. Such a semigroup is called *strict* and the class of all strict semigroups is denoted by $\mathcal{SR}$ (the modifier "regular" will be generally omitted since we consider only regular semigroups). Strict semigroups are precisely regular semigroups which are subdirect products of completely (0-) simple semigroups according to a result due to Lallement [6]. This is the principal characterization of strict semigroups; the main fact in the present context is that $\mathcal{SR}$ is an e-variety.

When completely regular or inverse semigroups are considered as varieties they include a (natural) unary operation. Stripped of this unary operation, the varieties of completely regular and inverse semigroups coincide with e-varieties all of whose members are either completely regular or inverse semigroups, respectively. Without serious threat of ambiguity these may then be identified and the same notation may employed for both concepts. It should be pointed out that the inverse unary operation in the case of inverse semigroups is unique whereas this is generally not the case for completely regular semigroups. In this context, we have sub-e-varieties of $\mathcal{SR}$: $\mathcal{NBG}$—normal cryptogroups and $\mathcal{SI}$—strict inverse semigroups. Hence $\mathcal{SR}$ provides a common roof for quite disparate looking (e-) varieties.

The purpose of this work is to provide bases for a number of sub-e-varieties of the e-variety $\mathcal{SR}$. Section 2 contains most of the needed notation and terminology. Here we list the notation for most of the e-varieties we shall study and depict them in two diagrams. In Section 3 we state a number of auxiliary results to be used throughout the paper. Section 4 comprises several bases for the e-variety $\mathcal{SR}$ as well as some structural properties of its members. Sections 5–11 contain bases for e-varieties which are intersections of $\mathcal{SR}$ with completely simple semigroups, normal cryptogroups, E-solid, orthodox, inverse, central, overabelian, combinatorial semigroups as well as with semigroups whose core is either overabelian or combinatorial. We conclude the paper in Section 12 with a brief discussion of sub-e-variaties of $\mathcal{SR}$ in general and suggest a few problems naturally arising in this context.

2 Terminology and notation

In addition to the concepts and symbolism introduced in Section 1, we will need a great number of definitions and notation. Some of these can be found in books [2], [9] and [12], others we now list.

Let S be a regular semigroup. Since we shall consider only regular semigroups, the attribute "regular" will generally be omitted. We denote by $E(S)$ the set of all idempotents of S and by $C(S)$ the *core* of S, that is the subsemigroup of S generated by $E(S)$. If $a \in S$ and a has an inverse with which it commutes, then a is completely regular; equivalently the $\mathcal{H}$-class H_a is a group; if so a^0 denotes the identity of H_a. For $a \in S$, $J(a)$ is the *principal ideal* generated by a, J_a and $I(a)$ are the sets of all generators and nongenerators of $J(a)$ in $J(a)$, and $J(a)/I(a)$ is the *principal factor* of a (or of S with $J(a)/\emptyset = J(a)$). The identity of a group is denoted by e.

The semigroup S is: *E-solid* if for any $e, f, g \in E(S)$ such that $e \;\mathcal{L}\; f \;\mathcal{R}\; g$, there exists $h \in E(S)$ such that $e \;\mathcal{R}\; h \;\mathcal{L}\; g$; *orthodox* if $E(S)$ is a subsemigroup of S; *inverse* if any two elements of $E(S)$ commute; *central* if for any $e, f \in E(S)$ and $a \in S$ such that $ef \;\mathcal{H}\; a \;\mathcal{H}\; a^2$, we have $efa = aef$; *overabelian* if all subgroups of S are abelian; *combinatorial* if $\mathcal{H}$ is the equality relation on S; *completely regular* if every element of S in completely regular; *rectangular (abelian) group* if S is a direct product of a left zero semigroup, a (abelian) group and a right zero semigroup.

Only the definition of "central" above is new. It extends [1, Definition 6] for E-solid semigroups which in turn generalizes the standard definition for the case of completely regular semigroups.

If $\mathcal{U}, \mathcal{V}, \mathcal{W}, \ldots$ are classes of semigroups, we write $\mathcal{U}\mathcal{V}\mathcal{W}\ldots$ for their intersection $\mathcal{U} \cap \mathcal{V} \cap \mathcal{W} \cap \ldots$. The classes $\mathcal{U}, \mathcal{V}, \mathcal{W}, \ldots$ may themselves be denoted by several letters so caution must be exercised in reading this notation.

For a unary word w, we denote by $w \in E$ the identity $w = w^2$; by $w \in G$ we denote the identity $w = w(w^2)'\,w^3\,(w^2)'\,w$. Following Hall [3], for unary words u and v, we denote by $u\ CW\ v$ the identity $uv = vu$. We shall interpret the reason for this notation in Section 3. For the moment, read these as w is *idempotent*, w is *completely regular* (or is in a *subgroup*) and u *commutes with* v.

If $u_1 = v_1,\ u_2 = v_2, \ldots$ are unary identities which form a basis of an e-variety $\mathcal{V}$, we write

$$\mathcal{V} = [u_1 = v_1,\ u_2 = v_2, \ldots].$$

In the terminology of [3], the identities $u_1 = v_1,\ u_2 = v_2, \ldots$ *strongly determine* the e-variety $\mathcal{V}$.

In order to prove that $\mathcal{V} = [u_1 = v_1,\ u_2 = v_2, \ldots]$, we must show that:

(i) for every $S \in \mathcal{V}$ and every inverse unary operation $a \to a'$ on S, $(S, ')$ satisfies all the identities $u_1 = v_1,\ u_2 = v_2, \ldots,$

(ii) if S is a semigroup such that for some inverse unary operation $a \to a'$ on S, $(S, ')$ satisfies all the identities $u_1 = v_1,\ u_2 = v_2, \ldots$, then $S \in \mathcal{V}$.

Let $\mathcal{V}$ be a class of semigroups. A semigroup S is *locally (in)* $\mathcal{V}$ if $e\mathcal{S}e \in \mathcal{V}$ for all $e \in E(S)$. Denote the class of all such semigroups by $C\mathcal{V}$. If we restrict our attention to regular semigroups and $\mathcal{V}$ is an e-variety, then according to [3, Theorem 4.6.3], $C\mathcal{V}$ is an e-variety.

We now give a list of classes of semigroups to be studied in the paper. They are mostly, but not all, e-varieties. Those all of whose members are strict semigroups are depicted in Diagram 1. An expansion of this diagram, including left and right concepts, is presented in Diagram 2. This is followed by some results needed later.

The list of classes of semigroups which play an important role in our considerations is too long to be stated in full. Instead, we define some of the typical symbols and explain some of the situations which ought to suffice for an easy and unambiguous reading of the notation.

$\mathcal{RUS}$ – regular unary semigroups,
$\mathcal{RS}$ – regular semigroups,
$\mathcal{ES}$ – E-solid semigroups,
$\mathcal{O}$ – orthodox semigroups,
$\mathcal{I}$ – inverse semigroups,
$\mathcal{SR}$ – strict semigroups,
$\mathcal{C}e$ – central semigroups,
$(\mathcal{AG})$ – (for abelian subgroups) overabelian semigroups,
$\mathcal{C}o$ – combinatorial semigroups,
$(\mathcal{AC})$ – semigroups with overabelian core,
$(\mathcal{C}o\mathcal{C})$ – semigroups with combinatorial core,
$\mathcal{RB}$ – rectangular bands,
$\mathcal{S}$ – semilattices,
$\mathcal{NB}$ - normal bands,
$\mathcal{G}$ – groups,
$\mathcal{R}e\mathcal{G}$ – rectangular groups,
$\mathcal{SG}$ – (for semilattices of groups) Clifford semigroups,
$\mathcal{NBG}$ – (for normal bands of groups) normal cryptogroups,
$\mathcal{AG}$ – abelian groups,
$\mathcal{R}e\mathcal{AG}$ - rectangular abelian groups,
$\mathcal{NBAG}$ – normal bands of abelian groups.

For the remaining symbols in Diagram 1 we should point out that except in $\mathcal{SR}$, we write only $\mathcal{S}$ for strict for the e-varieties not contained in $\mathcal{NBG}$. For example $\mathcal{SIC}o$ stands for strict, inverse, combinatorial. We shall discuss the e-varieties depicted in Diagram 1 throughout the paper but shall not mention those in Diagram 2 which do not appear in Diagram 1.

Briefly, in Diagram 2, $\mathcal{L}$ stands for left, $\mathcal{R}$ for right. In particular
$\mathcal{LZ}$ – left zero semigroups,
$\mathcal{LAG}$ – left abelian groups,

$\mathcal{LNO}$ – left normal orthodox semigroups (also called generalized left inverse semigroups) and their left-right duals.

3 Preliminaries

We state here a number of results which are either known or easy to prove and will be needed in the sequel.

Result 3.1. *The following conditions on a (arbitrary) semigroup S are equivalent.*
(i) S is completely regular.
(ii) For every $a \in S,\ a \in a^2\,Sa$.
(iii) For every $a \in S,\ a \in aSa^2$.

Proof: See [9, Theorem IV.1.6]. ■

Result 3.2. *Let T be a regular subsemigroup of a semigroup S and $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}\}$. Then $\mathcal{K}_S|_T = \mathcal{K}_T$.*

Proof: By duality and conjunction, it suffices to consider the case $\mathcal{K} = \mathcal{L}$. Let $a\,\mathcal{L}_S|_T\,b,\ a \neq b$. Then $a = xb$ and $b = ya$ for some $x, y \in S$. For inverses a' and b' of a and b, respectively, we obtain

$$a = xb(b'b) = ab'b,\ b = ya(a'a) = ba'a$$

and thus $a\,\mathcal{L}_T\,b$. Therefore $\mathcal{L}_S|_T \subseteq \mathcal{L}_T$ and the opposite inclusion is trivial. ■

We shall use the following symbolism extensively.

Notation 3.3. *For $S \in \mathcal{RUS}$ and $a \in S$, let $a^+ = a(a^2)'a$.*

Note that $aa^+a = a^2$.

Lemma 3.4. *Let $S \in \mathcal{RUS}$ and $a \in S$.*
(i) a is idempotent if and only if $a = a^+$.
(ii) a is completely regular if and only if $a = a^+aa^+$.

Proof: (i) Easy, see the proof of [3, Theorem 4.3.3].
(ii) *Necessity.* Let $x \in S$ be such that $a = axa$ and $ax = xa$. Then

$$a = xa^2 = xaa^+a = (axa)(a^2)'\,a^2 = a(a^2)'a^2 = a^+a$$

and similarly $a = aa^+$ so that $a = a^+aa^+$.

Sufficiency. By hypothesis, $a = a^+a = aa^+$ and thus

$$a = a(a^2)'a^2 = a^2(a^2)'a$$

which by [9, Proposition IV.1.2] yields that a is completely regular. ■

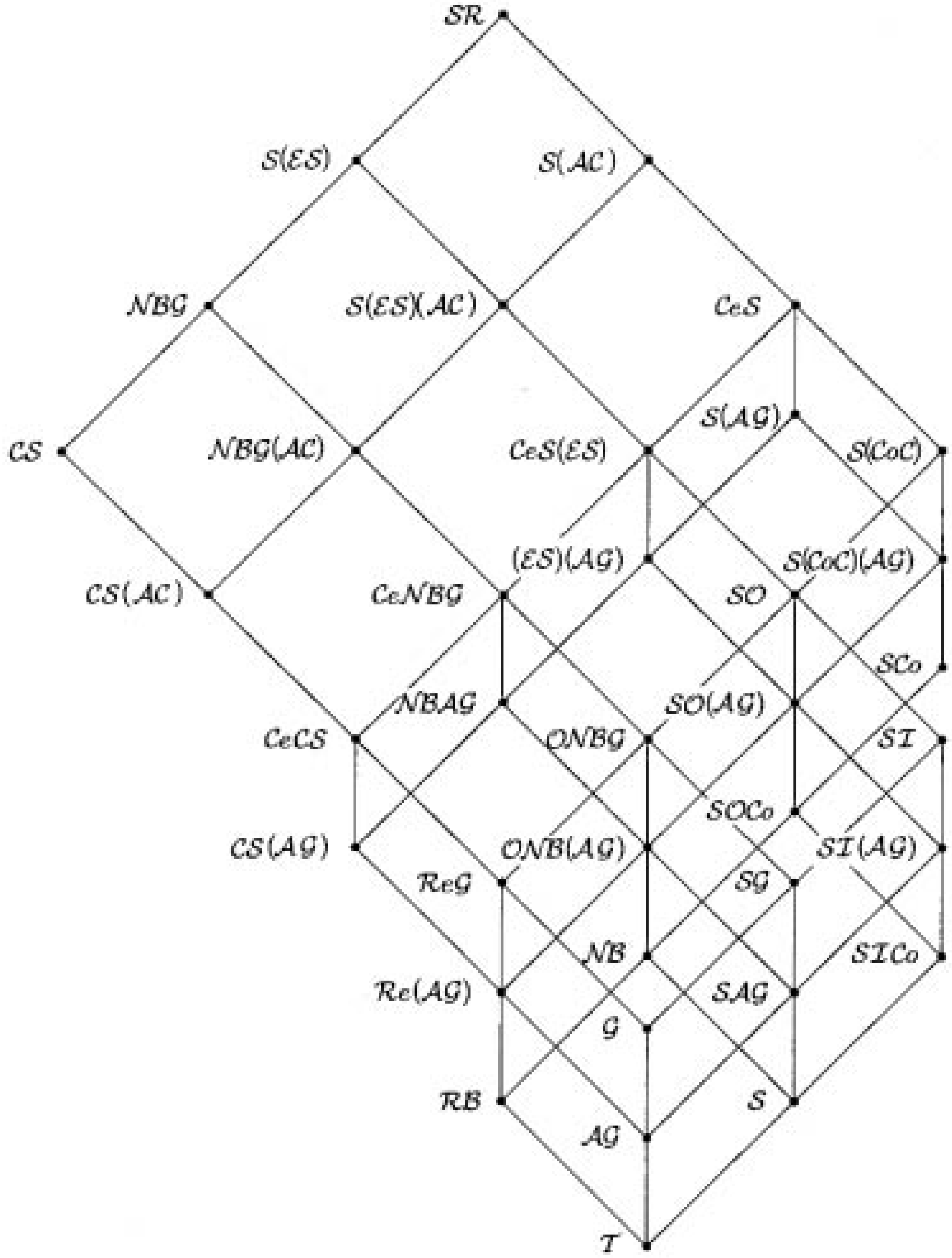

SR
S(ES)
S(AC)
NBG
S(ES)(AC)
CeS
CS
NBG(AC)
CeS(ES)
S(AG)
S(CoC)
CS(AC)
CeNBG
(ES)(AG)
SO
S(CoC)(AG)
SCo
NBAG
SO(AG)
CeCS
ONBG
ST
SOCo
CS(AG)
ONB(AG)
ST(AG)
ReG
SG
NB
STCo
Re(AG)
SAG
G
RB
S
AG
T

Diagram 1.

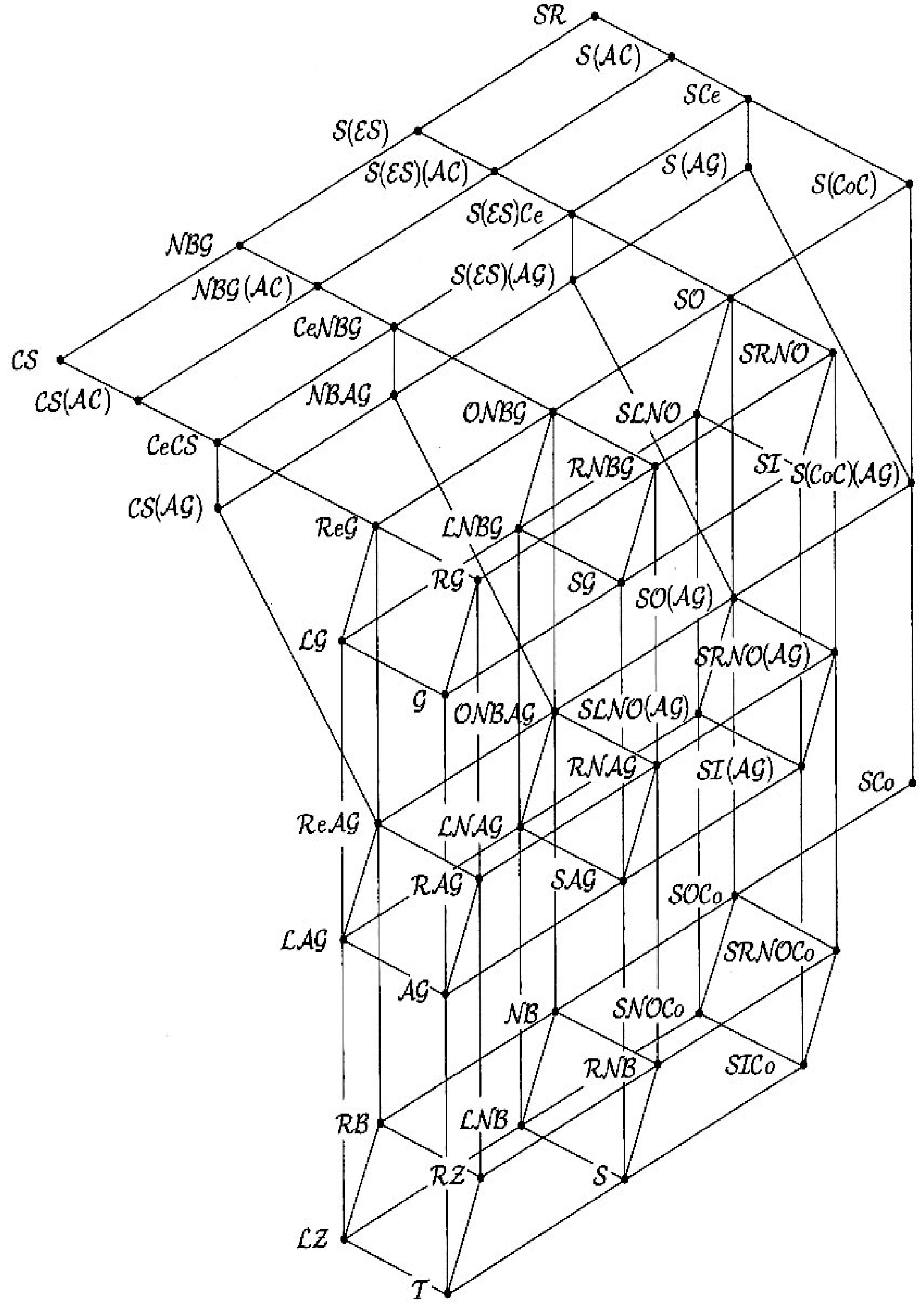

SR
S(AC)
SCe
S(ES)
S(ES)(AC)
S(AG)
S(CoC)
S(ES)Ce
NBG
NBG(AC)
S(ES)(AG)
SO
CeNBG
SRNO
CS
CS(AC)
NBAG
ONBG
SLNO
CeCS
RNBG
SI
S(CoC)(AG)
CS(AG)
ReG
LNBG
RG
SG
SO(AG)
LG
SRNO(AG)
G
ONBAG
SLNO(AG)
RNAG
SI(AG)
SCo
ReAG
LNAG
RAG
SAG
SOCo
LAG
SRNOCo
AG
NB
SNOCo
RNB
SICo
RB
LNB
RZ
S
LZ
T

Diagram 2.

We can interpret the notation $w \in G$ defined in Section 2 by means of Lemma 3.4 as $w = w^+ww^+$ which, as an element of $S \in \mathcal{RUS}$, means that w is in a subgroup of S. Similarly $w \in E$ means that w is an idempotent in S and can be written as $a = a^+$.

Lemma 3.5. *The following equalities hold.*
(i) $\mathcal{I} = [a^+ CW\, b^+]$
(ii) $\mathcal{CR} = [a = a^+a] = [a = aa^+]$.

Proof: (i) Easy, see [3, Remark 4.3.4].
(ii) By Lemma 3.4(ii), $\mathcal{CR} = [a = a^+aa^+]$. If $S \in \mathcal{RUS}$ and $a = a^+a$ for all $a \in S$, then $a = a(a^2)'a^2 \in aSa^2$ and Result 3.1 implies that $S \in \mathcal{CR}$. Similarly for $a = aa^+$. ∎

Lemma 3.6. *Let* $S = \mathcal{M}^0(I, G, \Lambda; P) \in \mathcal{RUS}$, $a = (i, g, \lambda)$ *and* $a' \in V(a)$.
(i) $a' = (i', p_{\lambda i'}^{-1} g^{-1} p_{\lambda' i}^{-1}, \lambda')$ *for some* $i' \in I$, $\lambda' \in \Lambda$ *such that* $p_{\lambda i'}, p_{\lambda' i} \neq 0$.
(ii) $aa' = (i, p_{\lambda' i}^{-1}, \lambda')$, $a'a = (i', p_{\lambda i'}^{-1}, \lambda)$ *with the notation of part (i).*

$$(iii) \quad a^+ = \begin{cases} a^0 & \text{if } p_{\lambda i} \neq 0 \\ 0 & \text{if } p_{\lambda i} = 0 \end{cases}.$$

Proof: (i) If $a' = (i', g', \lambda')$, then

$$(i, g, \lambda) = (i, g, \lambda)(i', g', \lambda')(i, g, \lambda) = (i, gp_{\lambda i'}\, g' p_{\lambda' i}\, g, \lambda)$$

whence the assertion.
(ii) This follows directly from part (ii).
(iii) If $p_{\lambda i} \neq 0$, then by part (i) for some i' and λ',

$$\begin{aligned} a^+ &= a(a^2)'a = (i, g, \lambda)(i, gp_{\lambda i}\, g, \lambda)'(i, g, \lambda) \\ &= (i, g, \lambda)(i', p_{\lambda i'}^{-1}\, g^{-1} p_{\lambda i}^{-1}\, g^{-1}\, p_{\lambda' i}^{-1}, \lambda')(i, g, \lambda) = (i, p_{\lambda i}^{-1}, \lambda) = a^0. \end{aligned}$$

If $p_{\lambda i} = 0$, then clearly $a^+ = 0$. ∎

We shall use the notation of Lemma 3.6(i) consistently generally without further comments.

4 Strict semigroups

We give manifold characterizations of strict semigroups and several bases for the e-variety they comprise. This is followed by a number of further properties of strict semigroups.

The theorem here is one of the principal results of the paper. The first two equalities in it are due to Lallement [6]. That $\mathcal{SR} = \mathcal{LSG}$ was stated by Hall [3, p. 76]. We shall often use this equality as an alternative to the original definition of a strict semigroup. In view of this, we include a short proof for it.

The appelation "strict" for this class of semigroups stems from the fact that they may be characterized among regular semigroups S as follows. For any $a \in S$, letting

$$P_a = \{x \in S \mid J_x \not> J_a\}, \qquad Q_a = \{x \in S \mid J_x \not\geq J_a\},$$

then the quotient semigroup S/Q_a is a strict (ideal) extension of the semigroup P_a/Q_a; the latter is isomorphic to the principal factor $J(a)/I(a)$. The proof of this assertion is essentially the same as for inverse semigroups in [12, Theorem II.4.5].

Theorem 4.1. *The following equalities hold.*

$$\begin{aligned}
\mathcal{SR} &= \{S \in \mathcal{RS} \mid \mathcal{S} \textit{ is a subdirect product of completely (0-) simple semigroups}\} \\
&= \{S \in \mathcal{RS} \mid S \textit{ is completely semisimple and a subdirect product} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{of its principal factors}\} \\
&= \mathcal{LSG} = [axa\ CW\ (aya)^+] = [a^+xa^+\ CW\ (aya)^+] \\
&= [axaya = (ay)^+axaya(xa)^+] = [axaya = ay(ay)'axaya(xa)'xa].
\end{aligned}$$

Proof: Denote these eight classes by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}$ and $\mathcal{H}$, respectively. The equality $\mathcal{A} = \mathcal{B}$ was proved in [6, Théorème 2.12]. The inclusion $\mathcal{A} \subseteq \mathcal{C}$ follows from the proof of [6, Théorème 2.17] whereas $\mathcal{C} \subseteq \mathcal{A}$ is a consequence of the inclusion $\mathcal{B} \subseteq \mathcal{A}$ so that $\mathcal{A} = \mathcal{C}$.

Let $S \in \mathcal{B}$. We assume that $S \subseteq \prod_{\alpha \in A} S_\alpha$ where S_α is a completely (0-) simple semigroup for every $\alpha \in A$. Let $e = (e_\alpha) \in E(S)$, $f = (f_\alpha) \in E(eSe)$ and $a = (a_\alpha) \in eSe$. Fix $\alpha \in A$. It follows that $f_\alpha \leq e_\alpha$ and $a_\alpha = a_\alpha e_\alpha = e_\alpha a_\alpha$. If S_α has no zero, then $f_\alpha = e_\alpha$ and hence $f_\alpha a_\alpha = a_\alpha f_\alpha$. If S_α has a zero 0_α, then either $e_\alpha = f_\alpha$ or $e_\alpha \neq 0 = f_\alpha$ and either $a_\alpha \in H_{e_\alpha}$ or $a_\alpha = 0_\alpha$. The only nontrivial case yields again $f_\alpha a_\alpha = a_\alpha f_\alpha$. Therefore $fa = af$ which proves that eSe is a Clifford semigroup so that $S \in \mathcal{D}$. Therefore $\mathcal{B} \subseteq \mathcal{D}$.

Let $S \in \mathcal{D}$ and $e, f, g \in E(S)$ be such that $e \geq f$, $e \geq g$ and $f\,\mathcal{D}\,g$. Then $f\,\mathcal{L}\,a\,\mathcal{D}\,g$ for some $a \in S$ and hence

$$a = af = a(fe) = (af)e = ae$$

and similarly $a = ga$ implies that $a = ea$. Hence $a, f, g \in eSe$ which by Result 3.2 implies that $f\ \mathcal{L}_{eSe}\ a\ \mathcal{R}_{eSe}\ g$. But eSe is a Clifford semigroup and thus $f = g$. Therefore $S \in \mathcal{A}$ which proves that $\mathcal{D} \subseteq \mathcal{A}$. Consequently $\mathcal{A} = \mathcal{B} = \mathcal{C} = \mathcal{D}$.

Using Lemma 3.6, straightforward checking shows that every Rees matrix semigroup satisfies the defining identities of $\mathcal{E}, \mathcal{F}, \mathcal{G}$ and $\mathcal{H}$. In view of the inclusion $\mathcal{A} \subseteq \mathcal{B}$, we conclude that $\mathcal{A} \subseteq \mathcal{E} \cap \mathcal{F} \cap \mathcal{G} \cap \mathcal{H}$. Let $S \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$. Letting $a \in E(S)$ and $x, y \in aSa$, we easily see that $xy^+ = y^+x$ so that $aSa \in \mathcal{SG}$. It follows that $S \in \mathcal{D}$. Therefore $\mathcal{E} \cup \mathcal{F} \cup \mathcal{G} \cup \mathcal{H} \subseteq \mathcal{D}$. ∎

The first two bases in Theorem 4.1 reflect the fact that strict semigroups are precisely regular semigroups which are locally Clifford semigroups since the latter are characterized by having idempotents in the center. The last two bases have the form of $axaya$ being equal to a word which (precisely) ensures that the resulting semigroups are strict. In the remainder of the paper, most bases represent variants of the four bases in Theorem 4.1.

In order to obtain further properties of strict semigroups, we must first recall of some of their structural characteristics.

From [6, Théorème 2.12] and Theorem 4.1, a regular semigroup S is strict if and only if for any $a, b \in S$ such that $J(a) \supseteq J(b)$ there exists a function $\varphi : J_a \to J_b$ satisfying

$$(\alpha) \quad x \in J_a,\ y \in J_b : xy \in J_b \Rightarrow xy = (x\varphi)y,\ yx \in J_b \Rightarrow yx = y(x\varphi),$$

$$(\beta) \quad E(J_a)\varphi \subseteq E(J_b).$$

Assuming that S is strict, for any $a, b \in S$ such that $J(a) \supseteq J(b)$, in view of the above and [8, Theorem 3.4], there is a (unique) function $\eta_{J_a,J_b} : J_a \to J_b$ satisfying (α) and (β) above with $\varphi = \eta_{J_a,J_b}$ and also the following:

$$(\gamma) \quad \text{if } J(a) \supseteq J(c),\ J(b) \supseteq J(c) \text{ and } ab \in J_c, \text{ then } ab = (a\eta_{J_a,J_c})\,(b\eta_{J_b,J_c}),$$

$$(\delta) \quad \text{if } J(a) \supseteq J(b) \supseteq J(c), \text{ then } \eta_{J_a,J_b}\,\eta_{J_b,J_c} = \eta_{J_a,J_c}.$$

Note that η_{J_a,J_a} is the identity mapping on J_a.

Lemma 4.2. *With the above notation, let $a_1, a_2, \ldots, a_n \in S,\ a = a_1a_2\ldots a_n,\ \varphi_i = \eta_{J_{a_i},J_a}$ for $i = 1, 2, \ldots, n$. Then $a = (a_1\varphi_1)(a_2\varphi_2)\ldots(a_n\varphi_n)$.*

Proof: The argument is by induction on n. The case $n = 2$ is given by (γ). Assuming the statement for $n-1$ and letting $b = a_1a_2\ldots a_{n-1}$, we have $b = \left(a_1\eta_{J_{a_1},J_b}\right)\ldots\left(a_{n-1}\,\eta_{J_{a_{n-1}},J_b}\right)$ whence, using (γ) and (δ), we obtain

$$a = ba_n = (b\eta_{J_b,J_a})\,(a_n\varphi_n) = (a_1\varphi_1)(a_2\varphi_2)\ldots(a_n\varphi_n),$$

as required. ∎

Lemma 4.3. *Let $\mathcal{P} \in \{(\mathcal{AG}), \mathcal{C}o, \mathcal{C}e, (\mathcal{AC}), (\mathcal{C}o\mathcal{C})\}$ and S be a strict semigroup. Then S has property $\mathcal{P}$ if and only if every principal factor of S has property $\mathcal{P}$.*

Proof: The direct part and the converse for $(\mathcal{AG})$ and $\mathcal{C}o$ require straightforward arguments.

Suppose that every principal factor of S is central. Let $e, f \in E(S)$ and $a \in S$ be such that $ef \in H_a$ and H_a is a group. Then $J(a) \subseteq J(e) \cap J(f)$; let $\varphi = \eta_{J_e,J_a}$ and $\psi = \eta_{J_f,J_a}$. By property (γ) above, we have $ef = (e\varphi)(f\psi)$ and by $(\beta), e\varphi, f\psi \in E(J_a)$. The hypothesis implies that $(e\varphi)(f\psi)a = a(e\varphi)(f\psi)$ whence $efa = aef$. Therefore S is central.

Assume next that every principal factor of S has overabelian core. Let $e_1, e_2, \ldots, e_m, f_1, f_2, \ldots, f_n \in E(S)$ be such that $e_1e_2\ldots e_m, f_1f_2\ldots f_n \in H_a$ where H_a is a group (relative to S or $C(S)$). In view of Lemma 4.2, we may let $\varphi_i = \eta_{J_{e_i},J_a},\ \psi_j = \eta_{J_{f_j},J_a}$ and get $e_i\varphi_i, f_j\psi_j \in E(J_a)$ by property (β) and

$$e_1e_2\ldots e_m = (e_1\varphi_1)(e_2\varphi_2)\ldots(e_m\varphi_m),$$
$$f_1f_2\ldots f_n = (f_1\psi_1)(f_2\psi_2)\ldots(f_n\psi_m).$$

The hypothesis implies that $(e_1\varphi_1)(e_2\varphi_2)\ldots(e_m\varphi_m)$ and $(f_1\psi_1)(f_2\psi_2)\ldots(f_n\psi_n)$ commute. But then also $e_1e_2\ldots e_m$ and $f_1f_2\ldots f_n$ commute. Therefore S is central.

The case of $(\mathcal{C}o\mathcal{C})$ is treated similarly. ∎

We shall discuss bases of virtually all e-varieties depicted in Diagram 1. Note that Diagram 2 is an expansion of Diagram 1 including left and right variants of those e-varieties in Diagram 1 which have them.

5 Normal Cryptogroups

A semigroup S is a normal cryptogroup if S is a cryptogroup, that is completely regular with $\mathcal{H}$ a congruence, and $S/\mathcal{H}$ is a normal band, that is $S/\mathcal{H}$ satisfies the identity $axya = ayxa$. Normal cryptogroups are precisely regular semigroups which are subdirect products of completely simple semigroups with a zero possibly adjoined. Their class is denoted by $\mathcal{NBG}$ (for normal bands of groups) even in the case that they are considered as unary semigroups. It then follows easily that

$$\mathcal{NBG} = \mathcal{CR} \cap \mathcal{SR}.$$

In view of the above, verifying that a normal cryptogroup satisfies some identity reduces to checking this identity in Rees matrix semigroups with normalized sandwich matrix.

We shall use the notation for varietites of normal cryptogroups to denote the corresponding e-varieties of normal cryptogroups even though the former designate unary semigroups and the latter do not. For the former we have bases in [11]. We shall construct some bases for the latter by modifying the corresponding ones for the former. The problem is to build into the identity the requirement that the resulting semigroups must also be completely regular.

We have divided the material of this section into four theorems. The first two treat completely simple semigroups, the latter two the rest. The second and the fourth theorem handle the overabelian case. Compare the next result with [11, Section 3].

Theorem 5.1. *The following equalities hold.*

- (i) $\mathcal{CS} = [a = (axa)^+ a] = [a = (axa)\,(axa)'\,a]$.
- (ii) $\mathcal{CS}(\mathcal{AC}) = [a = (ax^+a^+y^+a)\,(ay^+a^+x^+a)'\,a]$.
- (iii) $\mathcal{C}e\mathcal{CS} = [a = (a^+x^+a)\,(ax^+a^+)'\,a]$.
- (iv) $\mathcal{R}e\mathcal{G} = [a = a^+x^+a]$.
- (v) $\mathcal{G} = [x^+ = y^+] = [x'x = yy']$.

Proof: (i) Denote these three classes by $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, respectively. Let $S \in \mathcal{A}$. Then for $a \in S$, by Lemma 3.6(iii), we have $a^+ = a^0$ and hence, for any $x \in S$, $a^0 = (axa)^0$ implies that $a = (axa)^+ a$. Therefore $S \in \mathcal{B}$ and thus $\mathcal{A} \subseteq \mathcal{B}$. Conversely, let $S \in \mathcal{B}$. For $x = a'$, the given identity yields $a = a^+ a$ which by Lemma 3.5(ii) implies that $S \in \mathcal{CR}$. Again by Lemma 3.6(iii), we obtain that $a = (axa)^0 a$ which evidently implies that $S \in \mathcal{A}$. Therefore $\mathcal{B} \subseteq \mathcal{A}$.

Next let $S \in \mathcal{A}$. We may set $S = \mathcal{M}(I, G, \Lambda; P)$. Now for $a = (i, g, \lambda)$, by Lemma 3.6 (ii), we obtain

$$(axa)(axa)'\,a = (i, p^{-1}_{\lambda' i}, \lambda')(i, g, \lambda) = (i, g, \lambda) = a.$$

Therefore $S \in \mathcal{C}$ and thus $\mathcal{A} \subseteq \mathcal{C}$. Conversely, let $S \in \mathcal{C}$. For $x = a$ in the given identity, we deduce that $a \in a^2Sa$ which by Result 3.1 yields that $S \in \mathcal{CR}$. But then the given identity implies that $S \in \mathcal{A}$. Hence $\mathcal{C} \subseteq \mathcal{A}$.

(ii) Let $S \in \mathcal{CS}(\mathcal{AC})$. By [14, Proposition 7.4], we may let $S = \mathcal{M}(I, G, \Lambda; P)$ where P is normalized and its entries commute. By Lemma 3.6(i), we have $a^+ = a^0$ for any $a \in S$. Now let

$$a = (i, g, \lambda),\ x = (j, h, \mu),\ y = (k, t, \nu). \tag{1}$$

Then

$$c = ax^+a^+y^+a = (i, gp_{\lambda j}\, p_{\mu j}^{-1}\, p_{\mu i}\, p_{\lambda i}^{-1}\, p_{\lambda k}\, p_{\nu k}^{-1}\, p_{\nu i} g, \lambda),$$

$$\begin{aligned} d = (ay^+a^+x^+a)' &= \left(i, p_{\lambda k}\, p_{\nu k}^{-1}\, p_{\nu i}\, p_{\lambda i}^{-1}\, p_{\lambda j}\, p_{\mu j}^{-1}\, p_{\mu i}\, g, \lambda\right)' \\ &= \left(i', p_{\lambda i'}^{-1}\, g^{-1}\, p_{\mu i}^{-1}\, p_{\mu j}\, p_{\lambda j}^{-1}\, p_{\lambda i}\, p_{\nu i}^{-1}\, p_{\nu k}\, p_{\lambda k}^{-1}\, p_{\lambda' i}^{-1}, \lambda'\right) \end{aligned}$$

and thus

$$cda = \left(i, p_{\lambda' i}^{-1}, \lambda'\right)\, (i, g, \lambda) = (i, g, \lambda) = a,$$

as required.

Conversely let S be in the e-variety on the right in part (ii). For any $a \in S$ and $x = a$ in the given identity, we get $a \in a^2Sa$ which by Result 3.1 implies that $S \in \mathcal{CR}$ and thus also $S \in \mathcal{CS}$. Let $S = \mathcal{M}(I, G, \Lambda; P)$ where P is normalized. With the above notation and setting $i = \lambda = 1$, we get $e = p_{\mu j}^{-1}\, p_{\nu k}^{-1}\, p_{\mu j}\, p_{\nu k}$ and thus $p_{\nu k}\, p_{\mu j} = p_{\mu j}\, p_{\nu k}$. Now [14, Proposition 7.4] implies that $S \in \mathcal{CS}(\mathcal{AC})$.

(iii) Let $S \in \mathcal{C}e\mathcal{CS}$. By [14, Proposition 6.2], we may let $S = \mathcal{M}(I, G, \Lambda; P)$ where P is normalized and entries of P are in the center of G. With the notation in (1), we get

$$\begin{aligned} &c = a^+x^+a = \left(i, p_{\lambda i}^{-1}\, p_{\lambda j}\, p_{\mu j}^{-1}\, p_{\mu i}\, g, \lambda\right), \\ &d = (ax^+a^+)' = \left(i, gp_{\lambda j}\, p_{\mu j}^{-1}\, p_{\mu i}\, p_{\lambda i}^{-1}, \lambda\right)' = \left(i', p_{\lambda i'}^{-1}\, p_{\lambda i}\, p_{\mu i}^{-1}\, p_{\mu j}\, p_{\lambda j}^{-1}\, g^{-1}\, p_{\lambda' i}^{-1}, \lambda'\right) \end{aligned}$$

and thus

$$cda = \left(i, p_{\lambda' i}^{-1}, \lambda'\right) (i, g, \lambda) = (i, g, \lambda) = a,$$

as required.

Conversely, let S be in the e-variety on the right of part (iii). It follows that for any $a \in S$, we have $a = a^+a$ which by Lemma 3.5(ii) yields that $S \in \mathcal{CR}$ and thus also $S \in \mathcal{CS}$. Hence let $S = \mathcal{M}(I, G, \Lambda; P)$ where P is normalized. With the above notation and calculation, for $j = \mu = 1$, we get $p_{\lambda i}^{-1} g\, p_{\lambda i} g^{-1} g = g$ whence $gp_{\lambda i} = p_{\lambda i} g$. By [14, Proposition 6.2], we conclude that $S \in \mathcal{C}e\mathcal{CS}$.

(iv) A rectangular group obviously satisfies the identity $a = a^+x^+a$. Conversely, let $S \in [a = a^+x^+a]$. Then $e = efe$ for all $e, f \in E(S)$ whence all idempotents are primitive and S is completely simple. But then $ef = efef$ for all $e, f \in E(S)$ implies that $S \in \mathcal{R}e\mathcal{G}$.

(v) The first equality follows from the well-known fact that a regular semigroup with a single idempotent is a group. A group trivially satisfies the identity $x'x = yy'$.

Let S satisfy $x'x = yy'$. For any $a \in S$, we have

$$a = a(a'a)a'a = a(aa')a'a \in a^2Sa$$

and Result 3.1 implies that $S \in \mathcal{CR}$. But then $x = xyy'$ implies that $S \in \mathcal{CS}$. Hence let $S = \mathcal{M}(I, G, \Lambda; P)$. For $x = (i, g, \lambda)$ and $y = (j, h, \mu)$ by Lemma 3.6(ii), the hypothesis implies that $(i', p_{\lambda i'}^{-1}, \lambda) = (j, p_{\mu' j}^{-1}, \mu')$ whence $i' = j$ and $\lambda = \mu'$. Since this holds for arbitrary elements of S, we conclude that both I and Λ are trivial and S is a group. ∎

We treat next overabelian completely simple semigroups. Again compare the following result with [11, Section 3].

Theorem 5.2. *The following equalities hold.*

(*i*) $\mathcal{CS}(\mathcal{AG}) = [a = (a^+xa)(axa^+)'a]$.
(*ii*) $\mathcal{R}e\mathcal{AG} = [a = (axya)(ayxa)'a]$.
(*iii*) $\mathcal{AG} = [a = x'ax]$.
(*iv*) $\mathcal{RB} = [a = axa]$.
(*v*) $\mathcal{T} = [x = y]$.

Proof: (*i*) Let $S \in \mathcal{CS}(\mathcal{AG})$. We may set $= \mathcal{M}(I, G, \Lambda; P)$ where G in abelian. For $a = (i, g, \lambda)$ and $x = (j, h, \mu)$, we get

$$\begin{aligned} c &= a^+xa = \left(i, p_{\lambda i}^{-1}\, p_{\lambda j}\, h p_{\mu i} g, \lambda\right), \\ d &= (axa^+)' = \left(i, g p_{\lambda j} h\, p_{\mu i}\, p_{\lambda i}^{-1}, \lambda\right)' = \left(i', p_{\lambda i'}^{-1}\, p_{\lambda i}\, p_{\mu i}^{-1} h^{-1}\, p_{\lambda j}^{-1} g^{-1}\, p_{\lambda' i}^{-1}, \lambda'\right) \end{aligned}$$

and thus

$$cda = \left(i, p_{\lambda' i}^{-1}, \lambda'\right)(i, g, \lambda) = (i, g, \lambda) = a,$$

as required.

Conversely, let S be in the e-variety on the right in part (i). It follows that for any $a \in S$, we have $a = a^+a$ which by Lemma 3.5(ii) yields that $S \in \mathcal{CR}$ and thus also $S \in \mathcal{CS}$. Hence let $S = \mathcal{M}(I, G, \Lambda; P)$ where P is normalized. With the above notation and calculation, for $i = \lambda = 1$, we get $hg\, h^{-1}g^{-1}\, g = g$ whence $hg = gh$ and G is abelian. Therefore $S \in \mathcal{CS}(\mathcal{AG})$.

(*ii*) A rectangular abelian group clearly satisfies the given identity. Conversely, let S be in the e-variety on the right of part (ii). For any $a \in S$ and $x = a$, we get $a \in a^2Sa$ which by Lemma 3.5(ii) yields that $S \in \mathcal{CR}$ and thus also $S \in \mathcal{CS}$. Hence let $S = \mathcal{M}(I, G, \Lambda; P)$ with P normalized. For $a = (1, e, 1)$, $x = (1, g, \lambda)$ and $y = (j, h, 1)$, we obtain

$$c = axya = (1, gp_{\lambda j}h, 1),$$

$$d = (ayxa)' = (1, hg, 1)' = (1', g^{-1}\, h^{-1}, 1')$$

and the hypothesis $a = cda$ yields $e = gp_{\lambda j}hg^{-1}h^{-1}$. For $g = h = e$, we get $p_{\lambda j} = e$ so S is a rectangular group. But then also $gh = hg$ and G is abelian. Therefore $S \in \mathcal{R}e\,\mathcal{AG}$.

(*iii*) Any abelian group satisfies the given identity. Conversely, let S satisfy $a = x'ax$. For any $e, f \in E$, we get $e = f'ef = ef$ so S is a left group. Let $S = L \times G$ where L is a left zero semigroup and G is a group. For any $(i, g), (j, h) \in S$, we get $(j, h)' = (j', h^{-1})$ and thus

$$(i, g) = (j', h^{-1})(i, g)(j, h)$$

where $i = j'$ and $g = h^{-1}gh$. From the first equality, we deduce that L is trivial since i and j are arbitrary. Hence S is a group and the second equality yields that S is abelian.

(*iv*) This is well known.

(*v*) This is trivial. ∎

We now consider normal cryptogroups which are neither completely simple nor overabelian. Compare the next theorem with [11, Section 4].

Theorem 5.3. *The following equalities hold.*

(i) $\mathcal{NBG} = [(axya)^+ = (ayxa)^+] = [axaya = (ay)^+ axaya(axa)^+]$.

(ii) $\mathcal{NBG}(\mathcal{AC}) = [a = a^+a,\ ax^+a^+y^+a = ay^+a^+x^+a]$
$= [axaya = (ay^+a^+x^+a)(ax^+a^+y^+a)'axaya\,(axa)^+]$.

(iii) $\mathcal{C}e\mathcal{NBG} = [a = a^+a,\ ax^+a^+ya = aya^+x^+a]$
$= [axaya = (aya^+x^+a)(ax^+a^+ya)'axaya\ (axa)^+]$.

(iv) $\mathcal{ONBG} = [ax^+y^+b = ay^+x^+bb^+] = [axaya = (ay^+x^+a)(ax^+y^+a)'axaya(axa)^+]$.

(v) $\mathcal{SG} = [xy^+ = y^+x] = [x'xa = ax'x]$.

Proof: In each part, denote the classes by $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, respectively. Recall from Lemma 3.6(ii) that $a^+ = a^0$ if a is a completely regular element.

(*i*) The inclusion $\mathcal{A} \subseteq \mathcal{B}$ follows from [11, Lemma 4.7]. Conversely, let $S \in \mathcal{B}$. The substitution $a \to xx',\ y \to x'$ yields $(axya)^+ = xx'$ and $(ayxa)^+ = (xx'x'xxx')^+$ and the hypothesis implies that $x = (xx'x'xxx')^+x$. Hence

$$x = (xx'x'xxx')((xx'x'xxx')^2)'\,xx'xx \in Sx^2$$

which by Result 3.1 yields that $S \in \mathcal{CR}$. But then [11, Lemma 4.7] implies that $S \in \mathcal{A}$. Therefore $\mathcal{B} \subseteq \mathcal{A}$.

Any $\mathcal{M}(I, G, \Lambda; P)$ clearly satisfies the defining identity for $\mathcal{C}$ and thus $\mathcal{A} \subseteq \mathcal{C}$. Let $S \in \mathcal{C}$. For $x = y = a'$, the given identity yields $a = (aa')^+\,aa^+$ whence $a = aa^+$. Hence Result 3.1 implies that $S \in \mathcal{CR}$. It follows that S satisfies the identity $axaya = (ay)^0\,axaya(axa)^0$. Let $a \in E(S)$ and $x, y \in aSa$. The given identity implies that $xy = y^0\,xyx^0$. For $f = x \in\ E(aSa)$, we then get $fy = fyf$ and for $f = y \in E(aSa)$, $yf = fyf$. It follows that $fx = xf$ for all $x \in aSa$ and $f \in E(aSa)$. Therefore $aSa \in \mathcal{SG}$ and thus, by Theorem 4.1, $S \in \mathcal{SR}$. This together with $S \in \mathcal{CR}$ yields that $S \in \mathcal{A}$. Therefore $\mathcal{C} \subseteq \mathcal{A}$.

(*ii*) The equality $\mathcal{A} = \mathcal{B}$ follows from Lemma 3.5(ii) and [11, Lemma 4.6].

Let $S \in \mathcal{A}$. In view of [14, Proposition 7.4], to prove that $S \in \mathcal{C}$, it suffices to show that any $\mathcal{M}(I, G, \Lambda; P)$ with normalized P and whose entries commute satisfies the defining identity of $\mathcal{C}$ with ^{0} replacing ${}^+$. Hence let

$$a = (i, g, \lambda), \; x = (j, h, \mu), \; y = (k, t, \nu), \; c = axaya. \tag{2}$$

Then

$$\begin{aligned}
&(ay^0a^0x^0a)(ax^0a^0y^0a)'\,axaya\,(axa)^0 \\
&\quad = (ay^0a^0x^0a)\,\left(i, gp_{\lambda j}\,p_{\mu j}^{-1}\,p_{\mu i}\,p_{\lambda i}^{-1}\,p_{\lambda k}\,p_{\nu k}^{-1}\,p_{\nu i}g, \lambda\right)'\,c \\
&\quad = \left(i, gp_{\lambda k}\,p_{\nu k}^{-1}\,p_{\nu i}\,p_{\lambda i}^{-1}\,p_{\lambda j}\,p_{\mu j}^{-1}\,p_{\mu i}g\,p_{\lambda i'}\,p_{\lambda i'}^{-1}g^{-1}p_{\nu i}^{-1}\,p_{\nu k}\,p_{\nu k}^{-1}\,p_{\lambda i}\,p_{\mu i}^{-1}\,p_{\mu j}\,p_{\lambda j}^{-1}g^{-1}p_{\lambda' i}^{-1}, \lambda'\right)c \\
&\quad = \left(i, p_{\lambda' i}^{-1}, \lambda'\right)c = c,
\end{aligned}$$

as required. Therefore $\mathcal{A} \subseteq \mathcal{C}$.

Now let $S \in \mathcal{C}$. Let $a \in E(S)$ and $x, y \in aSa$. The given identity implies that $xy = y^+x^+(x^+y+)'\,xyx^+$. For $f = x \in E(aSa)$, we get $fy = fyf$ and for $f = y \in E(aSa)$, we obtain $xf = fxf$. Thus $xf = fx$ for all $x \in aSa$ and $f \in E(aSa)$ so that $aSa \in \mathcal{SG}$. But then $S \in \mathcal{SR}$ by Theorem 4.1.

For $x = y = a'$ in the given identity, we obtain $a = ua^+$ for some u and thus $a = aa^+$ which by Lemma 3.5(ii) gives that $S \in \mathcal{CR}$. Hence $S \in \mathcal{NBG}$ so that it remains to show that the given identity with ^{0} instead of ${}^+$ on a completely simple semigroup C implies that $C \in \mathcal{AC}$. In view of [14, Proposition 7.4], it suffices to let $C = \mathcal{M}(I, G, \lambda; P)$ with P normalized and prove that the entries of P commute. Hence let

$$a = (1, e, 1), \;\; x = (j, e, \mu), \;\; y = \left(1, p_{\nu k}^{-1}, \mu\right).$$

Then $axaya = \left(1, p_{\nu k}^{-1}, 1\right)$ and

$$\begin{aligned}
&(ay^0a^0x^0a)(ax^0a^0y^0a)'axaya(axa)^0 \\
&\quad = \left(1, p_{\nu k}^{-1}p_{\mu j}^{-1}, 1\right)(1', p_{\nu k}p_{\mu j}, 1')\left(1, p_{\nu k}^{-1}, 1\right)(1, e, 1) \\
&\quad = \left(1, p_{\nu k}^{-1}\,p_{\mu j}^{-1}\,p_{\nu k}\,p_{\mu j}\,p_{\nu k}^{-1}, 1\right)
\end{aligned}$$

and the hypothesis yields that $p_{\nu k}^{-1}\,p_{\mu j}\,p_{\nu k}\,p_{\mu j} = e$. Therefore the entries of P commute. We conclude that $\mathcal{C} \subseteq \mathcal{A}$.

(iii) The equality $\mathcal{A} = \mathcal{B}$ follows from Lemma 3.5(ii) and [11, Lemma 4.5].

Let $S \in \mathcal{A}$. In view of [14, Proposition 6.2], to prove that $S \in \mathcal{C}$, it suffices to show that any $\mathcal{M}(I, G, \Lambda; P)$ with normalized P and whose entries lie in the center of G satisfies the defining identities of $\mathcal{C}$ with ^{0} replacing ${}^+$. Let the notation be as in (2). Then

$$\begin{aligned}
&(aya^0x^0a)(ax^0a^0ya)'\,axaya(axa)^0 \\
&\quad = (aya^0x^0a)\,\left(i, gp_{\lambda j}\,p_{\mu j}^{-1}\,p_{\mu i}\,p_{\lambda i}^{-1}\,p_{\lambda k}t\,p_{\nu i}g, \lambda\right)'\,c \\
&\quad = \left(i, gp_{\lambda k}t\,p_{\nu i}\,p_{\lambda i}^{-1}\,p_{\lambda j}\,p_{\mu j}^{-1}\,p_{\mu i}g\,p_{\lambda i'}\,p_{\lambda i'}^{-1}g^{-1}p_{\nu i}^{-1}t^{-1}p_{\lambda k}^{-1}\,p_{\lambda i}p_{\mu i}^{-1}\,p_{\mu j}\,p_{\lambda j}^{-1}g^{-1}\,p_{\lambda' i}^{-1}, \lambda\right)c \\
&\quad = c,
\end{aligned}$$

as required. Therefore $\mathcal{A} \subseteq \mathcal{C}$.

Now let $S \in \mathcal{C}$. Let $a \in E(S)$ and $x, y \in aSa$. The given identity implies that $xy = yx^+(x^+y)'xyx^+$. As above, this leads to $S \in \mathcal{SR}$.

For $x = y = a'$ in the given identity, we obtain $a = ua^+$ for some u and thus $a = aa^+$ which by Lemma 3.5(ii) gives $S \in \mathcal{CR}$. Hence $S \in \mathcal{NBG}$ so that it remains to show that the given identity with ^{0} instead of $^+$ on a completely simple semigroup C implies that $C \in \mathcal{C}e$. In view of [14, Proposition 6.2], it suffices to let $C = \mathcal{M}(I, G, \Lambda; P)$ with P normalized and prove that the entries of P are in the center of G. Hence let

$$a = (1, e, 1), \quad x = (j, e, \mu), \quad y = (1, t^{-1}, 1).$$

Then $axaya = (1, t^{-1}, 1)$ and

$$\begin{aligned}(aya^0x^0a)(ax^0a^0ya)'\,axaya(axa)^0 &= \left(1, t^{-1}p_{\mu j}^{-1}, 1\right)\,\left(1, p_{\mu j}^{-1}t, 1\right)'(1, t, 1)\\ &= \left(1, t^{-1}p_{\mu j}^{-1}, 1\right)\,(1', t'p_{\mu j}, 1')\,(1, t^{-1}, 1) = \left(1, t^{-1}p_{\mu j}^{-1}\,tp_{\mu j}\,t^{-1}, 1\right)\end{aligned}$$

and the hypothesis yields that $t^{-1}p_{\mu j}^{-1}\,tp_{\mu j} = e$. Therefore the entries of P are in the center of G. We conclude that $\mathcal{C} \subseteq \mathcal{A}$.

(*iv*) In view of [9, Corollary IV.4.6], in order to prove that $\mathcal{A} \subseteq \mathcal{B}$, it suffices to show that a rectangular group satisfies the defining identity of $\mathcal{B}$ with ^{0} replacing $^+$. But this is trivial since a left zero semigroup, a group and a right zero semigroup obviously satisfy this identity. Therefore $\mathcal{A} \subseteq \mathcal{B}$.

Let $S \in \mathcal{B}$. The substitution $x, y, a \to bb'$ in the given identity yields $b = bb^+$ which by Lemma 3.5(ii) shows that $S \in \mathcal{CR}$. For $a = b$, the given identity becomes $ax^0y^0a = ay^0x^0a$ which by [11, Lemma 4.3] yields that $S \in \mathcal{A}$. Therefore $\mathcal{B} \subseteq \mathcal{A}$.

To see that $\mathcal{A} \subseteq \mathcal{C}$, it suffices to observe that every left zero semigroup, group and right zero semigroup satisfies the defining identity of $\mathcal{C}$ with ^{0} and $^{-1}$ replacing $^+$ and $'$, respectively. Therefore $\mathcal{A} \subseteq \mathcal{C}$.

Now let $S \in \mathcal{C}$. Let $a \in E(S)$ and $x, y \in aSa$. The given identity implies that $xy = y^+x^+(x^+y^+)'xyx^+$. As above, this leads to $S \in \mathcal{SR}$.

For $x = y = a'$ in the given identity, we obtain $a = ua^+$ for some u and thus $a = aa^+$ which by Lemma 3.5(ii) gives $S \in \mathcal{CR}$. Hence $S \in \mathcal{NBG}$ so that it remains to show that the given identity with ^{0} replacing $^+$ on a completely simple semigroup C implies that $C \in \mathcal{O}$. Let $C = \mathcal{M}(I, G, \Lambda; P)$ with P normalized satisfy the defining identity for $\mathcal{C}$. Hence let

$$a = (1, e, 1), \quad x = (j, e, 1), \quad y = (1, e, \nu).$$

Then $axaya = (1, e, 1)$ and

$$(ay^0x^0a)(ax^0y^0a)'\,axaya(axa)^0 = (1, p_{\nu j}, 1)\,(1, e, 1)'(1, e, 1) = (1, p_{\nu j}, 1)$$

and by hypothesis $e = p_{\nu j}$. Therefore C is orthodox and $\mathcal{C} \subseteq \mathcal{A}$.

(*v*) That $\mathcal{A} = \mathcal{B}$ follows from the characterization of Clifford semigroups as regular semigroups with idempotents in the center. This also implies that $\mathcal{A} \subseteq \mathcal{C}$. For $a \in S$ and $e \in E(S)$, we obtain

$$ea = e(e'ea) = e(ae'e) = eae.$$

For $e, f \in E(S)$, this gives that $ef = efe$ and hence $E(S)$ is a left regular band. For any $a \in S$, we also have $(a'a)a = a(a'a)$ and thus $a = a'a^2 = aa'a'a^2 \in aSa^2$. By Result 3.1, $S \in \mathcal{CR}$ and is thus a left regular orthogroup. Any $\mathcal{D}$-class of S is a left group, so we may consider $L \times G$ where L is a left zero semigroup and G is a group. For any $(i, g), (j, h) \in S$, by hypothesis

$$(j', h^{-1})(j, h)(i, g) = (i, g)(j', h^{-1})(j, h)$$

so that $j' = i$. Since i and j are arbitrary, L must be trivial so that S is a Clifford semigroup. Therefore $\mathcal{C} \subseteq \mathcal{A}$. ∎

We finally treat overabelian normal cryptogroups which are not completely simple. Again compare the next result with [11, Section 4].

Theorem 5.4. *The following equalities hold.*

(*i*) $\mathcal{NBAG} = [axaya = ayaxaa^+]$.
(*ii*) $\mathcal{ONBAG} = [axya = ayxa]$.
(*iii*) $\mathcal{SAG} = [xy = yx]$.
(*iv*) $\mathcal{NB} = [axya = ayxa^2]$.
(*v*) $\mathcal{S} = [xy = yx^2]$.

Proof: (*i*) If $S \in \mathcal{NBAG}$, then [11, Lemma 4.4] directly implies that S satisfies the identity $axaya = ayaxaa^+$. Conversely, let S satisfy the last identity. Substituting $x, y \to a'$, we get $a = aa^+$ and hence, by Lemma 3.5(ii), $S \in \mathcal{CR}$. Now [11, Lemma 4.4] yields that $S \in \mathcal{NBAG}$.

(*ii*) If $S \in \mathcal{ONBAG}$, then [11, Lemma 4.2] directly implies that S satisfies the identity $axya = ayxa$. Conversely, let S satisfy the last identity. Substituting $x \to a'a$, $y \to a'$, we get $a \in aSa^2$ which by Result 3.1 implies that $S \in \mathcal{CR}$. Now [11, Lemma 4.2] yields that $S \in \mathcal{ONBAG}$.

(*iii*) This is trivial.

(*iv*) If $S \in \mathcal{NB}$, then trivially S satisfies the identity $axya = ayxa^2$. Conversely, let S satisfy the last identity. Substituting $x \to a'$, $y \to aa'$ gives $a = a^2$ which evidently implies that $S \in \mathcal{NB}$.

(*v*) Any semilattice satisfies the identity $xy = yx^2$. Conversely, if S satisfies the last identity, then for $y = x'x$, we get $x \in xSx^2$ which by Result 3.1 gives that $S \in \mathcal{CR}$ whence easily that $S \in \mathcal{S}$. ∎

6 E-solid semigroups

We treat here the e-varieties of strict semigroups which are either E-solid, orthodox or inverse. The basis we give for the first one is a slight variant of the first basis in Theorem 4.1; indeed, it suffices to invert x^+ in the right place. For the second one we again have a variant of the first basis in Theorem 4.1 obtained by inserting $y^+x^+y^+$ in a suitable place. For the third case, we have a variant of the second basis in Theorem 4.1. Two lemmas handle the Rees matrix semigroup case.

We start with a simple statement.

Lemma 6.1. *Let S be a regular semigroup with the property that for any $a \in E(S)$, aSa is an inverse semigroup such that $xf = fxf$ for all $x \in aSa$ and $f \in E(aSa)$. Then S is strict.*

Proof: With the notation introduced, we get

$$x = x(x^{-1}x) = (x^{-1}x)x\,(x^{-1}x) = x^{-1}x^2 = xx^{-1}x^{-1}x^2$$

and Result 3.1 implies that aSa is completely regular. Hence aSa is a Clifford semigroup which by Theorem 4.1 implies that S is strict. ∎

And now for the E-solid case.

Lemma 6.2. *The following conditions on $S = \mathcal{M}^0(I, G, \Lambda; P)$ are equivalent.*
(i) S *is E-solid.*
(ii) *If any three of $p_{\lambda i}, p_{\lambda j}, p_{\mu j}, p_{\mu i}$ are nonzero, so is the fourth one.*
(iii) $S \in [x^2y^2 = x^2y^2(xy)^+]$.

Proof: *(i) implies (ii)*. This follows easily from the definition of E-solidity.

(ii) implies (iii). Let $x = (i, g, \lambda)$, $y = (j, h, \mu)$. If $x^2y^2 = 0$, then trivially $x^2y^2 = x^2y^2\,(xy)^+$. Suppose that $x^2y^2 \neq 0$. Then $p_{\lambda i}, p_{\lambda j}, p_{\mu j} \neq 0$ and the hypothesis implies that $p_{\mu i} \neq 0$ whence $(x^2y^2)^2 \neq 0$ and thus $x^2y^2 = x^2y^2(x^2y^2)^+$. But then $(x^2y^2)^+ = (xy)^+$ whence $x^2y^2 = x^2y^2(xy)^+$.

(iii) implies (i). Let $e, f, g \in E(S)$ be such that $e\,\mathcal{L}\,f\,\mathcal{R}\,g$. Hence $e = (i, p_{\lambda i}^{-1}, \lambda)$, $f = (j, p_{\lambda j}^{-1}, \lambda)$ and $g = (j, p_{\mu j}^{-1}, \mu)$ and thus $eg \neq 0$. By hypothesis $(eg)^+ \neq 0$ and hence $p_{\mu i} \neq 0$. But then $e\,\mathcal{R}\,(i, p_{\mu i}^{-1}, \mu)\,\mathcal{L}\,g$ and thus S is E-solid. ∎

Theorem 6.3. $\mathcal{S}(\mathcal{ES}) = \mathcal{SR} \cap [x^2y^2 = x^2y^2(xy)^+] = [(axa)(aya)^+ = (aya)^+(ax^+xa)]$.

Proof: Denote these three classes by $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, respectively. The equality $\mathcal{A} = \mathcal{B}$ follows from Theorem 4.1 and Lemma 6.2.

In order to prove that $\mathcal{A} \subseteq \mathcal{C}$, first let $S = \mathcal{M}^0(I, G, \Lambda; P)$ be E-solid and let

$$a = (i, g, \lambda),\ x = (j, h, \mu),\ y = (k, t, \nu), \tag{3}$$
$$c = (axa)(aya)^+,\ d = (aya)^+(ax^+xa). \tag{4}$$

Then

$$\begin{aligned} &c \neq 0 \ \Leftrightarrow\ p_{\lambda j}, p_{\mu i}, p_{\lambda i}, p_{\lambda k}, p_{\nu i} \neq 0, \\ &d \neq 0 \ \Leftrightarrow\ p_{\lambda k}, p_{\nu i}, p_{\lambda i}, p_{\lambda j}, p_{\mu i} \neq 0, \\ &p_{\lambda j}, p_{\lambda i}, p_{\mu i} \neq 0 \Rightarrow p_{\mu j} \neq 0 \end{aligned} \tag{5}$$

the last implication by E-solidity. Hence $c \neq 0$ if and only if $d \neq 0$. If $c \neq 0$, then $x^+x = x$ and thus $c = d$ follows from Theorem 4.1. In view of the last reference, we deduce that $\mathcal{A} \subseteq \mathcal{C}$.

Conversely, let $S \in \mathcal{C}$. Let $a \in E(S)$ and $x, y \in aSa$. The given identity implies that $xy^+ = y^+x^+x$. For $f = y \in E(aSa)$, we then obtain $xf = fxf$, and for $f = x$, we get $fy^+ = y^+f$. Therefore idempotents of aSa commute so aSa is an inverse semigroup in which $xf = fxf$ for all $x \in aSa$ and $f \in E(aSa)$. By

Lemma 6.1, we obtain $S \in \mathcal{SR}$. In view of Theorem 4.1, it remains to prove that $C = \mathcal{M}^0(I, G, \Lambda; P) \in \mathcal{C}$ is E-solid.

In this case, let $p_{\lambda i}, p_{\lambda j}, p_{\mu i} \neq 0$. With the notation (3) and (4) and $x = y$, we get by (5) that $c \neq 0$. Hence also $d \neq 0$ so that $x^+ \neq 0$ which yields $p_{\mu j} \neq 0$. In view of Lemma 6.2, we conclude that C is E-solid. Now Theorem 4.1 and the fact that E-solidity in an e-varietal property imply that $\mathcal{C} \subseteq \mathcal{A}$. ∎

We turn next to the orthodox case.

Lemma 6.4. *The following conditions on $S = \mathcal{M}^0(I, G, \Lambda; P)$ are equivalent.*

(i) S is orthodox.

(ii) $S \in [x^+y^+ = (x^+y)^+]$.

(iii) $S \in [xx^+y^+y = xy^+x^+y]$.

Proof: Let $x = (i, g, \lambda)$ and $y = (j, h, \mu)$.

(i) *implies* (ii). First

$$\begin{aligned} x^+y^+ \neq 0 \quad &\Leftrightarrow \quad p_{\lambda i}, p_{\lambda j}, p_{\mu j} \neq 0, \\ (x^+y)^+ \neq 0 \quad &\Leftrightarrow \quad p_{\lambda i}, p_{\lambda j}, p_{\mu i} \neq 0 \end{aligned}$$

and these two conditions are equivalent by orthodoxy. If $a^+b^+ \neq 0$, then by orthodoxy,

$$a^+b^+ = \left(i, p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1}, \mu\right) = \left(i, p_{\mu i}^{-1}, \mu\right) = (a^+b)^+.$$

(ii) *implies* (i). Immediate.

(i) *implies* (iii). First

$$\begin{aligned} xx^+y^+y \neq 0 \quad &\Leftrightarrow \quad p_{\lambda i}, p_{\lambda j}, p_{\mu j} \neq 0, \\ xy^+x^+y \neq 0 \quad &\Leftrightarrow \quad p_{\lambda j}, p_{\mu j}, p_{\mu i}, p_{\lambda i} \neq 0 \end{aligned}$$

and these two conditions are equivalent by orthodoxy. If $xx^+y^+y \neq 0$, then by orthodoxy

$$\begin{aligned} xy^+x^+y &= \left(i, gp_{\lambda j}\, p_{\mu j}^{-1}\, p_{\mu i}\, p_{\lambda i}^{-1}\, p_{\lambda j}h, \mu\right) \\ &= (i, gp_{\lambda j}h, \mu) = ab = xx^+y^+y. \end{aligned}$$

(iii) *implies* (i). Immediate. ∎

Note that Lemma 6.4(ii) has the dual $x^+y^+ = (xy^+)^+$ and that

$$xx^+y^+y = x^+x\, y^+y = xx^+yy^+ = x^+xyy^+$$

which provides variants of Lemma 6.2(iii).

Proposition 6.5. $\mathcal{SO} = \mathcal{SR} \,\cap\, [x^+y^+ = (x^+y)^+] = \mathcal{SR} \,\cap\, [xx^+y^+y = xy^+x^+y]$
$= [(axa)(aya)^+ = (aya)^+\, y^+x^+y^+\, (axa))]$.

Proof: Denote these four classes by $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$, respectively. The equalities $\mathcal{A} = \mathcal{B} = \mathcal{C}$ follow from Theorem 4.1 and Lemma 6.4.

In order to prove that $\mathcal{A} \subseteq \mathcal{D}$, we first let $S = \mathcal{M}^0(I, G, \Lambda; P)$ be orthodox and let

$$\begin{aligned} &a = (i, g, \lambda),\ x = (j, h, \mu),\ y = (k, t, \nu), \\ &c = (axa)(aya)^+,\ d = (aya)^+ x^+ y^+ x^+ (axa). \end{aligned}$$

Then

$$d \neq 0 \ \Leftrightarrow\ p_{\lambda k}, p_{\nu i}, p_{\lambda i}, p_{\lambda j}, p_{\mu j}, p_{\mu k}, p_{\nu k}, p_{\nu j}, p_{\mu i} \neq 0,$$

and by othodoxy, we have

$$\begin{aligned} p_{\lambda j}, p_{\lambda i}, p_{\mu i} \neq 0 \ &\Rightarrow\ p_{\mu j} \neq 0, \\ p_{\lambda k}, p_{\lambda i}, p_{\mu i} \neq 0 \ &\Rightarrow\ p_{\mu k} \neq 0, \\ p_{\lambda k}, p_{\lambda i}, p_{\nu i} \neq 0 \ &\Rightarrow\ p_{\nu k} \neq 0, \\ p_{\lambda j}, p_{\lambda i}, p_{\nu i} \neq 0 \ &\Rightarrow\ p_{\nu j} \neq 0, \end{aligned}$$

and thus

$$d \neq 0 \ \Leftrightarrow\ p_{\lambda j}, p_{\mu i}, p_{\lambda i}, p_{\lambda k}, p_{\nu i} \neq 0 \Leftrightarrow c \neq 0.$$

Assume that $c \neq 0$. By Lemma 3.6(iii), we get $c = (axa)\, a^0 = axa$, and also by orthodoxy, we obtain

$$d = a^0 x^0 y^0 x^0 (axa) = (ax)^0\, (axa) = axa.$$

Therefore $c = d$ and thus $S \in \mathcal{D}$. Now Theorem 4.1 and the fact that orthodoxy is an e-varietal property imply that $\mathcal{A} \subseteq \mathcal{D}$.

Conversely, let $S \in \mathcal{D},\ a \in E(S)$ and $x, y \in aSa$. The given identity implies that $xy^+ = y^+ x^+ y^+ x$. For $f = x \in E(aSa)$ and $g = y^+$, we obtain $fg = (gf)^2$ whence $fg = fgf = (fg)^2$ and aSa is orthodox. But then $fg = (gf)^2 = gf$ and aSa is an inverse semigroup. In addition $xf = fxf$ for all $x \in aSa$ and $f \in E(aSa)$. Now Lemma 6.1 implies that $S \in \mathcal{SR}$.

Next let $C = \mathcal{M}^0(I, G, \Lambda; P) \in \mathcal{D}$. Further let $e, f \in E(C)$ be such that $(ef)^2 \neq 0$. From the given identity, by the substitution $a, y \to e,\ x \to f$, we obtain that $efe = (efe)^2$. Now letting

$$e = \left(i, p_{\lambda i}^{-1}, \lambda\right),\ f = \left(j, p_{\mu j}^{-1}, \mu\right),\ z = p_{\lambda i}^{-1}\, p_{\lambda j}\, p_{\mu j}^{-1}\, p_{\mu i}\, p_{\lambda i}^{-1},$$

we get

$$efe = (i, z, \lambda),\ (efe)^2 = (i, z p_{\lambda i} z, \lambda)$$

and thus $z = z p_{\lambda i} z$ whence $z = p_{\lambda i}^{-1}$. It follows that $p_{\lambda i}^{-1}\, p_{\lambda j}\, p_{\mu j}^{-1} = p_{\mu i}^{-1}$ so that $ef = (ef)^2$ and C is orthodox.

In view of Theorem 4.1, we conclude that $\mathcal{D} \subseteq \mathcal{A}$. ∎

Finally we consider inverse semigroups.

Proposition 6.6. $\mathcal{SI} = \mathcal{SR} \cap [a^+ \; CW \; b^+] = [a^+ x a^+ \; CW \; (byb)^+]$.

Proof: Denote these three classes by $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, respectively. The equality $\mathcal{A} = \mathcal{B}$ follows from Lemma 3.5(i) and Theorem 4.1.

In order to prove that $\mathcal{A} \subseteq \mathcal{C}$, first let $S = B(G, I)$ be a Brandt semigroup. For $a, b, x, y \in S$ and

$$c = a^+ x a^+ \, (byb)^+, \; d = (byb)^+ a^+ x a^+,$$

we easily see that $c \neq 0$ if and only if $d \neq 0$, and if $c \neq 0$, then $c = x = d$. Therefore $c = d$ and $S \in \mathcal{D}$. Since being inverse is an e-varietal property, Theorem 4.1 implies that $\mathcal{A} \subseteq \mathcal{C}$.

Conversely, let $S \in \mathcal{C}$. For $a = b \in E(S)$, $x \in aSa$ and $y \in E(aSa)$, the given identity implies that $xy = yx$ so that aSa is a Clifford semigroup. By Theorem 4.1, we deduce that $S \in \mathcal{SR}$. Also for $x = a^+$ and $y = b'$ in the given identity, we obtain $a^+ b^+ = b^+ a^+$. Hence Lemma 3.5(i) implies that S is an inverse semigroup. Therefore $S \in \mathcal{C}$ and thus $\mathcal{C} \subseteq \mathcal{A}$. ∎

7 Central semigroups

We consider here the e-varieties of strict semigroups which are either central or central and E-solid. In each case, we need a lemma handling the Rees matrix semigroups. In the first case, we have variants of the second and third bases in Theorem 4.1. In the second case, we have a variant of the third basis in Theorem 4.1.

Lemma 7.1. *The following conditions on $S = \mathcal{M}^0(I, G, \Lambda; P)$ are equivalent.*

(i) S is central.

(ii) If $p_{\lambda i}, p_{\lambda j}, p_{\mu j}, p_{\mu i} \neq 0$ and $g \in G$, then

$$\left(p_{\lambda i}^{-1}\, p_{\lambda j}\, p_{\mu j}^{-1}\right) p_{\mu i}\, g = g p_{\mu i} \left(p_{\lambda i}^{-1}\, p_{\lambda j}\, p_{\mu j}^{-1}\right).$$

(iii) $S \in [ax^+a^+ = a^+x^+a]$.

(iv) $S \in [ax^+a^+y^+a = ay^+a^+xa]$.

(v) $S \in [xy \; CW \; x^+y^+]$.

Proof: (i) *implies* (ii). Assume the antecedent of part (ii) and let $e = (i, p_{\lambda i}^{-1}, \lambda)$, $f = (j, p_{\mu j}^{-1}, \mu)$. Then $(ef)^2 \neq 0$ and the hypothesis implies that

$$(i, p_{\lambda i}^{-1}, \lambda)(j, p_{\mu j}^{-1}, \mu)(i, g, \lambda) = (i, g, \mu)(i, p_{\lambda i}^{-1}, \lambda)(j, p_{\mu j}^{-1}, \mu)$$

whence the desired conclusion.

(ii) *and* (iii) *are equivalent.* This can be verified by straightforward calculation.

(ii) *implies* (iv). Let

$$a = (i, g, \lambda), \; x = (j, h, \mu), \; y = (k, t, \nu), \; c = axa^+y^+a, \; d = ay^+a^+xa.$$

It follows that $c \neq 0$ if and only if $d \neq 0$. If $c \neq 0$, then

$$c = \left(i, gp_{\lambda j}hp_{\mu i}\, p_{\lambda i}^{-1} p_{\lambda k}\, p_{\nu k}^{-1} p_{\nu i} g, \lambda\right),$$

$$d = \left(i, gp_{\lambda k}\, p_{\nu k}^{-1} p_{\nu i}\, p_{\lambda i}^{-1} p_{\lambda j} h p_{\mu i} g, \lambda\right)$$

and hence $c = d$ is equivalent to

$$p_{\lambda j} h\, p_{\mu i}\, p_{\lambda i}^{-1}\, p_{\lambda k}\, p_{\nu k}^{-1} p_{\nu i} = p_{\lambda k}\, p_{\nu k}^{-1}\, p_{\nu i}\, p_{\lambda i}^{-1}\, p_{\lambda j} h\, p_{\mu i}.$$

Letting $t = (p_{\lambda j} h p_{\mu i})^{-1}$ and taking inverses, this becomes

$$p_{\nu i}^{-1}\, p_{\nu k}\, p_{\lambda k}^{-1}\, p_{\lambda i}\, t = t p_{\lambda i}\, p_{\nu i}^{-1}\, p_{\nu k}\, p_{\lambda k}^{-1}$$

and the hypothesis implies that indeed $c = d$.

(iv) implies (v). First set $x = a'$ in the given identity. Then

$$x(yx^+y^+) = x(y^+x^+y) = (xy^+x^+)y = (x^+y^+x)y.$$

(v) implies (i). Let $e = (i, p_{\lambda i}^{-1}, \lambda),\ f = (j, p_{\mu j}^{-1}, \mu)$ and assume that $(ef)^2 \neq 0$. For $g \in G$, letting $x = (i, p_{\lambda j}^{-1}, \lambda)$ and $y = (j, g, \mu)$, we get $(i, g, \mu) = xy,\ x^+ = e$ and $y^+ = f$. The hypothesis implies that

$$(i, g, \mu)\, ef = xyx^+y^+ = x^+y^+xy = ef(i, g, \mu)$$

and S is central. ∎

Note that the condition in Lemma 7.1(iii) can be written as

$$p_{\lambda j}\, p_{\mu j}^{-1}\, p_{\mu i}\, p_{\lambda i}^{-1}\ CW\ p_{\lambda i} g\, p_{\mu i}\, p_{\lambda i}^{-1}.$$

Since we do not know whether $\mathcal{C}e$ is an e-variety or not, we must use Lemma 4.3 when dealing with strict semigroups.

Theorem 7.2. *$\mathcal{C}e\mathcal{S} = \mathcal{SR} \cap [ax^+a^2 = a^2x^+a] = \mathcal{SR} \cap [ax^+a^+ = a^+x^+a]$
$= \mathcal{SR} \cap [xy\ CW\ x^+y^+]$
$= [axa^+y^+a = ay^+a^+xa] = [a^+xa^+\ CW\ a^+y^+a^+]$.*

Proof: Denote these six classes by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ and $\mathcal{F}$, respectively. The equalities $\mathcal{A} = \mathcal{B} = \mathcal{C} = \mathcal{D}$ follow directly from Theorem 4.1 and Lemmas 4.3 and 7.1. Let $S \in \mathcal{A}$. By Theorem 4.1, S is a subdirect product of its principal factors, say $S_\alpha, \alpha \in A$. Hence S_α is a central completely (0-)simple semigroup which by Lemma 7.1 implies that $S_\alpha \in \mathcal{E}$. Since this holds for every $\alpha \in A$, it follows that $S \in \mathcal{E}$. Therefore $\mathcal{A} = \mathcal{E}$. The transition from the defining identity of $\mathcal{D}$ to that of $\mathcal{E}$ is effected by premultiplying the former by $a(a^2)'$ and postmultiplying by $(a^2)'a$. Therefore $\mathcal{E} \subseteq \mathcal{F}$.

It remains to prove that $\mathcal{F} \subseteq \mathcal{A}$. Hence let $S \in \mathcal{F}$. Let $a \in E(S)$ and $x, y \in aSa$. The given identity implies that $xy^+ = y^+x$ which shows that $aSa \in \mathcal{SG}$. By Theorem 4.1, we have that $S \in \mathcal{SR}$ and that S is a subdirect product of completely (0-) simple semigroups, say $S_\alpha, \alpha \in A$. But then S_α satisfies the identity $a^+xa^+\ CW\ a^+y^+a^+$ which by Lemma 7.1 yields that S_α is central; this holds for all $\alpha \in A$. Returning to S, we may suppose that $S \subseteq \prod_{\alpha \in A} S_\alpha$. Let $e = (e_\alpha), f = (f_\alpha) \in E(S)$ and $a = (a_\alpha) \in S$ be such that $ef\ \mathcal{H}\ a\ \mathcal{H}\ a^2$. For any $\alpha \in A$, we get $e_\alpha\, f_\alpha\ \mathcal{H}_\alpha\ a_\alpha\ \mathcal{H}_\alpha\ a_\alpha^2$ with $e_\alpha, f_\alpha \in E(S_\alpha)$. Since S_α is central, we obtain $e_\alpha f_\alpha a_\alpha = a_\alpha e_\alpha f_\alpha$. But this holds for all $\alpha \in A$; it follows that $efa = aef$ and S is central. Therefore $S \in \mathcal{A}$ which proves that $\mathcal{F} \subseteq \mathcal{A}$. ∎

We now consider the central E-solid case.

Lemma 7.3. *Let $S = \mathcal{M}^0(I, G, \Lambda; P)$. Then S is central and E-solid if and only if $S \in [x^+y^+ = xy^+x(x^2)'(xy)^+]$.*

Proof: *Necessity.* Let

$$x = (i, g, \lambda),\ y = (j, h, \mu),\ c = x^+y^+,\ d = xy^+x(x^2)'(xy)^+.$$

Then

$$\begin{aligned} c \neq 0 &\Leftrightarrow p_{\lambda i}, p_{\lambda j}, p_{\mu j} \neq 0, \\ d \neq 0 &\Leftrightarrow p_{\lambda j}, p_{\mu j}, p_{\mu i}, p_{\lambda i} \neq 0 \end{aligned}$$

and by E-solidity, we conclude that $c \neq 0$ if and only if $d \neq 0$.

Assume that $c \neq 0$. By Lemma 7.1(ii) and letting $t = p_{\mu j}^{-1}\, p_{\mu i}\, g$, we obtain

$$p_{\lambda i}^{-1}\, p_{\lambda j}\, t = p_{\mu i}^{-1}\, p_{\mu j}\, t p_{\mu i}\, p_{\lambda i}^{-1}\, p_{\lambda j}\, p_{\mu j}^{-1}$$

whence

$$p_{\mu j}^{-1}\, p_{\mu i}\, p_{\lambda i}^{-1}\, p_{\lambda j}\, t = t p_{\mu i}\, p_{\lambda i}^{-1}\, p_{\lambda j}\, p_{\mu j}^{-1}. \tag{6}$$

Now

$$c = \left(i, p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1}, \mu\right),$$

$$d = \left(i, g p_{\lambda j} p_{\mu j}^{-1} p_{\mu i}\, g p_{\lambda i'} p_{\lambda i'}^{-1} g^{-1} p_{\lambda' i}^{-1} p_{\lambda' i} p_{\mu i}^{-1}, \mu\right)$$

and thus

$$\begin{aligned} c = d \quad &\Leftrightarrow \quad p_{\lambda i}^{-1}\, p_{\lambda j} p_{\mu j}^{-1} = g p_{\lambda j}\, p_{\mu j}^{-1}\, p_{\mu i}\, p_{\lambda i}^{-1}\, g^{-1}\, p_{\mu i}^{-1} \\ &\Leftrightarrow \quad p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1} p_{\mu i} g = g p_{\lambda j} p_{\mu j}^{-1} p_{\mu i} p_{\lambda i}^{-1}. \end{aligned} \tag{7}$$

But writing g for t and interchanging $i \leftrightarrow j,\ \lambda \leftrightarrow \mu$ in (6) yields (7). Therefore $c = d$.

Sufficiency. The argument can be extracted from above by essentially reversing the steps. ∎

Proposition 7.4.

$$\mathcal{C}e\mathcal{S}(\mathcal{ES}) = \mathcal{SR} \cap \left[x^+y^+ = xy^+x(x^2)'(xy)^+\right] = \left[axa^+y^+a = ay^+a^+x^+xa\right].$$

Proof: Denote these three classes by $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, respectively. That $\mathcal{A} = \mathcal{B}$ is a direct consequence of Theorem 4.1 and Lemmas 4.3 and 7.3.

Let $S \in \mathcal{A}$. By Theorem 7.2, S satisfies the identity $axa^+y^+a^+ = ay^+a^+xa$. By Theorem 4.1, every principal factor S_α of S satisfies the last identity. Let $a, x, y \in S_\alpha$ and

$$c = axa^+y^+a,\ d = ay^+a^+x^+xa.$$

The hypothesis implies that S_α is E-solid. If $c \neq 0$, then $axa^+ \neq 0$ which by Lemma 6.2 yields that $x^+ \neq 0$ which by Lemma 3.6(iii) gives $x = x^+x$ and thus $c = d$. If $d \neq 0$, then clearly $c \neq 0$ and hence $c = d$. Therefore S_α satisfies the identity $c = d$ which proves that $S_\alpha \in \mathcal{C}$. By Theorem 4.1, S is a subdirect product of S_α's and hence $S \in \mathcal{C}$. Therefore $\mathcal{A} \subseteq \mathcal{C}$.

Converesely, let $S \in \mathcal{C}$. Let $a \in E(S)$ and $x, y \in aSa$. The given identity implies that $x^+y = yy^+x^+$. For $f = x \in E(aSa)$, we get $fy = yy^+f$ whence $fy = fyf$, and for $f = y \in E(aSa)$ we obtain $x^+f = fx^+$. Thus aSa is an inverse semigroup with $fx = fxf$ for all $x \in aSa$ and $f \in E(aSa)$. Hence by Lemma 6.1, we have $S \in \mathcal{SR}$.

For $y = a$ in the given identity, we get the identity

$$axa^+a = aa^+x^+xa. \tag{8}$$

This identity evidently implies that $a^+xa^+ = a^+x^+xa^+$ which together with the given identity yields $a^+xa^+ \,CW\, a^+y^+a^+$. By Theorem 7.2, S is central.

Theorem 4.1 implies that S in a subdirect product of its principal factors, say S_α, $\alpha \in A$. Let $S_\alpha = \mathcal{M}^0(I, G, \Lambda; P)$ and suppose that $p_{\lambda i}, p_{\lambda j}, p_{\mu i} \neq 0$. For $a = (i, g, \lambda)$ and $x = (j, h, \mu)$, we get $axa^+a \neq 0$ which by (8) yields that $x^+ \neq 0$ so that $p_{\mu j} \neq 0$. By Lemma 6.2, S_α is solid. Since this holds for all $\alpha \in A$, we conclude that S is E-solid. Therefore $\mathcal{C} \subseteq \mathcal{A}$. ∎

8 Overabelian case

We treat here strict semigroups all of whose subgroups are abelian. For this e-variety, we devise bases for identities some of which use the operation $+$, others only the given operation $'$ and yet some use neither. As special cases, we consider e-varieties of E-solid, orthodox and inverse overabelian strict semigroups. We encounter here variants of the second, third and fourth bases in Theorem 4.1. As before, lemmas handle the Rees matrix semigroup cases.

Lemma 8.1. *The following conditions on $S = \mathcal{M}^0(I, G, \Lambda; P)$ are equivalent*

(i) *S is overabelian.*

(ii) $S \in [axaya = ayaxa]$.

(iii) $S \in [a^+xa = axa^+]$.

Proof: (i) *implies* (ii). Noting that then G is abelian, simple verification shows that S satisfies the identity $axaya = ayaxa$.

(ii) *implies* (iii). It suffices to set $y = (a^2)'$.

(iii) *implies* (i). Let $p_{\lambda i} \neq 0$. Then

$$(i, p_{\lambda i}^{-1}, \lambda)(i, h, \lambda)(i, g, \lambda) = (i, g, \lambda)(i, h, \lambda)(i, p_{\lambda i}^{-1}, \lambda)$$

whence $hp_{\lambda i}g = gp_{\lambda i}h$. For $h = e$, we get $p_{\lambda i}g = gp_{\lambda i}$ so that $hgp_{\lambda i} = ghp_{\lambda i}$ whence $hg = gh$. Therefore G in abelian and S is overabelian. ∎

Theorem 8.2. $\mathcal{S}(\mathcal{AG}) = L\mathcal{SAG} = \mathcal{SR} \cap [a^+xa = axa^+] = [axa\,CW\,aya]$
$= [a^+xa^+\,CW\,a^+ya^+] = [axaya = ayaxa]$.

Proof: Denote these six classes by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ and $\mathcal{F}$, respectively. The equality $\mathcal{A} = \mathcal{B}$ is an easy consequence of Theorem 4.1. The same reference and Lemmas 4.3 and 8.1 imply that $\mathcal{A} = \mathcal{C}$.

Let $S \in \mathcal{A}$. By Theorem 4.1, S is a subdirect product of its principal factors, say $S_\alpha, \alpha \in A$. The hypothesis also implies that S_α is overabelian for every $\alpha \in A$. For any $a \in S_\alpha, axa \in H_a \cup \{0\}$ and hence $axa\, \mathcal{C}W\, aya$ since H_a is abelian if it is a group. Therefore $S_\alpha \in \mathcal{D}$ for every $\alpha \in A$ and thus also $S \in \mathcal{D}$ by Theorem 4.1. Therefore $\mathcal{A} \subseteq \mathcal{D}$.

The substitution $a \to a^+$ in the defining identity of $\mathcal{D}$ yields that of $\mathcal{E}$. Thus $\mathcal{D} \subseteq \mathcal{E}$. The defining identity of $\mathcal{E}$ clearly implies the property of local commutativity. Hence $\mathcal{E} \subseteq \mathcal{B}$. Let $S \in \mathcal{A}$. As above, S is a subdirect product of overabelian completely (0-) simple semigroups $S_\alpha, \alpha \in A$. Clearly $S_\alpha \in \mathcal{F}$ for every $\alpha \in A$ and hence $S \in \mathcal{F}$. Therefore $\mathcal{A} \subseteq \mathcal{F}$. If $S \in \mathcal{F}$, we see immediately that S is locally commutative and hence $S \in \mathcal{B}$. Thus $\mathcal{F} \subseteq \mathcal{B}$. ∎

We now consider E-solid overabelian semigroups.

Lemma 8.3. *Let $S = \mathcal{M}^0(I, G, \Lambda; P)$. Then S is E-solid and overabelian if and only if $S \in [a^+xa = ax^+xa^+]$.*

Proof: *Necessity.* Let

$$a = (i, g, \lambda),\ x = (j, h, \mu),\ c = a^+xa,\ d = ax^+xa^+.$$

Then

$$\begin{aligned} c \neq 0 \quad &\Leftrightarrow \quad p_{\lambda i}, p_{\lambda j}, p_{\mu i} \neq 0, \\ d \neq 0 \quad &\Leftrightarrow \quad p_{\lambda j}, p_{\mu j}, p_{\mu i}, p_{\lambda i} \neq 0. \end{aligned}$$

Since S is E-solid, it follows that $c \neq 0$ if and only if $d \neq 0$. If $c \neq 0$, then

$$c = \left(i, p_{\lambda i}^{-1} p_{\lambda j} h p_{\mu i} g, \lambda\right),\ d = \left(i, g p_{\lambda j} h p_{\mu i} p_{\lambda i}^{-1}, \lambda\right)$$

and since G is abelian, we get $c = d$.

Sufficiency. Reversing the above argument, we see that $p_{\lambda i}, p_{\lambda j}, p_{\mu i} \neq 0$ implies that $p_{\mu j} \neq 0$ which by Lemma 6.2 yields that S is E-solid. Essentially the same argument as in the proof of Lemma 8.1 shows that S is overabelian. ∎

Proposition 8.4. $\mathcal{S}(\mathcal{ES})(\mathcal{AG}) = \mathcal{SR} \cap [a^+xa = ax^+xa^+] = [axa^+ya = ayax^+xa^+]$.

Proof: Denote these three classes by $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, respectively. The equality $\mathcal{A} = \mathcal{B}$ follows directly from Theorem 4.1 and Lemmas 4.3 and 8.3.

To prove that $\mathcal{A} \subseteq \mathcal{C}$, we first let $S = \mathcal{M}^0(I, G, \Lambda; P) \in \mathcal{A}$ and

$$\begin{aligned} &a = (i, g, \lambda),\ x = (j, h, \mu), \quad y = (k, t, \nu), \\ &c = axa^+ya,\ d = ayax^+xa^+. \end{aligned} \tag{9}$$

Then

$$\begin{aligned} c \neq 0 \quad &\Leftrightarrow \quad p_{\lambda j}, p_{\mu i}, p_{\lambda i}, p_{\lambda k}, p_{\nu i} \neq 0, \\ d \neq 0 \quad &\Leftrightarrow \quad p_{\lambda k}, p_{\nu i}, p_{\lambda j}, p_{\mu j}, p_{\mu i}, p_{\lambda i} \neq 0, \end{aligned}$$

and the hypothesis implies that

$$p_{\lambda j}, p_{\lambda i}, p_{\mu i} \neq 0 \ \Rightarrow \ p_{\mu j} \neq 0.$$

Therefore $c \neq 0$ if and only if $d \neq 0$. If $c \neq 0$, then

$$c = \left(i, gp_{\lambda j}hp_{\mu i}p_{\lambda i}^{-1}p_{\lambda k}tp_{\nu i}g, \lambda\right), \ d = \left(i, gp_{\lambda k}tp_{\nu i}gp_{\lambda j}hp_{\mu i}p_{\lambda i}^{-1}, \lambda\right)$$

and thus, since G is abelian, we get $c = d$. In view of Theorem 4.1 and Lemmas 4.3 and 8.3, we conclude that $\mathcal{A} \subseteq \mathcal{C}$.

Conversely, let $S \in \mathcal{C}$. Let $a \in E(S)$ and $x, y \in aSa$. The defining identity implies that $xy = yx^+x$. For $y = x'x$, this yields $x = x'xx^+x$ so that $x = xx'x'x(x^2)'x^2$ and by Result 3.1, $aSa \in \mathcal{CR}$. But then $xy = yx$ and $S \in \mathcal{SAG}$. Now Theorem 4.1 yields that $S \in \mathcal{S}(\mathcal{AG})$. For $C = \mathcal{M}^0(I, G, \Lambda; P) \in \mathcal{C}$, a and x as in (9) and $p_{\mu i}, p_{\lambda i}, p_{\lambda j} \neq 0$, we get $axa^+xa \neq 0$ and thus $x^2 \neq 0$ so that $p_{\mu j} \neq 0$. By Lemma 6.2 this implies that S is E-solid. By Theorem 4.1, we get that $S \in \mathcal{A}$. Therefore $\mathcal{C} \subseteq \mathcal{A}$. ∎

Next we treat orthodox overabelian semigroups.

Lemma 8.5. *Let $S = \mathcal{M}^0(I, G, \Lambda; P)$. Then S is orthodox and overabelian if and only if $S \in [x^+xyy^+ = x^+yxy^+]$.*

Proof: *Necessity.* Let

$$x = (i, g, \lambda), \ y = (j, h, \mu), \ c = x^+xyy^+, \ d = x^+yxy^+.$$

Then

$$c \neq 0 \ \Leftrightarrow \ p_{\lambda i}, p_{\lambda j}, p_{\mu j} \neq 0,$$

$$d \neq 0 \ \Leftrightarrow \ p_{\lambda i}, p_{\lambda j}, p_{\mu i}, p_{\mu j} \neq 0.$$

If $c \neq 0$, then $0 \neq a^+x^+$ and hence $a^+x^+ = (a^+x^+)^2$ by orthodoxy which yields that $x^+a^+ \neq 0$. It follows that $c \neq 0$ if and only if $d \neq 0$. If $c \neq 0$, then

$$c = (i, gp_{\lambda j}h, \mu), \ d = (i, p_{\lambda i}^{-1}p_{\lambda j}hp_{\mu i}gp_{\lambda j}p_{\mu j}^{-1}, \mu).$$

By orthodoxy, $p_{\mu i}^{-1} = p_{\lambda i}^{-1}p_{\lambda j}p_{\mu j}^{-1}$ whence it follows that $c = d$ since G is abelian.

Sufficiency. For $e, f \in E(S)$, the given identity yields $ef = efef$ so S is orthodox. For $p_{\lambda i} \neq 0$, we get

$$(i, g, \lambda)(i, h, \lambda) = (i, p_{\lambda i}^{-1}, \lambda)(i, h, \lambda)(i, g, \lambda)(i, p_{\lambda i}^{-1}, \lambda)$$

whence $gp_{\lambda i}h = hp_{\lambda i}g$ which as in the proof of Lemma 8.1 implies that S is overabelian. ∎

Proposition 8.6.

$$\mathcal{SO}(\mathcal{AG}) = \mathcal{SR} \cap [x^+xyy^+ = x^+yxy^+] = [axa^+ya = aya^+x^+y^+a^+xa].$$

Proof: Denote these three classes by $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, respectively. The equality $\mathcal{A} = \mathcal{B}$ follows directly from Theorem 4.1 and Lemmas 4.3 and 8.5.

To prove that $\mathcal{A} \subseteq \mathcal{C}$, we first let $S = \mathcal{M}^0(I, G, \Lambda; P) \in \mathcal{A}$ and

$$a = (i, g, \lambda),\ x = (j, h, \mu),\ y = (k, t, \nu),\ c = axa^+ya,\ d = aya^+x^+y^+a^+xa.$$

Then

$$\begin{aligned} c \neq 0 \ &\Leftrightarrow\ p_{\lambda j}, p_{\mu i}, p_{\lambda i}, p_{\lambda k}, p_{\nu i} \neq 0, \\ d \neq 0 \ &\Leftrightarrow\ p_{\lambda k}, p_{\nu i}, p_{\lambda i}, p_{\lambda j}, p_{\mu j}, p_{\mu k}, p_{\nu k}, p_{\mu i} \neq 0, \end{aligned}$$

and by orthodoxy,

$$\begin{aligned} p_{\lambda j}, p_{\lambda i}, p_{\mu i} \neq 0 \ &\Rightarrow\ p_{\mu j} \neq 0, \\ p_{\lambda k}, p_{\lambda i}, p_{\mu i} \neq 0 \ &\Rightarrow\ p_{\mu k} \neq 0, \\ p_{\lambda k}, p_{\lambda i}, p_{\nu i} \neq 0 \ &\Rightarrow\ p_{\nu k} \neq 0, \end{aligned}$$

so that $c \neq 0$ if and only if $d \neq 0$. If $c \neq 0$, by orthodoxy of S and commutativity of G, we obtain

$$\begin{aligned} d &= \left(i, gp_{\lambda k}tp_{\nu i}p_{\lambda i}^{-1}p_{\lambda j}p_{\mu j}^{-1}(p_{\mu k}p_{\nu k}^{-1}p_{\nu i})p_{\lambda i}^{-1}p_{\lambda j}hp_{\mu i}g, \lambda\right) \\ &= \left(i, gp_{\lambda j}hp_{\mu i}p_{\lambda i}^{-1}\, p_{\lambda k}tp_{\nu i}g, \lambda\right)\left(i, p_{\lambda i}^{-1}(p_{\lambda i}^{-1}p_{\lambda j}p_{\mu j}^{-1})p_{\mu i}, \lambda\right) \\ &= c\left(i, p_{\lambda i}^{-1}p_{\mu i}^{-1}p_{\mu i}, \lambda\right) = c. \end{aligned}$$

Therefore $S \in \mathcal{C}$. Now Theorem 4.1 and Lemmas 4.3 and 8.5 imply that $\mathcal{A} \subseteq \mathcal{C}$.

Conversely, let $S \in \mathcal{C}$. Let $a \in E(S)$ and $x, y \in aSa$. The given identity implies that $xy = yx^+y^+x$. For $f = x \in E(aSa)$, we obtain $fy = yfy^+f$ whence $fy = fyf$; for $f = y \in E(aSa)$, we get $xf = fx^+fx$ so that $xf = fxf$. It follows that $xf = fx$ for all $x \in aSa$ and $f \in E(aSa)$ and thus $aSa \in \mathcal{SG}$. By Theorem 4.1, we have that $S \in \mathcal{SR}$. In a group, the given identity yields commutativity. Therefore $S \in \mathcal{S}(\mathcal{AG})$.

In view of Theorem 4.1, we may prove orthodoxy by showing it for $S = \mathcal{M}^0(I, G, \lambda; P) \in \mathcal{C}$. Hence let $p_{\lambda i}, p_{\lambda k}, p_{\nu k} \neq 0$. Recalling that G is abelian, we obtain

$$\begin{aligned} &(i, e, \lambda)(i, e, \lambda)(i, p_{\lambda i}^{-1}, \lambda)(k, e, \nu)(i, e, \lambda) = (i, p_{\lambda i}^2 p_{\lambda k} p_{\nu i}, \lambda), \\ &(i, e, \lambda)(k, e, \nu)(i, p_{\lambda i}^{-1}, \lambda)(i, p_{\lambda i}^{-1}, \lambda)(k, p_{\nu k}^{-1}, \nu)(i, p_{\lambda i}^{-1}, \lambda)(i, e, \lambda)(i, e, \lambda) \\ = &\ (i, p_{\lambda k}\, p_{\nu i} p_{\lambda i}^{-1}\, p_{\lambda k}\, p_{\nu k}^{-1}\, p_{\nu i}\, p_{\lambda i}, \lambda) = (i, p_{\lambda k}^2\, p_{\nu k}^{-1}\, p_{\nu i}^2\, p_{\lambda i}, \lambda) \end{aligned}$$

which by hypothesis implies that $p_{\lambda i}^2\, p_{\lambda k}\, p_{\nu i} = p_{\lambda k}^2\, p_{\nu k}^{-1} p_{\nu i}^2\, p_{\lambda i}$. But then $p_{\nu i}^{-1} = p_{\lambda i}^{-1} p_{\lambda k} p_{\nu k}^{-1}$ and thus S is orthodox. ∎

Finally we handle inverse overabelian semigroups.

Lemma 8.7. *Let $S = \mathcal{M}^0(I, G, \Lambda; P)$. Then S is inverse and overabelian if and only if $S \in [x^+xy^+y = y^+yx^+x]$.*

Proof: Straightforward. ∎

Note that the above S satisfies the identity $x^+x = xx^+$ which gives several variants of the identity in Lemma 8.7.

Proposition 8.8. $\mathcal{SI}(\mathcal{AG}) = \mathcal{SR} \cap [x^+xy^+y = y^+yx^+x] = [a^+xa^+ \, CW \, b^+yb^+]$.

Proof: Denote these three classes by $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, respectively. The equality $\mathcal{A} = \mathcal{B}$ follows from Theorem 4.1 and Lemmas 4.3 and 8.7.

To prove that $\mathcal{A} \subseteq \mathcal{C}$, it suffices to show that any Brandt semigroup $B(G, I)$ with G abelian satisfies the defining identity of $\mathcal{C}$. This consists of simple checking. Therefore $\mathcal{A} \subseteq \mathcal{C}$.

Conversely, let $S \in \mathcal{C}$. For $a = b$, by Theorem 8.2, we obtain $S \in \mathcal{S}(\mathcal{AG})$. For $x = a^+$ and $y = b^+$, we get $a^+b^+ = b^+a^+$. Hence idempotents of S commute and S is also an inverse semigroup. Thus $S \in \mathcal{A}$ and therefore $\mathcal{C} \subseteq \mathcal{A}$. ■

9 Combinatorial case

For the e-variety of combinatorial strict semigroups we find several bases for its identities. It is of some interest that none of these bases uses the unary operation of $^+$ and some of them do not use the given unary operation $'$ either. We also treat the special cases of orthodox and inverse combinatorial strict semigroups. These bases are already rather distant variants of those in Theorem 4.1 even though there is some resemblence left. Lemmas again take care of the Rees matrix semigroup case.

Lemma 9.1. *The following conditions on $S = \mathcal{M}^0(I, G, \Lambda; P)$ are equivalent.*

(i) *S is combinatorial.*
(ii) $S \in [axaya = aya(xa)^2]$.
(iii) $S \in [axa = axaxa]$.
(iv) $S \in [x^2 = x^3]$.

Proof: Straightforward. ■

By [5, Theorem 3.3], the e-variety $\mathcal{SC}o$ is generated by the semigroup $\mathcal{M}^0(2, \{1\}, 2; P)$ with $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. According to Trahtman [15], the semigroup variety generated by this semigroup has $\{x^2 = x^3, \ axa = axaxa, \ axaya = ayaxa\}$ as a basis. We shall now see some variants of these identities.

Theorem 9.2. $\mathcal{SC}o = L\mathcal{S} = \mathcal{SR} \cap [x^2 = x^3] = \mathcal{SR} \cap [axa = axaxa]$
$= [axaya = aya(xa)^2] = [(axa)(aya) = (aya)(axa)^2]$.

Proof: Denote these six classes by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ and $\mathcal{F}$, respectively. The equality $\mathcal{A} = \mathcal{B}$ follows easily from Theorem 4.1 and Lemma 4.3. From the same references and Lemma 9.1 follow the equalities $\mathcal{A} = \mathcal{C} = \mathcal{D}$.

Let $S \in \mathcal{A}$. By Theorem 8.2, S satisfies the identity $axaya = ayaxa$. From $\mathcal{D}$ we have that $axa = axaxa$ and hence $axaya = aya(xa)^2$. Therefore $S \in \mathcal{E}$ and thus $\mathcal{A} \subseteq \mathcal{E}$.

Assume the defining identity of $\mathcal{E}$. The substitution $x \to ax$ yields $a^2xaya = (aya)(axa)^2$ and the substitution $y \to ay$ gives $axa^2ya = a^2ya(xa)^2$. Hence

$$(axa)(aya) = a^2ya(xa)^2 = a^2xaya = (aya)(axa)^2.$$

Therefore $\mathcal{E} \subseteq \mathcal{F}$.

Let $S \in \mathcal{F}$. For $a \in E(S)$ and $x, y \in aSa$, the defining identity implies that $xy = yx^2$. By Theorem 5.4(v), aSa is a semilattice. Therefore $S \in \mathcal{B}$ and thus $\mathcal{F} \subseteq \mathcal{B}$. ∎

We consider next E-solid combinatorial semigroups. If G in $S = \mathcal{M}^0(I, G, \Lambda; P)$ is trivial, we write (i, λ) for $(i, 1, \lambda)$.

Lemma 9.3. *The following conditions on $S = \mathcal{M}^0(I, G, \Lambda; P)$ are equivalent.*

(i) *S is E-solid and combinatorial.*

(ii) *S is orthodox and combinatorial.*

(iii) *$S \in [axa^2 = ax^2a]$.*

Proof: *(i) implies (ii).* Straightforward.

(ii) implies (iii). Let $a = (i, \lambda)$ and $x = (j, h)$. Then

$$\begin{aligned} axa^2 \neq 0 \quad &\Leftrightarrow \quad p_{\lambda j}, p_{\mu i}, p_{\lambda i} \neq 0, \\ ax^2a \neq 0 \quad &\Leftrightarrow \quad p_{\lambda j}, p_{\mu j}, p_{\mu i} \neq 0 \end{aligned}$$

and by orthodoxy, we get that $axa^2 \neq 0$ if and only if $ax^2a \neq 0$ whence $axa^2 = ax^2a$ by the combinatorial property.

(iii) implies (i). The given identity in a group implies triviality. If $p_{\lambda i}, p_{\lambda j}, p_{\mu i} \neq 0$, then for $a = (i, \lambda), x = (j, \mu), axa^2 \neq 0$ and thus $x^2 \neq 0$ so $p_{\mu j} \neq 0$. By Lemma 6.2, S is E-solid. ∎

Proposition 9.4. $\mathcal{S}(\mathcal{ES})\mathcal{C}o = \mathcal{SOC}o = \mathcal{SR} \cap [axa^2 = ax^2a]$
$= [(axa)(aya) = (aya)(ax^2a)] = [axa^2ya = ayax^2a]$.

Proof: Denote these five classes by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and $\mathcal{E}$, respectively. Equalities $\mathcal{A} = \mathcal{B} = \mathcal{C}$ follow directly from Theorem 4.1 and Lemmas 4.3 and 9.3. To show that $\mathcal{B} \subseteq \mathcal{D}$, first let $S = \mathcal{M}^0(I, \{1\}, \Lambda; P)$ be orthodox and let

$$a = (i, \lambda),\ x = (j, \mu),\ y = (k, \nu).$$

Then

$$axa^2ya \neq 0 \quad \Leftrightarrow \quad p_{\lambda j}, p_{\mu i}, p_{\lambda i}, p_{\lambda k}, p_{\nu i} \neq 0, \tag{10}$$

$$aya^2x^2a \neq 0 \quad \Leftrightarrow \quad p_{\lambda k}, p_{\nu i}, p_{\lambda i}, p_{\lambda j}, p_{\mu j}, p_{\mu i} \neq 0, \tag{11}$$

$$ayax^2a \neq 0 \quad \Leftrightarrow \quad p_{\lambda k}, p_{\nu i}, p_{\lambda j}, p_{\mu j}, p_{\mu i} \neq 0, \tag{12}$$

and by Lemma 6.2,

$$p_{\lambda j}, p_{\lambda i}, p_{\mu i} \neq 0 \quad \Rightarrow \quad p_{\mu j} \neq 0, \tag{13}$$

$$p_{\lambda j}, p_{\mu j}, p_{\mu i} \neq 0 \quad \Rightarrow \quad p_{\lambda i} \neq 0. \tag{14}$$

Now (13) shows that (10) implies (11) and the converse is trivial. Also (13) shows that (10) implies (12) and (14) shows that the converse holds.

In view of Theorem 4.1, it follows that $\mathcal{B} \subseteq \mathcal{D} \cap \mathcal{E}$.

Next let $S \in \mathcal{D}$. Let $a \in E(S)$ and $x, y \in aSa$. The defining identity implies that $xy = yx^2$ which by Theorem 5.4(v) yields that aSa is a semilattice. In view of Theorem 9.2, we conclude that $S \in \mathcal{SC}o$. Let $C = \mathcal{M}^0(I, \{1\}, \Lambda; P) \in \mathcal{D}$ and suppose that $p_{\lambda i}, p_{\lambda j}, p_{\mu i} \neq 0$. Then $((i,\lambda)(j,\mu)(i,\lambda))^2 \neq 0$ and the hypothesis implies that $(i,\lambda)(j,\mu)^2(i,\lambda) \neq 0$ whence $p_{\mu j} \neq 0$. It follows that C is orthodox. Since orthodoxy is an e-varietal property, by Theorem 4.1 we conclude that S is orthodox. Therefore $S \in \mathcal{B}$ and thus $\mathcal{D} \subseteq \mathcal{B}$.

The proof that $\mathcal{E} \subseteq \mathcal{B}$ is the same as for $\mathcal{D} \subseteq \mathcal{B}$. ■

Finally we arrive at inverse combinatorial semigroups.

Lemma 9.5. *Let $S = \mathcal{M}^0(I, G, \Lambda; P)$. Then S is inverse and combinatorial if and only if $S \in [x^3y^2 = y^2x^2]$.*

Proof: *Necessity*. Straightforward.

Sufficiency. For $e, f \in E(S)$, we get $ef = fe$ so S is an inverse semigroup. For $x = (i, g, i)$ and $y = (i, 1, i)$, we obtain from the given identity that $g^3 = g^2$ so that $g = 1$. Therefore S is also combinatorial. ■

Proposition 9.6. $\mathcal{SIC}o = \mathcal{SR} \cap [x^3y^2 = y^2x^2] = [(a^+xa^+)(byb)^+ = (byb)^+(a^+x^2a^+)]$.

Proof: Denote these three classes by $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, respectively. The equality $\mathcal{A} = \mathcal{B}$ follows from Theorem 4.1 ad Lemma 9.5. Clearly any combinatorial Brandt semigroup satisfies the defining identity for $\mathcal{C}$. In view of Theorem 4.1, we conclude that $\mathcal{B} \subseteq \mathcal{C}$.

Finally, let $S \in \mathcal{C}$. For $x = a^+$ and $y = b'$, we get $a^+ \, CW \, b^+$ and thus S is an inverse semigroup by Lemma 3.5(i). Let $a = b \in E(S)$ and $x, y \in aSa$. The given identity yields $xy^+ = y^+x^2$. For $y = x'x$, we obtain $x = x'x^3$ whence $x^2 = x^3$. Hence

$$x = x'x^3 = (x'x^3)x = x^2$$

and aSa is a semilattice. By Theorem 9.2, we obtain that $S \in \mathcal{SC}o$ and hence $S \in \mathcal{A}$. Therefore $\mathcal{C} \subseteq \mathcal{A}$. ■

10 Overabelian core

We consider here the case of strict semigroups for which the core $C(S)$ is overabelian, that is, has all subgroups abelian. For bases of identities for this e-variety of regular semigroups we obtain several versions, however, all countably infinite. To this end, we need the following concept.

Definition 10.1. *Let $S = \mathcal{M}^0(I, G, \Lambda; P)$. Following [7], we call the expressions of the form*

$$p^{-1}_{\lambda_1 i_1} p_{\lambda_1 i_2} \, p^{-1}_{\lambda_2 i_2} p_{\lambda_2 i_3} \cdots p_{\lambda_{n-1} i_n} p^{-1}_{\lambda_n i_n} p_{\lambda_n i_1}$$

polygonal products *where all* $p_{\lambda_k i_\ell} \neq 0$. *In addition, we say that this polygonal product has* pivot (i_1, λ_1).

We shall also use the following notation. For the variables $a, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$, *let*

$$X_n = a^+ x_1^+ x_2^+ \ldots x_n^+ a^+, \quad Y_n = a^+ y_1^+ y_2^+ \ldots y_n^+ a^+,$$

for $n \geq 1$.

Again we start with Rees matrix semigroups.

Lemma 10.2. *The following conditions on* $S = \mathcal{M}^0(I, G, \Lambda; P)$ *are equivalent.*

(*i*) $C(S)$ *is overabelian.*

(*ii*) *Any two polygonal products with the same pivot commute.*

(*iii*) $S \in [(axa)\, X_m Y_n (ax'xa)(aya)^+ = (aya)^+(axa)\, Y_n X_m\, (ax'xa)]_{m,n \geq 1}$.

(*iv*) $S \in [X_m\, CW\, Y_n]_{m,n \geq 1}$.

Proof: (*i*) *implies* (*ii*). Let g and h be polygonal products with the same pivot, say

$$g = p_{\lambda i}^{-1} p_{\lambda j_1} p_{\mu_1 j_2} \ldots p_{\mu_m j_m}^{-1} p_{\mu_m i}, \tag{15}$$

$$h = p_{\lambda i}^{-1} p_{\lambda k_1} p_{\nu_1 k_1}^{-1} p_{\nu_1 k_2} \ldots p_{\nu_n k_n}^{-1} p_{\nu_n i}. \tag{16}$$

For $s = 1, 2, \ldots, m$ and $t = 1, 2, \ldots, n$, let

$$a = (i, g, \lambda),\ x_s = (j_s, g_s, \mu_s),\ y_t = (k_t, h_t, \nu_t), \tag{17}$$
$$c = a^+ x_1^+ x_2^+ \ldots x_m^+ a^+,\ d = a^+ y_1^+ y_2^+ \ldots y_n^+ a^+. \tag{18}$$

Then $c\,\mathcal{H}\,d\,\mathcal{H}\,c^2 \neq 0$ and by hypothesis $cd = dc$. Moreover

$$c = (i, gp_{\lambda i}^{-1}, \lambda),\ d = (i, hp_{\lambda i}^{-1}) \tag{19}$$

and hence $cd = dc$ implies that $ghp_{\lambda i}^{-1} = hgp_{\lambda i}^{-1}$ whence $gh = hg$.

(*ii*) *implies* (*iii*). We show first that $X_m Y_n = Y_n X_m$. Let $a \in C(S)$, $a^2 \neq 0$. Then $a = e_1 e_2 \ldots e_n$ for some $e_1, e_2, \ldots, e_n \in E(S)$. If $a = (i, g, \lambda)$, then $p_{\lambda i} \neq 0$ and hence $a = a^+ e_1 e_2 \ldots e_n a^+$. Let $c, d \in C(S)$, $c\,\mathcal{H}\,d\,\mathcal{H}\,c^2 \neq 0$. Then c and d are of the form (18). In the notation (17) we arrive at polygonal products (15) and (16). By hypothesis $gh = hg$ which implies that

$$cd = (i, gp_{\lambda i}^{-1}, \lambda)(i, hp_{\lambda i}^{-1}, \lambda) = (i, ghp_{\lambda i}^{-1}, \lambda)$$

$$= (i, hgp_{\lambda i}^{-1}, \lambda) = (i, hp_{\lambda i}^{-1}, \lambda)(i, gp_{\lambda i}^{-1}, \lambda) = dc.$$

Therefore S satisfies the identity $X_m Y_n = Y_n X_m$.

The identity in part (iii) holds trivially when $(aya)^2 = 0$. If $(aya)^2 \neq 0$, then $(aya)^+ = a^0$ and hence

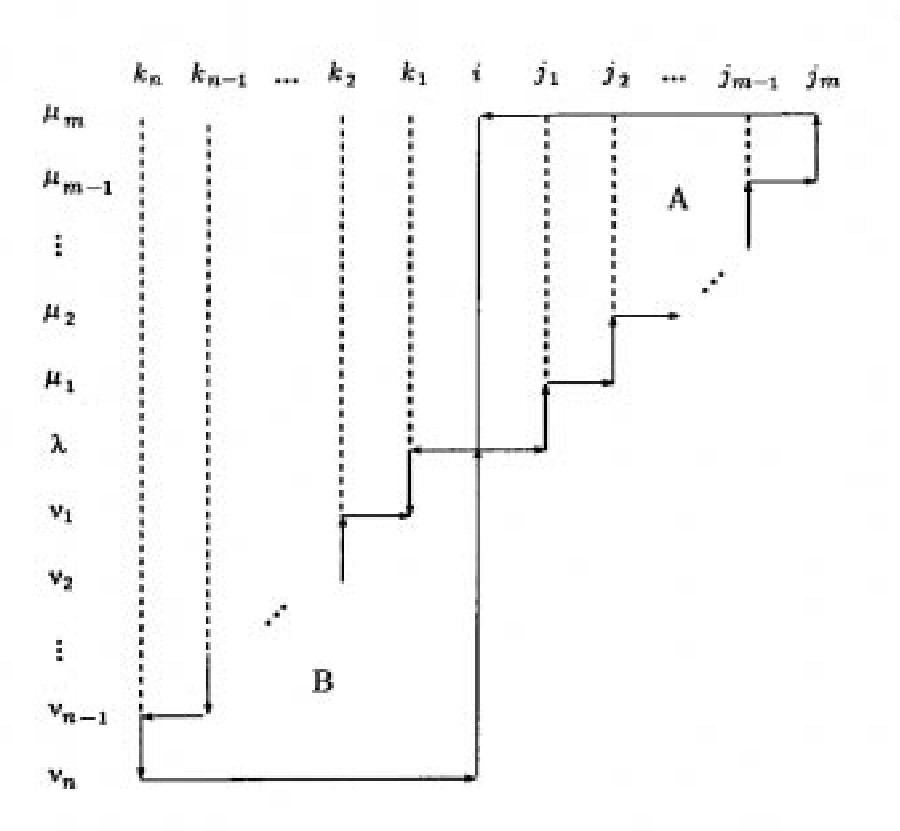
$$axa = (aya)^+(axa),\ (ax'xa)(aya)^+ = ax'xa$$

which together with $X_m Y_n = Y_n X_m$ implies that the identity in part (iii) holds.

(*iii*) *implies* (*iv*). The substitutions $a \to a^+$, $x \to a^+$, $y \to a'$ in the given identities yield $X_m\, CW\, Y_n$.

(*iv*) *implies* (*i*). The steps in the proof above of "(ii) implies (iii)" leading to $X_m\, CW\, Y_n$ may be reversed showing that the latter implies that any two polygonal products with the same pivot comute. Now the steps of the proof of "(i) implies (ii)" may be reversed showing that $C(S)$ is overabelian. ∎

We may illustrate two polygonal products A and B with the common pivot (i, λ) as follows.



o

Theorem 10.3.
$$\begin{aligned}\mathcal{S}(\mathcal{AC}) &= \mathcal{SR} \cap [X_m\, CW\, Y_n]_{m,n\geq 1}\\ &= \Big[(axa)\, X_m Y_n (ax'xa)(aya)^+ = (aya)^+(axa)\, Y_n X_m (ax'xa)\Big]_{m,n\geq 1}\\ &= [a^+x\, X_m Y_n\, x'xa^+(aya)^+ = (aya)^+\, a^+x\, Y_n X_m x'xa^+]_{m,n\geq 1}.\end{aligned}$$

Proof: Denote these four classes by $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$, respectively. The equality $\mathcal{A} = \mathcal{B}$ and the inclusion $\mathcal{A} \subseteq \mathcal{C}$ follow from Theorem 4.1 and lemmas 4.3 and 10.2. The substitution $a \to a^+$ in the defining identity of $\mathcal{C}$ yields that of $\mathcal{D}$. Therefore $\mathcal{C} \subseteq \mathcal{D}$.

Let $S \in \mathcal{D}$. Letting $x_i = y_j = a$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, the given identity yields
$$a^+xa^+x'xa^+(aya)^+ = (aya)^+\, a^+xa^+x'xa^+.$$
Let $a \in E(S)$ and $x, y \in aSa$. The last identity implies that $xy^+ = y^+x$ which shows that $aSa \in \mathcal{SG}$. Hence Theorem 4.1 yields that $S \in \mathcal{SR}$ and that S is a subdirect product of its principal factors, say $S_\alpha, \alpha \in A$. For the substitutions $x \to a^+$ and $y \to a'$ in the given identity, we easily get the identity $X_m\, CW\, Y_n$. Now Theorem 4.1 and Lemmas 4.3 and 10.2, imply that S is a subdirect product of completely (0-)simple semigroups, say $S_\alpha,\ \alpha \in A$, with $C(S_\alpha)$ overabelian. We may assume that $S \subseteq \prod_{\alpha\in A} S_\alpha$. Let $e_1, e_2, \ldots, e_m, f_1, f_2, \ldots, f_n, g \in E(S)$ be such that
$$e_1e_2\ldots e_m\ \mathcal{H}\ f_1f_2\ldots f_n\ \mathcal{H}\ g.$$

Let $e_1 = (e_{\alpha i}), f_j = (f_{\alpha j}), g = (g_\alpha)$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then for every $\alpha \in A$, we have

$$e_{\alpha 1}, e_{\alpha 2} \ldots e_{\alpha m} \, \mathcal{H}_{S_\alpha} \, f_{\alpha 1} \, f_{\alpha 2} \ldots f_{\alpha n} \, \mathcal{H}_{S_\alpha} \, g_\alpha$$

and since $C(S_\alpha)$ is overabelian, it follows that $e_{\alpha 1} e_{\alpha 2} \ldots e_{\alpha m}$ and $f_{\alpha 1} f_{\alpha 2} \ldots f_{\alpha n}$ commute. But then also $e_1 e_2 \ldots e_m$ and $f_1 f_2 \ldots f_n$ commute. Therefore $C(S)$ is overabelian and thus $S \in \mathcal{A}$ proving that $\mathcal{D} \subseteq \mathcal{A}$. ∎

We consider now the case of E-solid S and overabelian $C(S)$.

Lemma 10.4. *Let $S = \mathcal{M}^0(I, G, \Lambda; P)$. Then S is E-solid and $C(S)$ is overabelian if and only if $S \in [X_m Y_n \, z^+ = Y_m X_n \, z^+(az)^+]_{m,n \geq 1}$.*

Proof: *Necessity.* By Lemma 10.2, S satisfies the identities $X_m Y_n = Y_n X_m$. Let

$$a = (i, g, \lambda), z = (j, h, \mu), \; c = X_m Y_n \, z^+, \; d = Y_n X_m \, z^+(az)^+.$$

Suppose that $c \neq 0$. Then $X_m Y_n = (i, t, \lambda)$ for some $t \in G$ and hence

$$c = (i, t, \lambda)(j, p_{\mu j}^{-1}, \mu) = (i, t p_{\lambda j} \, p_{\mu j}^{-1}, \mu).$$

Since $p_{\lambda i}, p_{\lambda j}, p_{\mu j} \neq 0$, by Lemma 6.2, we get $p_{\mu i} \neq 0$ and thus

$$d = (i, t, \lambda)(j, p_{\mu j}^{-1}, \mu) = (i, p_{\mu i}^{-1}, \mu) = c.$$

Conversely, if $d \neq 0$, then clearly $c \neq 0$. Therefore $c = d$.

Sufficiency. For $z = a^+$, we get $X_m Y_n = Y_n X_m$ and by Lemma 10.2, $C(S)$ is overabelian. For the substitutions $x_i \to a$ and $y_j \to a$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, we get $a^+ z^+ = a^+ z^+ (az)^+ (az)^+$. In the above notation, if $p_{\lambda i}, p_{\lambda j}, p_{\mu j} \neq 0$, then $a^+ z^+ \neq 0$ and hence $(az)^+ \neq 0$ whence $p_{\mu i} \neq 0$. Now Lemma 6.2 implies that S is E-solid. ∎

Proposition 10.5. $\mathcal{S}(\mathcal{ES})(\mathcal{AC}) = \mathcal{SR} \cap [X_m Y_n \, z^+ = Y_n X_m \, z^+(az)^+]$
$= \Big[(axa) \, X_m Y_n \, z^+(ax'xa)(aya)^+ = (aya)^+(axa) \, Y_n X_m z^+(az)^+(ax'xa)\Big]_{m,n \geq 1}.$

Proof: Denote these three classes by $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, respectively. The equality $\mathcal{A} = \mathcal{B}$ and the inclusion $\mathcal{A} \subseteq \mathcal{C}$ follow by Theorem 4.1 and Lemmas 4.3 and 10.4.

Now let $S \in \mathcal{C}$. For $z = a^+$ in the given identity, by Theorem 10.3 we obtain that $S \in \mathcal{S}(\mathcal{AC})$. For the substitutions $x_i \to a$ and $y_j \to a$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, we get $a^+ z^+ = a^+ z^+ (az)^+$. This identity then holds in every principal factor of S. Now as in the last part of the proof of Lemma 10.4 we conclude that every principal factor, and thus also S, is E-solid. Therefore $S \in \mathcal{A}$ and thus $\mathcal{C} \subseteq \mathcal{A}$. ∎

11 Combinatorial core

We consider here the case of strict semigroups S for which $C(S)$ is combinatorial. This case is similar to the one we studied in the preceding section. Polygonal products are again useful; indeed, they were introduced in [7] just to handle this situation. In order to put the case under consideration in proper perspective, we digress somewhat by quoting in full the relevant result from [7]. The case of overabelian S with combinatorial core concludes the section.

Lemma 11.1. *The following conditions on* $S = \mathcal{M}^0(I, G, \Lambda; P)$ *are equivalent.*

(i) $C(S)$ *is combinatorial.*

(ii) All polygonal products are equal to e.

(iii) $S \in [X_n\, axa = axa\, X_n^2]$.

(iv) $S \in [X_n \in E]$.

Proof: (i) *implies* (ii). Let (15) be a polygonal product and let a and x_s be defined as in (17) and c as in (18). Then $c\,\mathcal{H}\,a\,\mathcal{H}\,a^2 \neq 0$ and hence, by hypothesis, $c = a$. Since $c = (i, gp_{\lambda i}^{-1}, \lambda)$ and $a = (i, p_{\lambda i}^{-1}, \lambda)$, we obtain $g = e$.

(ii) *implies* (iii). Starting with $a \in C(S)$, $a^2 \neq 0$, as in the proof of Lemma 10.2, we see that c in (18) is a general element of $C(S)$ for which $c^2 \neq 0$. Now c gives rise to the polygonal product g as in (15) with $c = (i, gp_{\lambda i}^{-1}, \lambda)$ as in (19). By hypothesis $g = e$ so that $c \in E(S)$. But then $c = a^0$ which together with $a^0(axa) = (axa)a^0$ implies that S satisfies the identity in part (iii).

(iii) *implies* (iv). Set $x = (a^2)'$.

(iv) *implies* (i). The steps in the proof above of "(ii) implies (iii)" leading to $c \in E(S)$, that is $X_n \in E$, may be reversed showing that the latter implies that any polygonal product is equal to e. Now the steps of the proof of "(i) implies (ii)" may be reversed showing that $C(S)$ is combinatorial. ∎

A semigroup S is said to be *rectangular* if for any $a, b, x, y \in S,\ ax = bx = ay = m$ implies $by = m$. Hence a band is rectangular band if and only if it is rectangular (as a semigroup). For the case when the semigroup S has a zero, these concepts were generalized in [7] as follows.

A semigroup S with zero is 0-*rectangular* if for any $a_i, x_i \in S,\ i = 1, 2, \ldots, n$,

$$a_1x_1 = a_1x_2 = a_2x_2 = \ldots = a_{n-1}x_n = a_nx_n = m \neq 0$$

and $a_nx_1 \neq 0$ imply that $a_nx_1 = m$. Also, a combinatorial completely 0-simple semigroup is a *rectangular 0-band*.

The direct product can be adapted to the situation when one of the factors has a zero. We state here only the special case which we require. Let S be a semigroup with zero and T a semigroup without zero. Then

$$S \times_0 T = (S \times T)/(\{0\} \times T)$$

is the 0-*product* of S and T.

We now combine [7, Corollaire 4.10 and Théorème 4.13] into the following result part of which is crucial for the characterization of members of $\mathcal{SR}(\mathcal{CC})$. For general edification we state it in full.

Result 11.2. *The following condition on $S = \mathcal{M}^0(I, G, \Lambda; P)$ are equivalent.*

(*i*) *$S \cong C \times_0 G$ where C is a rectangular 0-band.*

(*ii*) *There exist invertible $I \times I$-matrix U and $\Lambda \times \Lambda$-matrix V over G^0 such that $Q = VPU$ is a regular matrix all of whose nonzero entries are equal to e.*

(*iii*) *There exist mappings $\alpha : I \to G$ and $\beta : \Lambda \to G$ such that $p_{\lambda i} = (\lambda\beta)(i\alpha)$ or $p_{\lambda i} = 0$.*

(*iv*) *S is 0-rectangular.*

(*v*) *All polygonal products are equal to e.*

(*vi*) *S has a subsemigroup which intersects every $\mathcal{H}$-class of S exactly once.*

We observe that Result 11.2 is the generalization of the theorem which says that an orthodox completely simple semigroup is a rectangular group to the case of completely 0-simple semigroups. In part (i), we can clearly take $C = \mathcal{M}^0(I, \{e\}, \Lambda; Q)$ where $q_{\lambda i} = e$ if $p_{\lambda i} \neq 0$ and $q_{\lambda i} = 0$ otherwise. In particular, $S/\mathcal{H} \cong C$. We are now ready for the desired result.

Theorem 11.3. *The following conditions on a regular semigroup S are equivalent.*

(*i*) *S is a subdirect product of a combinatorial strict semigroup and a Clifford semigroup.*

(*ii*) *S is strict and $C(S)$ is combinatorial.*

(*iii*) *$S \in [X_n\, axa = axa\, X_n^2]_{n \geq 1}$.*

Proof: (*i*) *implies* (*ii*). We may suppose that $S \subseteq A \times B$ is a subdirect product where A is a combinatorial strict semigroup and B is a Clifford semigroup. Since also B is strict, we deduce that S is strict. Let $e_1, \ldots, e_m, f_1, \ldots, f_n \in E(S)$ be such that $e_1 \ldots e_m \,\mathcal{H}\, f_1 \ldots f_n$. Then $e_i = (e_i', e_i'')$ and $f_j = (f_j', f_j'')$ where $e_i', f_j' \in E(A)$ and $e_i'', f_j'' \in E(B)$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, and

$$e_1' \ldots e_m' \,\mathcal{H}_A\, f_1' \ldots f_n', \quad e_1'' \ldots e_m'' \,\mathcal{H}_B\, f_1'' \ldots f_n''.$$

Since A is combinatorial and B is orthodox, it follows that

$$e_1' \ldots e_m' = f_1' \ldots f_n', \quad e_1'' \ldots e_m'' = f_1'' \ldots f_n''$$

and thus $e_1 \ldots e_m = f_1 \ldots f_n$. Therefore $C(S)$ is combinatorial.

(*ii*) *implies* (*iii*). By Theorem 4.1, S is a subdirect product of its principal factors, say $S_\alpha, \alpha \in A$. Since $C(S)$ is combinatorial, so are $C(S_\alpha)$ for all $\alpha \in A$. Hence by Lemma 11.1, S_α satisfies all the identities in part (iii) for all $\alpha \in A$ and so does S.

(*iii*) *implies* (*i*). In particular, S satisfies the identity $(a^+y^+a^+)\, axa = axa(a^+y^+a^+)^2$. Letting $a \in E(S)$ and $x, y \in aSa$, this identity implies that $y^+x = xy^+$ and hence $aSa \in \mathcal{SG}$. For $x = (a^2)'$, the given identities yield that S satisfies the identities $X_n \in E$.

In view of Theorem 4.1, we may assume that $S \subseteq \prod_{\alpha \in A} S_\alpha$ is a subdirect product where S_α satisfies the identities $X_n \in E$ for every $\alpha \in A$. By Lemma 11.1 and Result

11.2, we may also suppose that for every $\alpha \in A$, we have either $S_\alpha \cong C_\alpha \times_0 G_\alpha$ or $S_\alpha \cong C_\alpha \times G_\alpha$ where C_α is a rectangular (0-) band and G_α is a group. Let $C = \prod_{\alpha \in A} C_\alpha$ and $G = \prod_{\alpha \in A} K_\alpha$ where $K_\alpha = G_\alpha^0$ if $S_\alpha = C_\alpha \times_0 G_\alpha$ and $K_\alpha = G_\alpha$ otherwise. In the light of Theorem 4.1, we conclude that C is a combinatorial strict semigroup whereas G in a Clifford semigroup.

Let

$$T = \{(c, g) \in C \times G \mid c = (c_\alpha),\ g = (g_\alpha) \quad \text{and} \quad (c_\alpha, g_\alpha) \in S\}\,.$$

The mapping

$$(c_\alpha, g_\alpha) \longrightarrow ((c_\alpha), (g_\alpha)) \qquad ((c_\alpha, g_\alpha) \in S)$$

is clearly an isomorphism of S onto T. Now T is a subdirect product of its projections in C and G, the former is a combinatorial strict semigroup, in view of Theorem 4.1, and the latter in a Clifford semigroup. In fact, it is easy to verify that these projections are equal to C and G, respectively. ∎

Corollary 11.4. $\mathcal{S}(\mathcal{C}o\mathcal{C}) = \mathcal{G} \vee \mathcal{SC}o = \mathcal{SR} \cap [X_n \in E]_{n \geq 1} = [X_n\, axa = axa\, X_n^2]_{n \geq 1}$
$= [X_n\, xa^+ = a^+ x\, X_n^2]_{n \geq n}$.

Proof: Denote these five classes by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and $\mathcal{E}$, respectively. Since $\mathcal{G} \vee \mathcal{SC}o = \mathcal{SG} \vee \mathcal{SC}o$, Theorem 11.3 implies that $\mathcal{A} = \mathcal{B}$. The equality $\mathcal{A} = \mathcal{C}$ follows from Theorem 11.3 and Lemmas 4.3 and 11.1. The equality $\mathcal{A} = \mathcal{D}$ follows easily from Theorem 11.3. The substitution $a \to a^+$ shows that $\mathcal{D} \subseteq \mathcal{E}$. Let $S \in \mathcal{E}$. The proof of "(iii) implies (i)" in Theorem 11.3 carries over to this case with the sole modification that the substitution now is $x \to a^+$. Hence Theorem 11.3 implies that $S \in \mathcal{D}$. Therefore $\mathcal{E} \subseteq \mathcal{D}$. ∎

Remark [1, p. 208] in our notation states that $\mathcal{G} \vee \mathcal{SC}o \subseteq \mathcal{S}(\mathcal{C}o\mathcal{C})$. We have one more special case.

Lemma 11.5. *Let* $S = \mathcal{M}^0(I, G, \Lambda; P)$. *Then* $S \in (\mathcal{AG})(\mathcal{C}o\mathcal{C})$ *if and only if* $S \in \left[axX_n = a^+ xaX_n^2\right]$.

Proof: *Necessity.* By Lemmas 8.1 and 11.1, we have $a^+xa = axa^+$ and $X_n \in E$ whence

$$axX_n = axa^+X_n = a^+xaX_n^2.$$

Sufficiency. The substitution $x_i \to a$ for $i = 1, 2, \ldots, n$ yields $axa^+ = a^+xaa^+$ which in S implies that $axa^+ = a^+xa$ and thus $S \in (\mathcal{AG})$ by Lemma 8.1. The substitution $x, a \to a^+$ yields $X_n \in E$ which by Lemma 11.1 gives $S \in (\mathcal{C}o\mathcal{C})$. ∎

Proposition 11.6.

$$\mathcal{S}(\mathcal{AG})(\mathcal{C}o\mathcal{C}) = \mathcal{SR} \cap \left[axX_n = a^+xaX_n^2\right]_{n \geq n} = \left[axX_n\, ya = ay\, X_n^2\, xa\right]_{n \geq 1}.$$

Proof: Denote these three classes by $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, respectively. Then $\mathcal{A} = \mathcal{B}$ by Theorem 4.1 and Lemmas 4.3 and 11.5.

For $S \in \mathcal{A}$, we obtain

$$\begin{aligned} a^+xX_nya^+ &= (a^+xa^+)X_n(a^+ya^+) \\ &= (a^+ya^+)X_n\,(a^+xa^+) && \text{by Theorem 8.2} \\ &= (a^+ya^+)X_n^2\,(a^+xa^+) && \text{by Corollary 11.4} \\ &= a^+y\,X_n^2\,xa^+. \end{aligned}$$

As in the proof of Lemma 11.5, the last identity implies that $S \in \mathcal{C}$. Therefore $\mathcal{A} \subseteq \mathcal{C}$.

Conversely, let $S \in \mathcal{C}$. Substituting $x_i \to a$ for $i = 1, 2, \ldots, n$, we get $axa^+ya = aya^+xa$. Multiplying on the left by $a(a^2)'$ and by $(a^2)'a$ on the right yields the identity $a^+xa^+\,CW\,a^+ya^+$. Now Theorem 8.2 implies that $S \in \mathcal{S}(\mathcal{AG})$. Substituting $x, y \to a'$ in the given identity yields the identity $X_n \in E$ so that, by Corollary 11.4, we have $S \in \mathcal{S}(\mathcal{C}o\mathcal{C})$. Therefore $\mathcal{C} \subseteq \mathcal{A}$. ∎

12 The lattice of sub-e-varieties of $\mathcal{SR}$

We discuss here the fragmentary information available on this lattice. First we deduce the following consequences of Theorems 7.2, 8.2, 9.2, 10.3 and Corollary 11.4.

Proposition 12.1. *Let S be a strict semigroup and $\mathcal{P} \in \{\mathcal{C}e, (\mathcal{AG}), \mathcal{C}o, (\mathcal{AC}), (\mathcal{C}o\mathcal{C})\}$. Then $S \in \mathcal{P}$ if and only if S is a subdirect product of completely (0-)simple semigroups in $\mathcal{P}$.*

About the joins in Diagram 1, we have the following sporadic information.

From [1, Theorem 4.6 (2)(3)] and Corollary 11.4, we have

$$\mathcal{CS} \vee \mathcal{SC}o = \mathcal{SR},\ \mathcal{CS} \vee \mathcal{SIC}o = \mathcal{S}(\mathcal{ES}),\ \mathcal{G} \vee \mathcal{SC}o = \mathcal{S}(\mathcal{C}o\mathcal{C}).$$

To this, we add the following statement.

Proposition 12.2. $\mathcal{AG} \vee \mathcal{SC}o = \mathcal{S}(\mathcal{AG})(\mathcal{C}o\mathcal{C})$.

Proof: Let $S \in \mathcal{S}(\mathcal{AG})(\mathcal{C}o\mathcal{C})$. By Theorem 11.3, we know that S is a subdirect product of a strict combinatorial semigroup A and a Clifford semigroup B. By Proposition 11.6, S satisfies the identity $axX_n = a^+xaX_n^2$ for all $n \geq 1$ and thus so does B being a homomorphic image of S. For the substitution $x_i \to a$ in this identity, we obtain $axa^+ = a^+xa$ which in a group implies commutativity. Hence $B \in \mathcal{SAG}$ and is thus a subdirect product of abelian groups with a zero possibly adjoined. It follows that $B \in \mathcal{AG} \vee \mathcal{S}$ and thus

$$S \in (\mathcal{AG} \vee \mathcal{S}) \vee \mathcal{SC}o = \mathcal{AC} \vee \mathcal{SC}o.$$

Therefore $\mathcal{S}(\mathcal{AG})(\mathcal{C}o\mathcal{C}) \subseteq \mathcal{AG} \vee \mathcal{SC}o$ and the opposite inclusion is obvious. ∎

The remainder of the joins in Diagram 1 will either follow from the results on sub-e-varieties of $\mathcal{SR}$ discussed below or will be left as open problems at the end of the section. As usual, the meets (intersections) are easier to handle so most of them in Diagram 1 present no challenge. Nevertheless, we prove the following simple statement.

Proposition 12.3. *$\mathcal{S}(\mathcal{ES})(\mathcal{C}o\mathcal{C}) = \mathcal{SO}$.*

Proof: Let $S = \mathcal{M}^0(I, G, \Lambda; P) \in (\mathcal{CS})(\mathcal{C}o\mathcal{C})$. Then $S \in (\mathcal{C}o\mathcal{C})$ by Lemma 11.1 and Result 11.2 imply that $S \cong C \times_0 G$ where $C \cong S/\mathcal{H}$. Since $S \in \mathcal{ES}$, we obtain that $C \in \mathcal{ES}$. But C is combinatorial and thus $C \in \mathcal{O}$. Hence $S \in \mathcal{O}$ as well. In view of Theorem 4.1 and Lemma 4.3, we conclude that $\mathcal{S}(\mathcal{ES})(\mathcal{C}o\mathcal{C}) \subseteq \mathcal{SO}$. The opposite inclusion is trivial. ■

Further meets will follow from the results on sub-e-varieties of $\mathcal{SR}$ or can be verified without difficulty.

We denote by $\mathcal{L}(\mathcal{V})$ the lattice of all sub-e-varieties of an e-variety $\mathcal{V}$. In order to facilitate our review, we introduce the following concept.

Definition 12.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be e-varieties. Then $(\mathcal{A}, \mathcal{B})$ is a* direct pair *if the mappings*

$$\mathcal{V} \longrightarrow (\mathcal{V} \cap \mathcal{A},\ \mathcal{V} \cap \mathcal{B}), \qquad (\mathcal{U}, \mathcal{W}) \longrightarrow \mathcal{U} \vee \mathcal{W}$$

are mutually inverse isomorphisms between $\mathcal{L}(\mathcal{A} \vee \mathcal{B})$ and $\mathcal{L}(\mathcal{A}) \times \mathcal{L}(\mathcal{B})$.

We now summarize the highlights of the existing knowledge about the lattices of sub-e-varieties of $\mathcal{SR}$.

By [1, Theorem 5.14 and Corollary 5.15], we have a direct pair $(\mathcal{C}e\mathcal{CS}, \mathcal{SIC}o)$ with $\mathcal{C}e\mathcal{CS} \vee \mathcal{SIC}o = \mathcal{C}e\mathcal{S}(\mathcal{ES})$.

In [4, p. 110] we have a diagram of $\mathcal{L}(\mathcal{SC}o)$ as the bottom of Diagram 2.

From [5, Theorem 4.4(ii)] we deduce that $(\mathcal{RB}, \mathcal{SI})$ is a direct pair with $\mathcal{RB} \vee \mathcal{SI} = \mathcal{SO}$.

Reference [10, Theorem 4.7] asserts that $(\mathcal{CS}, \mathcal{S})$ is a direct pair with $\mathcal{CS} \vee \mathcal{S} = \mathcal{NBG}$ within completely regular semigroups with the usual unary operation but the result remains valid for e-varieties.

By [12, Theorem XII.4.16], we deduce that $(\mathcal{G}, \mathcal{SIC}o)$ is a direct pair with $\mathcal{G} \vee \mathcal{SIC}o = \mathcal{SI}$ in the same way as in the preceding case.

The reference [13, Theorem 3.11] describes $\mathcal{L}(\mathcal{C}e\mathcal{CS})$ in terms of $\mathcal{L}(\mathcal{RB}), \mathcal{L}(\mathcal{AG})$ and $\mathcal{L}(\mathcal{G})$.

The above list, with a certain dose of wishful thinking, seems to be leading to a complete determination of the lattice of sub-e-varieties of $\mathcal{SR}$, possibly in the form of a direct pair. This impression is quickly dispelled by the example in [1, Theorem 5.10] which implies that $(\mathcal{CS}, \mathcal{SIC}o)$ is not a direct pair and has further negative consequences [1, Corollaries 5.11 and 5.12]. We deduce that the lattice $\mathcal{L}(\mathcal{SR})$ is more complex than one would expect in view of the structure of $\mathcal{L}(\mathcal{V})$ for the above e-varieties $\mathcal{V}$.

In the definition of a direct pair, we can more generally substitute $\mathcal{L}(\mathcal{A} \vee \mathcal{B})$ by the interval $[\mathcal{A} \cap \mathcal{B}, \mathcal{A} \vee \mathcal{B}]$. Also note that there are further examples of direct pairs of (e-)varieties in the literature. It is of interest here that the classes of overabelian and of combinatorial regular semigroups are not closed under homomorphic images as seen on the example of a free inverse semigroup on a set which is combinatorial, see [12, Proposition VII.1.14]. Therefore these two classes do not form e-varieties. Neretheless, Proposition 12.1 is valid for them.

Our discussion suggests a variety of problems a sample of which follows.

1. Are $\mathcal{C}e, (\mathcal{AC}), (\mathcal{C}o\mathcal{C})$ e-varieties? If some of them are, find bases for them.

2. Note that if a strict semigroup S has a subsemigroup which intersects each $\mathcal{H}$-class of S exactly once, then $C(S)$ is combinatorial. Is the converse true?

3. Are the following joins correct

$$\mathcal{CS}(\mathcal{AC}) \vee \mathcal{SIC}o = \mathcal{S}(\mathcal{ES})(\mathcal{AC}),$$

$$\mathcal{CS}(\mathcal{AC}) \vee \mathcal{SC}o = \mathcal{S}(\mathcal{AC}),$$

$$\mathcal{C}e\,\mathcal{CS} \vee \mathcal{SC}o = \mathcal{C}e\mathcal{S}?$$

4. Determine the lattice $\mathcal{L}(\mathcal{SR})$ of sub-e-varieties of the e-variety of inverse semigroups. One possible approach is through fully invariant congruences on a bifree object.

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