On fibrations with Grassmannian fibers

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To Peter D. Zvengrowski on his sixtieth birthday

1 Introduction

Let F be either the field $\mathbb{R}$ of reals or the field $\mathbb{C}$ of complex numbers, and let $FG_{n,k}$ be the Grassmann manifold of all k-dimensional vector subspaces in F^n. In particular, $\mathbb{R}G_{n,1} = \mathbb{R}P^{n-1}$, real projective space of dimension $n-1$, and $\mathbb{C}G_{n,1} = \mathbb{C}P^{n-1}$, complex projective space of complex dimension $n-1$. Since one can identify $FG_{n,k}$ with $FG_{n,n-k}$, we may suppose that $2k \leq n$ in the sequel.

There are situations, mainly in topology or geometry, where one needs to obtain cohomological information on the total space of a fibration (Hurewicz or Serre, a locally trivial fibration, or a fiber bundle) if the cohomological data of its fiber are given. Of course, in general, when the fiber is "wild enough", one hardly can find anything reasonable about the cohomology of the total space. But in spite of that, there are certain results for some types of fibers. For instance, J. C. Becker and D. H. Gottlieb in [3, Corollary 9] proved that for any Hurewicz fibration $p : E \to B$ with fiber $\mathbb{R}P^{2n}$ (with E path connected, locally path connected, and semi-locally 1-connected), the fiber inclusion $i : \mathbb{R}P^{2n} \to E$ induces an epimorphism, $i^* : H^*(E;\mathbb{Z}_2) \to H^*(\mathbb{R}P^{2n};\mathbb{Z}_2)$, in $\mathbb{Z}_2$-cohomology. In other words, here the fiber $\mathbb{R}P^{2n}$ is totally non-homologous to zero in E with respect to $\mathbb{Z}_2$; as is well known, then the Leray-Hirsch theorem applies, and one has that

$$H^*(E;\mathbb{Z}_2) \cong H^*(\mathbb{R}P^{2n};\mathbb{Z}_2) \otimes H^*(B;\mathbb{Z}_2)$$

as $\mathbb{Z}_2$-vector spaces.

For smooth fiber bundles, we proved the following more general result in [14].

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Theorem A. *([14]). Let $p : E \to B$ be a smooth fiber bundle with E a closed connected manifold and with fiber the Grassmann manifold $\mathbb{R}G_{n,k}$ $(2 \le 2k \le n)$. If n is odd, then the fiber $\mathbb{R}G_{n,k}$ is totally non-homologous to zero in E with respect to $\mathbb{Z}_2$.*

The present paper is a kind of addendum to this theorem.

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2 On fibrations with fiber $\mathbb{C}G_{n,k}$ or with fiber $\mathbb{R}G_{n,2}$

Suppose that $p : E \to B$ is a locally trivial fibration (not necessarily smooth) with fiber F such that E and F are compact (we understand that this includes T_2) and path connected, and B is locally path connected. It is known (see e.g. A. Borel [7]) that if the fundamental group $\pi_1(B)$ acts trivially on $H^*(F;\mathbb{Z})$ (in other words, if the fibration $p : E \to B$ is orientable over $\mathbb{Z}$), if $H^+(F;\mathbb{Z}) = \{x \in H^*(F;\mathbb{Z}); \deg(x) > 0\}$ is generated by its elements of minimal positive degree, and if $H^*(F;\mathbb{Z})$ is torsion-free, then the fiber F is totally non-homologous to zero in E with respect to $\mathbb{Z}$. In particular, this conclusion is true for locally trivial fibrations as above if the fiber is $\mathbb{C}P^n$. Indeed, recall that $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$, where $x = c_1(\gamma) \in H^2(\mathbb{C}P^n;\mathbb{Z})$ is the first Chern class of the canonical complex line bundle over $\mathbb{C}P^n$.

More generally, one can consider Serre fibrations with fiber a homogeneous Kähler manifold (note that the complex flag manifolds $U(n_1+\cdots+n_s)/U(n_1)\times\cdots\times U(n_s)$, in particular the complex Grassmannians $\mathbb{C}G_{n,k} = U(n)/U(k) \times U(n-k)$, are homogeneous kählerian). Fibrations of this type (also with still more general fibers) were studied e.g. by A. Blanchard [4] (equivalently [5]), or by W. Meier [16], [17]: roughly speaking, the real or rational cohomology algebra of such a homogeneous kählerian fiber turns out to have no nontrivial derivations of negative degrees, therefore the Leray-Serre spectral sequence collapses at its E_2-term, and the fiber is totally non-homologous to zero with respect to $\mathbb{R}$ or $\mathbb{Q}$ if the fibration is orientable over $\mathbb{R}$ or $\mathbb{Q}$, correspondingly.

Similarly, H. Shiga and M. Tezuka in [21] study fibrations with special type of fibers for which they also are able to show that they have only trivial $\mathbb{Q}$-derivations in negative degrees. More precisely, they study (Serre) fibrations $p : E \to B$ (under suitable hypotheses on E and B) with fiber G/U, where G is a compact connected Lie group and U is a closed connected subgroup having the same rank as G. The result is that if the fibration p is $\mathbb{Q}$-orientable, then its fiber G/U is totally non-homologous to zero in E with respect to the field $\mathbb{Q}$. Another theorem of the same paper states that the result is also valid with coefficients $\mathbb{Z}_q$ if q is a prime number not dividing the order of the Weyl group $W(G)$. Note that here also one may in particular take $\mathbb{C}G_{n,k}$, or more generally $U(n_1+\cdots+n_s)/U(n_1)\times\cdots\times U(n_s)$, in the rôle of G/U.

Shiga and Tezuka remark at the end of their paper [21, p. 105] that the condition that the prime q should not divide the order of the Weyl group $W(G)$ need not be best possible. Here is an indication (not mentioned by them): Gottlieb proved in [12, Corollary 7] that if $p : E \to B$ is a $\mathbb{Z}_q$-orientable Hurewicz fibration with fiber $\mathbb{C}P^n$ such that $n+1 \not\equiv 0 \pmod{q}$, then the fiber $\mathbb{C}P^n$ is totally non-homologous to zero in E with respect to $\mathbb{Z}_q$. Observe that the order of the Weyl group $W(U(n))$ is $n!$ (see e.g. Husemoller [15]), hence the condition in the Shiga-Tezuka theorem [21, Theorem C] is more restrictive, namely $(n+1)! \not\equiv 0 \pmod{q}$.

We now show that for $q = 2$ and smooth fiber bundles, Gottlieb's result can be strengthened to the first part of the following theorem; its second part will provide another indication that the above mentioned divisibility condition of Shiga and Tezuka might be too strong.

Theorem B. *(1) Let $p : E \to B$ be a smooth fiber bundle with E a closed connected manifold and with fiber the Grassmann manifold $\mathbb{C}G_{n,k}$ $(2 \le 2k \le n)$. If n is odd, then the fiber $\mathbb{C}G_{n,k}$ is totally non-homologous to zero in E with respect to $\mathbb{Z}_2$.*

(2) Let $p : E \to B$ be a Serre fibration with E compact, with B a connected finite CW-complex, and with fiber either the complex Grassmannian $\mathbb{C}G_{2^s+1,2}$ $(s \ge 2)$ or the real Grassmannian $\mathbb{R}G_{2^s+1,2}$ $(s \ge 2)$. Then the fiber is totally non-homologous to zero in E with respect to $\mathbb{Z}_2$.

Remarks (a) Note that, as compared to the result of Gottlieb cited above, we do not have any orientability assumption in Theorem B(1). On the other hand, an obvious corollary is that every smooth fiber bundle considered in Theorem B(1) is $\mathbb{Z}_2$-orientable if n is odd. But in fact, for $q = 2$, the $\mathbb{Z}_2$-orientability assumption can be dropped from Gottlieb's result, too. Indeed, since $H^1(\mathbb{C}P^n; \mathbb{Z}_2) = 0$ and $H^2(\mathbb{C}P^n; \mathbb{Z}_2)$ has only one nonzero element, a, any degree preserving automorphism of the algebra $H^*(\mathbb{C}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[a]/(a^{n+1})$ maps a to a, and is the identity. Therefore the action of the fundamental group of the base space on $H^*(\mathbb{C}P^n; \mathbb{Z}_2)$ is trivial.

(b) The proof of Theorem B(1) will be geometrical in flavour, based on properties of tangent bundles and their characteristic classes. The proof of Theorem B(2) will be more cohomological in spirit, following the same line which was roughly outlined above when speaking about the papers by Blanchard and Meier. Note that the assumptions of Theorem B(2) can be relaxed. But if one wishes to state it under the weakest restrictions on E and B, then one should take care to distinguish whether one uses singular cohomology or cohomology with compact supports; in the latter case, B is necessarily compact (see Borel [6, §4(c)]).

I gratefully acknowledge that Theorem B(2) with its proof is a modification and generalization of that what was suggested by the referee in case of fibrations with fiber $\mathbb{C}G_{5,2}$. After having seen an earlier version of Theorem B(2), P. Sankaran conjectured that the (originally present) assumption of $\mathbb{Z}_2$-orientability of the fibration $p : E \to B$ might be deleted. This turned out to be true, as shown by Proposition 3.

(c) Theorem B(1) has applications similar to those of Theorem A. For instance, when one is interested in the question of whether a given manifold can be the total space of a smooth fiber bundle with fiber $\mathbb{R}G_{n,k}$ or $\mathbb{C}G_{n,k}$ with n odd, then one can use Theorem A or Theorem B(1) for obtaining some necessary conditions (see [14] for examples of such conditions). Theorem B(2) can be applied in an analogous way.

Note that for instance R. J. D. Ferdinands, W. Meier, and R. E. Schultz ([10], [11], [18], [20]) have studied a different but related question: When can a Grassmann manifold be the total space of a fibration?

(d) In contrast to Theorem B(1), there are many smooth fiber bundles with fiber $\mathbb{C}G_{n,k}$ (also with n odd) not totally non-homologous to zero with respect to $\mathbb{Z}$ or $\mathbb{Z}_q$, $q \neq 2$; more precisely, there are many such fiber bundles *non-orientable* over $\mathbb{Z}$ or $\mathbb{Z}_q$, $q \neq 2$. An example of this type will be presented in Remark (f) in Section 3. Hence, modifying Theorem B(1) by just changing the coefficients from $\mathbb{Z}_2$ to $\mathbb{Z}$ or $\mathbb{Z}_q$, $q \neq 2$, would not lead to a correct result. On the other hand, by the result of Shiga and Tezuka [21, Theorem C], the situation is different if we *assume orientability*: if the prime number q does not divide $(n+1)!$, then for any $\mathbb{Z}_q$-*orientable* Serre fibration (in particular, $\mathbb{Z}_q$-orientable fiber bundle) with fiber $\mathbb{C}G_{n,k}$ the fiber is totally non-homologous to zero with respect to $\mathbb{Z}_q$. As already remarked, the divisibility condition can probably be weakened.

After some preparations, Theorem B(1) will be proved essentially along the same lines as Theorem A in [14].

2.1 Preparations for the proof of Theorem B

As is known (see e.g. Borel [6]), there is an isomorphism

$$\varphi : H^*(\mathbb{R}G_{n,k}; \mathbb{Z}_2) \to H^*(\mathbb{C}G_{n,k}; \mathbb{Z}_2)$$

of the cohomology algebras which is doubling the degrees. More precisely, the algebra $H^*(\mathbb{R}G_{n,k}; \mathbb{Z}_2)$ is generated by the Stiefel-Whitney characteristic classes $w_i(\xi_k) \in H^i(\mathbb{R}G_{n,k}; \mathbb{Z}_2)$ $(i = 1, \ldots, k)$ of the canonical k-plane bundle over $\mathbb{R}G_{n,k}$, the ring $H^*(\mathbb{C}G_{n,k}; \mathbb{Z})$ is generated (see e.g. A. Dold [9]) by the Chern classes $c_i(\gamma_k) \in H^{2i}(\mathbb{C}G_{n,k}; \mathbb{Z})$ $(i = 1, \ldots, k)$ of the canonical complex k-plane bundle over $\mathbb{C}G_{n,k}$, and the isomorphism φ maps $w_i(\xi_k) \in H^i(\mathbb{R}G_{n,k}; \mathbb{Z}_2)$ to the mod 2 reduction of $c_i(\gamma_k)$, hence to the Stiefel-Whitney class $w_{2i}(\mathrm{r}(\gamma_k)) \in H^{2i}(\mathbb{C}G_{n,k}; \mathbb{Z}_2)$. Here $\mathrm{r}(\gamma_k)$ means the realification of γ_k. We shall just write $w_{2i}(\gamma_k)$ instead of $w_{2i}(\mathrm{r}(\gamma_k))$ in the sequel.

For the proof of Theorem B, we shall need the following.

Lemma. *If n is odd and $2k \leq n$, then the cohomology algebra $H^*(\mathbb{C}G_{n,k}; \mathbb{Z}_2)$ is generated by the Stiefel-Whitney classes $w_{2i}(\mathbb{C}G_{n,k})$ $(i = 1, \ldots, k)$ of the realification of the complex tangent bundle of $\mathbb{C}G_{n,k}$.*

Proof. We have $2k \leq n$ so that (see Dold [9]) the Chern classes $c_1(\gamma_k), \ldots, c_k(\gamma_k)$ in $H^*(\mathbb{C}G_{n,k}; \mathbb{Z})$ are algebraically independent in dimensions $\leq 2k$. Similarly, the Stiefel-Whitney classes $w_1(\xi_k), \ldots, w_k(\xi_k)$ in $H^*(\mathbb{R}G_{n,k}; \mathbb{Z}_2)$ satisfy no algebraic relations in dimensions $\leq k$, and then also their φ-images $w_2(\gamma_k), \ldots, w_{2k}(\gamma_k)$, are algebraically independent in dimensions $\leq 2k$.

Now let γ_k^* be the conjugate bundle of the canonical vector bundle γ_k over $\mathbb{C}G_{n,k}$. Formally (as in A. Borel and F. Hirzebruch [8]) factorize the total Chern class

$$c(\gamma_k) = 1 + c_1(\gamma_k) + \cdots + c_k(\gamma_k) = \prod_{i=1}^{k}(1 + x_i),$$

hence $c_j(\gamma_k)$ is the j-th elementary symmetric function $\sigma_j(x_1,\dots,x_k)$ in the variables $x_1,\dots,x_k$. Then

$$c(\gamma_k^*) = \prod_{i=1}^{k}(1-x_i),$$

and by [8, p. 522] we have

$$c(\mathbb{C}G_{n,k}) \cdot \prod_{1\le i<j\le k}(1-(x_i-x_j)^2) = \prod_{i=1}^{k}(1-x_i)^n. \qquad (*)$$

Therefore

$$w(\mathbb{C}G_{n,k}) \cdot (\prod_{1\le i<j\le k}(1+x_i+x_j))^2 = (\prod_{i=1}^{k}(1+x_i))^n \pmod 2.$$

Since

$$\prod_{i=1}^{k}(1+x_i) = 1 + w_2(\gamma_k) + \cdots + w_{2k}(\gamma_k) \pmod 2,$$

we obtain

$$w(\mathbb{C}G_{n,k}) \cdot (\prod_{1\le i<j\le k}(1+x_i+x_j))^2 = (1 + w_2(\gamma_k) + \cdots + w_{2k}(\gamma_k))^n \pmod 2. \quad (**)$$

Now $\prod_{1\le i<j\le k}(1+x_i+x_j) \pmod 2$, being symmetric in $x_1,\dots,x_k$, is of the form

$$1 + \sum_{i\ge 1} P_{2i}(w_2(\gamma_k),\dots,w_{2k}(\gamma_k)),$$

where P_{2i} are $\mathbb{Z}_2$-polynomials, $P_{2i}(w_2(\gamma_k),\dots,w_{2k}(\gamma_k)) \in H^{2i}(\mathbb{C}G_{n,k};\mathbb{Z}_2)$. So $(**)$ implies

$$w(\mathbb{C}G_{n,k}) \cdot (1 + \sum_{i\ge 1} P_{2i}(w_2(\gamma_k),\dots,w_{2k}(\gamma_k)))^2 = (1 + w_2(\gamma_k) + \cdots + w_{2k}(\gamma_k))^n,$$

or, in other words,

$$w(\mathbb{C}G_{n,k}) = (1 + w_2(\gamma_k) + \cdots + w_{2k}(\gamma_k))^n \cdot (1 + \sum_i P_{2i}^2 + \sum_i P_{2i}^4 + \dots).$$

From this (recall that the Stiefel-Whitney classes $w_2(\gamma_k),\dots,w_{2k}(\gamma_k)$ satisfy no algebraic relations in dimensions $\le 2k$) then

$$w_{2i}(\mathbb{C}G_{n,k}) = n w_{2i}(\gamma_k) + \text{terms without } w_{2i}(\gamma_k)$$

for $i = 1,\dots,k$. Hence if n is odd, then

$$w_{2i}(\mathbb{C}G_{n,k}) = w_{2i}(\gamma_k) + \text{terms without } w_{2i}(\gamma_k)$$

for $i = 1,\dots,k$. Since the classes $w_{2i}(\gamma_k)$ $(i = 1,\dots,k)$ generate the algebra $H^*(\mathbb{C}G_{n,k};\mathbb{Z}_2)$, an obvious induction (similar to that used in [14]) shows that the classes $w_2(\mathbb{C}G_{n,k}),\dots,w_{2k}(\mathbb{C}G_{n,k})$ also generate $H^*(\mathbb{C}G_{n,k};\mathbb{Z}_2)$ if n is odd. This closes the proof of the lemma. ∎

Having the lemma, we are now able to prove Theorem B(1).

2.2 Proof of Theorem B(1)

Now, denoting by TM the tangent bundle of a manifold M, we have

$$TE \cong p^*(TB) \oplus \kappa,$$

where κ is the vector bundle along the fibers, and

$$i^*(TE) \cong \varepsilon^{\dim(B)} \oplus T\mathbb{C}G_{n,k},$$

where $T\mathbb{C}G_{n,k}$ is the realification of the complex tangent bundle of $\mathbb{C}G_{n,k}$. Here $i : \mathbb{C}G_{n,k} \to E$ is the fiber inclusion and ε^t is the trivial t-plane bundle.

Hence

$$i^*(w_{2j}(TE)) = w_{2j}(\mathbb{C}G_{n,k})$$

for $j = 1, \dots, k$. Since by the lemma $w_2(\mathbb{C}G_{n,k}), \dots, w_{2k}(\mathbb{C}G_{n,k})$ generate the algebra $H^*(\mathbb{C}G_{n,k}; \mathbb{Z}_2)$ if n is odd, one sees that $i^* : H^*(E; \mathbb{Z}_2) \to H^*(\mathbb{C}G_{n,k}; \mathbb{Z}_2)$ is an epimorphism in this case. This closes the proof of Theorem B(1).

Once again we shall first derive some preparatory results, and then pass to the proof of Theorem B(2).

2.3 Preparations for the proof of Theorem B(2)

As we already have recalled, the cohomology algebra $H^*(\mathbb{R}G_{n,2}; \mathbb{Z}_2)$ is generated by the Stiefel-Whitney classes $w_1(\xi_2), w_2(\xi_2)$. But now we need the following detailed description (see e.g. H. Hiller [13, Theorem 1]). One has

$$H^*(\mathbb{R}G_{n,2}; \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1(\xi_2), w_2(\xi_2)]/J(2, n-2),$$

where the ideal $J(2, n-2)$ is generated by the homogeneous relations $f_{1,n-2}, f_{2,n-2}$ given by

$$\begin{pmatrix} f_{1,n-2} \\ f_{2,n-2} \end{pmatrix} = \begin{pmatrix} w_1(\xi_2) & 1 \\ w_2(\xi_2) & 0 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Using this, we prove the following result on cup products.

Proposition 1. *In the cohomology algebra $H^*(\mathbb{R}G_{2^s+1,2}; \mathbb{Z}_2)$ $(s \geq 2)$ one has that $w_1^{2^{s+1}-5}(\xi_2) w_2(\xi_2) = 0 \in H^{d-1}(\mathbb{R}G_{2^s+1,2}; \mathbb{Z}_2) \cong \mathbb{Z}_2$, where $d = \dim(\mathbb{R}G_{2^s+1,2}) = 2^{s+1} - 2$. Consequently, in the cohomology algebra $H^*(\mathbb{C}G_{2^s+1,2}; \mathbb{Z}_2)$ $(s \geq 2)$ one has that $w_2^{2^{s+1}-5}(\gamma_2) w_4(\gamma_2) = 0$.*

Proof. In view of the isomorphism

$$\varphi : H^*(\mathbb{R}G_{n,k}; \mathbb{Z}_2) \to H^*(\mathbb{C}G_{n,k}; \mathbb{Z}_2),$$

it is enough to verify the claim for $H^*(\mathbb{R}G_{2^s+1,2}; \mathbb{Z}_2)$. More precisely, we assert that already $w_1^{2^s-1}(\xi_2) w_2(\xi_2)$ vanishes (which then clearly implies the claim of the proposition since $s \geq 2$).

To prove this, observe that the relation $f_{2,2^s-1}$ in $H^*(\mathbb{R}G_{2^s+1,2};\mathbb{Z}_2)$ is the element of the second row and the first column in the matrix

$$\begin{pmatrix} w_1(\xi_2) & 1 \\ w_2(\xi_2) & 0 \end{pmatrix}^{2^s}.$$

Then the following fact enables us to identify $w_1^{2^s-1}(\xi_2)w_2(\xi_2)$ with the relation $f_{2,2^s-1}$ in the algebra $H^*(\mathbb{R}G_{2^s+1,2};\mathbb{Z}_2)$.

Fact. *Abbreviate $a = w_1(\xi_2)$, $b = w_2(\xi_2)$. Then for any positive integer s we have*

$$\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}^{2^s} = \begin{pmatrix} a^{2^s} + x(s) & a^{2^s-1} \\ a^{2^s-1}b & x(s) \end{pmatrix}$$

for some element $x(s) \in H^*(\mathbb{R}G_{2^s+1,2};\mathbb{Z}_2)$.

This can readily be verified by induction on s, and the proof of Proposition 1 is complete. ∎

Recall (see e.g. Meier [17]) that a (graded) derivation of degree s ($s \in \mathbb{Z}$) in a commutative graded $\mathbb{Z}_2$-algebra A is a linear map θ of degree s such that $\theta(a \cdot b) = \theta(a) \cdot b + a \cdot \theta(b)$, where a, b are homogeneous elements of the algebra A. For any given n, let $\mathrm{Der}_{<n}(A)$ denote the graded $\mathbb{Z}_2$-vector space of all derivations in A of degree smaller than n.

Recall further that the height of the Stiefel-Whitney class $w_1(\xi_k)$ in the algebra $H^*(\mathbb{R}G_{n,k};\mathbb{Z}_2)$ ($2k \leq n$) is

$$\mathrm{height}(w_1(\xi_k)) := \sup\{m; w_1(\xi_k)^m \neq 0\},$$

and for $2^s < n \leq 2^{s+1}$ one has (see Stong [22])

$$\mathrm{height}(w_1(\xi_k)) = \begin{cases} n-1 & \text{if } k = 1, \\ 2^{s+1} - 2 & \text{if } k = 2 \text{ or if } k = 3 \text{ and } n = 2^s + 1, \\ 2^{s+1} - 1 & \text{otherwise.} \end{cases}$$

For $H^*(\mathbb{C}G_{n,k};\mathbb{Z}_2)$ ($2k \leq n$), we then analogously define $\mathrm{height}(w_2(\gamma_k))$, and due to the isomorphism

$$\varphi : H^*(\mathbb{R}G_{n,k};\mathbb{Z}_2) \to H^*(\mathbb{C}G_{n,k};\mathbb{Z}_2),$$

the value of $\mathrm{height}(w_2(\gamma_k))$ in $H^*(\mathbb{C}G_{n,k};\mathbb{Z}_2)$ is the same as the above cited value of $\mathrm{height}(w_1(\xi_k))$ in $H^*(\mathbb{R}G_{n,k};\mathbb{Z}_2)$.

Now we shall need the following.

Proposition 2. *For any $s \geq 2$, one has $\mathrm{Der}_{<0}(H^*(\mathbb{C}G_{2^s+1,2};\mathbb{Z}_2)) = 0$ and also $\mathrm{Der}_{<0}(H^*(\mathbb{R}G_{2^s+1,2};\mathbb{Z}_2)) = 0$.*

Proof. We know that the algebra $H^*(\mathbb{C}G_{2^s+1,2};\mathbb{Z}_2)$ is generated by the Stiefel-Whitney classes $w_2(\gamma_2)$ and $w_4(\gamma_2)$. Hence to show that a derivation is trivial, it is enough to verify that it vanishes on $w_2(\gamma_2)$ and $w_4(\gamma_2)$.

It is clear that the $\mathbb{Z}_2$-cohomology of $\mathbb{C}G_{n,k}$ vanishes in odd degrees. Therefore any of those elements in $\mathrm{Der}_{<0}(H^*(\mathbb{C}G_{2^s+1,2};\mathbb{Z}_2))$ which are nontrivially evaluated on $w_2(\gamma_2)$ must be of degree -2. Let us suppose that there exists a derivation θ with $\theta(w_2(\gamma_2)) = 1 \in H^0(\mathbb{C}G_{2^s+1,2};\mathbb{Z}_2)) = \mathbb{Z}_2$. Then of course $w_2^{2^{s+1}-1}(\gamma_2) = 0$, and therefore $0 = \theta(w_2^{2^{s+1}-1}(\gamma_2)) = (2^{s+1}-1)w_2^{2^{s+1}-2}(\gamma_2)\theta(w_2(\gamma_2)) = w_2^{2^{s+1}-2}(\gamma_2)$. But this is a contradiction, because $w_2^{2^{s+1}-2}(\gamma_2) \neq 0$ by the above mentioned height result. This means that each negative-degree derivation vanishes on the class $w_2(\gamma_2)$.

Further, any of those elements in $\mathrm{Der}_{<0}(H^*(\mathbb{C}G_{2^s+1,2};\mathbb{Z}_2))$ which are non-trivially evaluated on $w_4(\gamma_2)$ must be of degree -2 or of degree -4. Hence if θ is a non-vanishing derivation of degree -2, we must have $\theta(w_4(\gamma_2)) = w_2(\gamma_2)$, because the cohomology group $H^2(\mathbb{C}G_{2^s+1,2};\mathbb{Z}_2)$ has just two elements, its generator being $w_2(\gamma_2)$. Similarly, if $\tilde{\theta}$ is a non-vanishing derivation of degree -4, then one has $\tilde{\theta}(w_4(\gamma_2)) = 1 \in H^0(\mathbb{C}G_{2^s+1,2};\mathbb{Z}_2)$. In either case we come to a contradiction.

Indeed, we know by Proposition 1 that $w_2^{2^{s+1}-5}(\gamma_2)w_4(\gamma_2) = 0$. Hence, since both θ and $\tilde{\theta}$ vanish on $w_2(\gamma_2)$, we have $0 = \theta(w_2^{2^{s+1}-5}(\gamma_2)w_4(\gamma_2)) = w_2^{2^{s+1}-4}(\gamma_2)$, and $0 = \tilde{\theta}(w_2^{2^{s+1}-5}(\gamma_2)w_4(\gamma_2)) = w_2^{2^{s+1}-5}(\gamma_2)$. This contradicts the fact that the height of $w_2(\gamma_2)$ is $2^{s+1}-2$.

For the real Grassmannians, the proof is an obvious modification of the proof given above for the complex case. The proof of Proposition 2 is complete. ∎

Solving the $\mathbb{Z}_2$-orientability question for certain fibrations, A. H. Back in [1] proved that if n is even and not congruent to 64 mod 192, then $H^*(G_{n+1,2};\mathbb{Z}_2)$ admits no nontrivial degree preserving automorphisms. To show that the fibrations considered in Theorem B(2) are $\mathbb{Z}_2$-orientable, we shall prove the following.

Proposition 3. *Let $p : E \to B$ be a Serre fibration with B path connected and with one of the following spaces as fiber:*

(i) $\mathbb{R}G_{n,2}$ $(n \geq 5)$;

(ii) $\mathbb{C}G_{n,2}$ $(n \geq 5)$.

Then the fibration is $\mathbb{Z}_2$-orientable.

Proof. Those automorphisms of $H^*(\mathbb{R}G_{n,2};\mathbb{Z}_2)$ or of $H^*(\mathbb{C}G_{n,2};\mathbb{Z}_2)$ which correspond to the action of the fundamental group of the base space are induced by continuous maps. Hence all such automorphisms must commute with the Steenrod squares. To prove the claim, it is then enough to prove that the only degree preserving $\mathbb{Z}_2$-algebra automorphism of $H^*(\mathbb{R}G_{n,2};\mathbb{Z}_2)$ or of $H^*(\mathbb{C}G_{n,2};\mathbb{Z}_2)$ $(n \geq 5$ in both cases) that commutes with the Steenrod squares is the identity.

Consider the real case. Suppose that f^* is an automorphism of $H^*(\mathbb{R}G_{n,2};\mathbb{Z}_2)$ $(n \geq 5)$ commuting with the Steenrod squares. Since in $H^1(\mathbb{R}G_{n,2};\mathbb{Z}_2)$ the only nonzero element is $w_1(\xi_2)$, we must have $f^*(w_1(\xi_2)) = w_1(\xi_2)$. Now $H^2(\mathbb{R}G_{n,2};\mathbb{Z}_2)$ has three nonzero elements: $w_1^2(\xi_2), w_2(\xi_2)$, and $w_1^2(\xi_2) + w_2(\xi_2)$. We show that $f^*(w_2(\xi_2)) = w_2(\xi_2)$, which then yields the result.

Indeed, first suppose that $f^*(w_2(\xi_2)) = w_1^2(\xi_2)$. Then, using the Wu formula, we obtain that

$$Sq^1(f^*(w_2(\xi_2))) = 0 = f^*(Sq^1(w_2(\xi_2))) = f^*(w_1(\xi_2)w_2(\xi_2)) = w_1^3(\xi_2).$$

But this is a contradiction: note that the classes $w_1(\xi_2), w_2(\xi_2)$ satisfy no algebraic relations in dimension 3 since $n-2 \geq 3$. Similarly, supposing that $f^*(w_2(\xi_2)) = w_1^2(\xi_2) + w_2(\xi_2)$, we derive that

$$Sq^1(f^*(w_2(\xi_2))) = w_1(\xi_2)w_2(\xi_2) = f^*(Sq^1(w_2(\xi_2))) = w_1^3(\xi_2) + w_1(\xi_2)w_2(\xi_2),$$

hence again $w_1^3(\xi_2) = 0$, which is a contradiction.

In the complex case, one proceeds analogously, replacing $w_1(\xi_2)$ with $w_2(\gamma_2)$, $w_2(\xi_2)$ with $w_4(\gamma_2)$, and using the Steenrod square Sq^2 instead of Sq^1.

This completes the proof of Proposition 3. ∎

2.4 Proof of Theorem B(2)

It is known (it is enough to change $\mathbb{Q}$ to $\mathbb{Z}_2$ in Meier [16, Lemma 2.5], or cf. Shiga and Tezuka [21, Lemma 6.1]) that if $p : E \to B$ is a $\mathbb{Z}_2$-orientable Serre fibration with fiber F (under suitable hypotheses on the spaces E, B, F, which are all satisfied in our case) such that $\mathrm{Der}_{<0}(H^*(F;\mathbb{Z}_2)) = 0$, then the Leray-Serre spectral sequence collapses, and the fiber is therefore totally non-homologous to zero with respect to $\mathbb{Z}_2$. In view of this, Theorem B(2) now follows from Propositions 2 and 3.

3 Comments and observations

In [14, Remark 3] we showed, using an example due to Robert E. Stong, that Theorem A cannot be extended to cover also all smooth fiber bundles having fiber $\mathbb{R}G_{n,k}$ with n even and k odd ($(n,k) \neq (2,1)$). Here we present another example, which we also owe to Stong, showing that Theorem A cannot be extended to cover all smooth fiber bundles with fiber $\mathbb{R}G_{n,k}$ with n and k both even.

Example Let $k \geq 2$ and $T : \mathbb{R}G_{2k,k} \to \mathbb{R}G_{2k,k}$ be the involution sending $D \in \mathbb{R}G_{2k,k}$ to its orthogonal complement in $\mathbb{R}^{2k}$. Let $a : S^m \to S^m$, $a(x) = -x$, be the antipodal involution on the m-dimensional sphere; we shall suppose that $m \geq 1$. Then the map

$$p : \frac{\mathbb{R}G_{2k,k} \times S^m}{T \times a} \longrightarrow \frac{S^m}{a} (= \mathbb{R}P^m),$$
$$p([D,x]) = [x],$$

defines a smooth fiber bundle with fiber $\mathbb{R}G_{2k,k}$. About this one can prove that the fiber $\mathbb{R}G_{2k,k}$ is not totally non-homologous to zero in E with respect to $\mathbb{Z}_2$. This can be done for instance in the following way.

For any $s \in S^m$, the map

$$i_s : \mathbb{R}G_{2k,k} \to \frac{\mathbb{R}G_{2k,k} \times S^m}{T \times a},$$

$$i_s(D) = [D,s],$$

is the inclusion of the fiber over $[s] \in \mathbb{R}P^m$.

Now for any $D \in \mathbb{R}G_{2k,k}$ one has

$$i_s \circ T(D) = [T(D), s] = [D, a(s)] = i_{a(s)}(D),$$

hence $i_s \circ T = i_{a(s)}$. Since the sphere S^m is path connected (and therefore i_s and $i_{a(s)}$ are homotopic), one has $i_s^* = i_{a(s)}^* := i^*$ for the induced homomorphisms in cohomology. Hence $T^* \circ i^* = i^*$ which means that the image of

$$i^* : H^*(\mathbb{R}G_{2k,k} \times S^m/T \times a; \mathbb{Z}_2) \longrightarrow H^*(\mathbb{R}G_{2k,k}; \mathbb{Z}_2)$$

must be contained in the set of elements which are invariant under T^*.

On the other hand, the pullback $T^*(\xi)$ of the canonical k-plane bundle over $\mathbb{R}G_{2k,k}$ is the complementary k-plane bundle $\xi^\perp$. Since $\xi \oplus \xi^\perp$ is the trivial $2k$-plane bundle, one has $w(\xi)w(\xi^\perp) = 1$ for the total Stiefel-Whitney classes, and then

$$T^*(w_2(\xi)) = w_2(\xi^\perp) = w_1^2(\xi) + w_2(\xi) \neq w_2(\xi).$$

The latter is true, because $k \geq 2$, and there are no relations among the classes $w_1(\xi), \ldots, w_k(\xi)$ in the $\mathbb{Z}_2$-cohomology of $\mathbb{R}G_{2k,k}$ up to dimension k.

Hence $w_2(\xi) \notin \mathrm{Im}(i^*)$ which means that i^* is not surjective. Therefore the fiber $\mathbb{R}G_{2k,k}$ is not totally non-homologous to zero in the total space $\mathbb{R}G_{2k,k} \times S^m/T \times a$ as claimed.

Gathering the present knowledge, the following conjecture seems plausible.

Conjecture. *Let $p : E \to B$ be a smooth fiber bundle with E a closed connected manifold and with fiber the Grassmann manifold $\mathbb{R}G_{n,k}$. If the fiber is not bordant to* 0 *(in other words (see [19] or [2]), if each power of* 2 *dividing n also divides k), then it is totally non-homologous to zero in E with respect to $\mathbb{Z}_2$.*

Remarks (e) If the conjecture is valid, then the Leray-Hirsch theorem applies. As a consequence, a necessary condition for the validity of the conjecture is the following: $p^* : H^*(B; \mathbb{Z}_2) \to H^*(E; \mathbb{Z}_2)$ is a monomorphism in case that each power of 2 dividing n also divides k. By [14, Theorem (a)], this necessary condition is fulfilled.

(f) Let $T : \mathbb{C}G_{n,k} \to \mathbb{C}G_{n,k}$ be the involution induced by complex conjugation. Then one has $T^*(c_1(\gamma_k)) = -c_1(\gamma_k)$. So reasoning analogous to that in the above example shows that the fiber $\mathbb{C}G_{n,k}$ of the fiber bundle

$$\frac{\mathbb{C}G_{n,k} \times S^m}{T \times a} \longrightarrow \mathbb{R}P^m$$

(with $m \geq 1$) is not totally non-homologous to zero with respect to $\mathbb{Z}$ or $\mathbb{Z}_q$, $q \neq 2$. This is the example promised in Remark (d) after Theorem B.

(g) It seems natural to ask whether Theorem A or Theorem B(1) remains valid (without adding orientability or other assumptions) when one passes from smooth fiber bundles to continuous locally trivial fibrations or even to Hurewicz or Serre fibrations. Theorem B(2) provides a partial (positive) answer to this question.

(h) Observe that the first part of the proof of Proposition 2, which shows that $\theta(w_2(\gamma_2)) = 0$, requires only that the smallest r for which ${w_2}^r(\gamma_2) = 0$ is odd. Therefore one can prove analogously that $\mathrm{Der}_{<0}(H^*(\mathbb{C}P^{2k}; \mathbb{Z}_2)) = 0$, and so one obtains another proof of the above cited Gottlieb's result ([12, Corollary 7]) for $q = 2$.

We also know (see the result on the height) that the smallest r for which $w_2{}^r(\gamma_3) = 0$ is odd for $\mathbb{C}G_{2^s+1,3}$ ($s \geq 3$). But in this case I do not know if all odd-degree derivations vanish on $w_4(\gamma_3)$ and on $w_6(\gamma_3)$.

An inspection of Blanchard's argument (see Meier [17, p. 474]) evokes a feeling that a "source of difficulties" in case of $\mathbb{Z}_2$-coefficients, as compared to the situation with real or rational coefficients, is that for the manifolds $\mathbb{C}G_{n,k}$ ($4 \leq 2k \leq n$) with $(n, k) \neq (2^s + 1, 2)$ there is no $\mathbb{Z}_2$-version of the hard Lefschetz theorem. To see the latter, it is enough to look at the above cited height result for the generator $w_2(\gamma_2) \in H^*(\mathbb{C}G_{n,k}; \mathbb{Z}_2)$.

(i) For a smooth closed connected manifold M and a stable characteristic class ψ (in the sense of Husemoller [15, Chap. 20, Sec. 2]), put $\psi(M) := \psi(TM) \in H^*(M; R)$, where TM denotes the tangent bundle of M and R denotes a fixed coefficient ring. Using the same methods as in the proofs of Theorems A and B(1), it is clear that the following generalization can be proved.

Theorem C. *Let F be a smooth closed connected manifold such that the characteristic classes $\psi_j(F)$ generate the cohomology ring $H^*(F; R)$. If $p : E \to B$ is a smooth fiber bundle with E a closed connected manifold and with fiber F, then the fiber inclusion $i : F \to E$ induces an epimorphism, $i^* : H^*(E; R) \to H^*(F; R)$, in cohomology with coefficients in the ring R.*

Bibliography

[1] A. Back, *Involutions on Grassmann manifolds*, Thesis (Univ. of California), Berkeley, 1977;

[2] V. Bartík, J. Korbaš, *Stationary point free elementary abelian 2-group actions on Grassmannians – an elementary approach*, Arch. Math. (Basel) **60** (1993), 563–567;

[3] J. C. Becker, D. H. Gottlieb, *Coverings of fibrations*, Compositio Math. **26** (1973), 119–128;

[4] A. Blanchard, *Sur les variétés analytiques complexes*, Ann. Sci. Ecole Norm. Sup. (3) **73** (1956), 157–202;

[5] A. Blanchard, *Sur les variétés analytiques complexes*, Thèses (Univ. Paris), Gauthier-Villars, Paris, 1956;

[6] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. **57** (1953), 115–207;

[7] A. Borel, *Cohomologie des espaces localement compacts d'après J. Leray*, Lecture Notes in Math., Springer-Verlag, Berlin **2** (1964);

[8] A. Borel, F. Hirzebruch, *Characteristic classes and homogeneous spaces. I*, Amer. J. Math. **80** (1958), 458–538;

[9] A. Dold, *Lectures on Algebraic Topology*, Springer-Verlag, Berlin (1972);

[10] R. J. D. Ferdinands, *$G_{n,3}(C)$ is prime if n is odd*, Topology Appl. **86** (1998), 233–251;

[11] R. J. D. Ferdinands, R. E. Schultz, *Algebraic fiberings of Grassmann varieties*, Illinois J. Math. **41** (1997), 31–53;

[12] D. H. Gottlieb, *Witnesses, transgressions, and the evaluation map*, Indiana Univ. Math. J. **24** (1975), 825–836;

[13] H. Hiller, *On the cohomology of real Grassmannians*, Trans. Amer. Math. Soc. **257** (1980), 521–533;

[14] Ľ. Horanská, J. Korbaš, *On cup products in some manifolds*, Bull. Belgian Math. Soc. **7** (2000), 21–28;

[15] D. Husemoller, *Fibre Bundles*, Third edition, Springer-Verlag, New York, 1994;

[16] W. Meier, *Rational universal fibrations and flag manifolds*, Math. Ann. **258** (1982), 329–340;

[17] W. Meier, *Some topological properties of Kähler manifolds and homogeneous spaces*, Math. Z. **183** (1983), 473–481;

[18] W. Meier, *On fibre inclusions and Kähler manifolds*, Proc. Amer. Math. Soc. **88** (1983), 173–176;

[19] P. Sankaran, *Determination of Grassmannians which bound*, Canad. Math. Bull. **34** (1991), 119–122;

[20] R. Schultz, *Compact fiberings of homogeneous spaces. I*, Compositio Math. **43** (1981), 181–215, *Correction,* 419–421;

[21] H. Shiga, M. Tezuka, *Rational fibrations, homogeneous spaces with positive Euler characteristics and Jacobians*, Ann. Inst. Fourier **37** (1987), 81–106;

[22] R. E. Stong, *Cup products in Grassmannians*, Topology Appl. **13** (1982), 103–113;

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