On the stability of orthogonal bases in non-archimedean metrizable locally convex spaces

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Abstract

It is proved that an orthogonal basis in a non-archimedean metrizable locally convex space E (in particular, a Schauder basis in a non-archimedean Fréchet space E) is stable if and only if there is a continuous norm on E. Many others results are also obtained.

1 Introduction

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot| : \mathbb{K} \to [0, \infty)$. For fundamentals of locally convex Hausdorff spaces (lcs) and normed spaces we refer to [8], [6] and [7]. Orthogonal bases and orthogonal sequences in locally convex spaces are studied in [2], [3], [4] and [9].

If (x_n) is an orthogonal basis (a basic orthogonal sequence) in a Banach space $(E, ||\cdot||)$, then any sequence $(y_n) \subset E$ with $||x_n - y_n|| \le t||x_n||, n \in \mathbb{N}$ for some $t \in (0, 1)$ (with $||x_n - y_n|| < ||x_n||, n \in \mathbb{N}$) is also an orthogonal basis (a basic orthogonal sequence, respectively) in $(E, ||\cdot||)$ ([7], p.183).

In ([5], p.2) it is showed that for any Schauder basis (x_n) in a Banach space $(E, ||\cdot||)$ with the sequence (f_n) of coefficient functionals any sequence $(y_n) \subset E$ with $\sup_n ||f_n|| \, ||x_n - y_n|| < 1$ is also a Schauder basis in $(E, ||\cdot||)$.

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In this article we study the problem of the stability of orthogonal bases and basic orthogonal sequences in metrizable locally convex spaces (in particular, Schauder bases and basic sequences in Fréchet spaces). An orthogonal basis (x_n) in a lcs E will be called stable, if there exists a sequence (U_n) of neighborhoods of the zero in E such that any sequence $(y_n) \subset E$ with $y_n \in (x_n + U_n), n \in \mathbb{N}$ is an orthogonal basis in E which is equivalent to (x_n).

We prove that any orthogonal basis in a metrizable lcs with a continuous norm is stable (Proposition 5(b)); in particular, any Schauder basis in a Fréchet space with a continuous norm is stable (Corollary 6). Hence, if a metrizable lcs E with a continuous norm has an orthogonal basis, then every dense subspace of E has an orthogonal basis (Corollary 7). On the other hand a metrizable lcs E without a continuous norm has no stable orthogonal basis (Proposition 5(a)) and if E is not of finite type, then it contains a dense subspace without an orthogonal basis (Proposition 8(b)). In particular, the Fréchet space $c_0 \times \mathbb{K}^{\mathbb{N}}$ with an orthogonal basis contains a dense subspace without an orthogonal basis (Example 9). (Recall that c_0 is the Banach space of all sequences in $\mathbb{K}$ converging to zero with the sup--norm, and $\mathbb{K}^{\mathbb{N}}$ is the Fréchet space of all sequences in $\mathbb{K}$ with the topology of pointwise convergence.)

It is still unknown whether any Fréchet space of countable type has an orthogonal basis (cf.[3]). The above example shows that there exist metrizable locally convex spaces of countable type without an orthogonal basis. (In [9], we proved that any infinite-dimensional metrizable lcs has a basic orthogonal sequence.)

2 Preliminaries

The linear span of a subset A of a linear space E is denoted by lin$A$. The topological dual of a lcs E is indicated by E'.

Let E, F be locally convex spaces. A map $T : E \to F$ is called a *linear homeomorphism* if T is linear, one-to-one, surjective and the maps T, T^{-1} are continuous.

Sequences (x_n) and (y_n) in a lcs E are *equivalent* if there exists a linear homeomorphism P between the linear spans of (x_n) and (y_n), such that $Px_n = y_n$ for all $n \in \mathbb{N}$. Sequences (x_n) and (y_n) in a Fréchet space E are equivalent iff there exists a linear homeomorphism P between the closed linear spans of (x_n) and (y_n), such that $Px_n = y_n$ for all $n \in \mathbb{N}$.

A sequence (x_n) in a lcs E is a *Schauder basis* in E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n x_n$ with $(\alpha_n) \subset \mathbb{K}$ and the coefficient functionals $f_n : E \to \mathbb{K}, x \to \alpha_n \, (n \in \mathbb{N})$ are continuous. A sequence in a lcs E is a *basic sequence* in E if it is a Schauder basis in its closed linear span in E.

By a *seminorm* on a linear space E we mean a function $p : E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}, x \in E$ and $p(x + y) \leq \max\{p(x), p(y)\}$ for all $x, y \in E$. A seminorm p on E is a *norm* if Ker $p := \{x \in E : p(x) = 0\} = \{0\}$.

The set of all continuous seminorms on a lcs E is denoted by $\mathcal{P}(E)$. A family $\mathcal{B} \subset \mathcal{P}(E)$ is a *base* in $\mathcal{P}(E)$ if for every $p \in \mathcal{P}(E)$ there exists $q \in \mathcal{B}$ with $p \leq q$.

A lcs E is *of finite type* if for each $p \in \mathcal{P}(E)$ the quotient space $E/\text{Ker}\, p$ is finite-dimensional. A metrizable lcs E is *of countable type* if it contains a linearly dense countable set. A *Fréchet space* is a metrizable complete lcs.

If F is a subspace of a lcs E, then for every $p \in \mathcal{P}(F)$ there exists $q \in \mathcal{P}(E)$ with $q|F = p$ ([4], Lemma 1.1).

Every metrizable lcs E has a non-decreasing sequence of continuous seminorms (p_k) which forms a (non-decreasing) base in $\mathcal{P}(E)$.

Let $t \in (0, 1]$ and p be a seminorm on a linear space E. A sequence $(x_n) \subset E$ is *t-orthogonal with respect to p* if

$$p(\sum_{i=1}^{n} \alpha_i x_i) \geq t \max_{1 \leq i \leq n} p(\alpha_i x_i)$$

for all $n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in \mathbb{K}$.

A sequence (x_n) in a lcs E is *orthogonal* in E if the family $\mathcal{B}$ of all continuous seminorms p on E for which (x_n) is 1-orthogonal with respect to p forms a base in $\mathcal{P}(E)$. (In [7], a sequence (x_n) in a normed space $(E, ||\cdot||)$ is called orthogonal if it is 1-orthogonal with respect to the norm $||\cdot||$.)

An orthogonal sequence (x_n) of non-zero elements in a lcs E is a *basic orthogonal sequence* in E. A linearly dense basic orthogonal sequence in a lcs E is an *orthogonal basis* in E.

An orthogonal basis (a basic orthogonal sequence, a Schauder basis, a basic sequence) (x_n) in a lcs E will be called *stable* if there exists a sequence (U_n) of neighborhoods of the zero in E such that any sequence $(y_n) \subset E$ with $y_n \in (x_n + U_n)$, $n \in \mathbb{N}$ is an orthogonal basis (a basic orthogonal sequence, a Schauder basis, a basic sequence) in E which is equivalent to (x_n).

Let $(t_\alpha) \subset (0, 1]$. A sequence (x_n) in a lcs E is (t_α)*-orthogonal with respect to* $(p_\alpha) \subset \mathcal{P}(E)$ if (x_n) is t_α-orthogonal with respect to p_α for every α.

A sequence (x_n) in a lcs E is orthogonal in E iff it is (t_α)-orthogonal with respect to (p_α) for some base (p_α) in $\mathcal{P}(E)$ and some $(t_\alpha) \subset (0, 1]$ (cf. [3], Proposition 2.6).

Let F be a subspace of a lcs E. A sequence $(x_n) \subset F$ is orthogonal in F iff it is orthogonal in E ([3], Remark 1.2(i)).

Every basic orthogonal sequence in a lcs is a basic sequence ([3], Proposition 1.4) and every basic sequence in a Fréchet space is a basic orthogonal sequence ([3], Proposition 1.7).

The results of this paper concern infinite-dimensional metrizable lcs. Nevertheless, some spaces, which are constructed in the proof of Theorem 1, may be finite-dimensional. Therefore we recall the following.

A finite sequence $(x_1, \ldots, x_k)$ in a k-dimensional lcs E is a *Schauder basis* in E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{k} \alpha_n x_n$ with $(\alpha_1, \ldots, \alpha_k) \subset \mathbb{K}$ and the coefficient functionals $f_n : E \to \mathbb{K}, x \to \alpha_n$ $(1 \leq n \leq k)$ are continuous. (Every linear basis in a finite-dimensional lcs E is a Schauder basis in E.)

Let $t \in (0, 1]$ and p be a seminorm on a linear space E. A finite sequence $(x_1, \ldots, x_n) \subset E$ is *t-orthogonal with respect to p* if $p(\sum_{i=1}^{n} \alpha_i x_i) \geq t \max_{1 \leq i \leq n} p(\alpha_i x_i)$ for all $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$.

3 Results

First, we prove the following stability theorem for basic sequences in Fréchet spaces.

Theorem 1. *Let (x_n) be a basic sequence in a Fréchet space E and (p_k) a non--decreasing base in $\mathcal{P}(E)$ such that*

(1) *(x_n) is (s_k)-orthogonal with respect to (p_k) for some $(s_k) \subset (0,1]$;*

(2) *$p_k(x_k) > 0$ for all $k \in \mathbb{N}$.*

Assume that $(y_n) \subset E$ and $p_k(x_n - y_n) < s_k p_k(x_n)$ or $p_k(x_n - y_n) = p_k(x_n) = 0$ for all $k, n \in \mathbb{N}$ with $k \leq n$. Then (y_n) is a basic sequence in E which is equivalent to (x_n) and (r_k)-orthogonal with respect to (p_k) for some $(r_k) \subset (0,1]$.

Proof. Let $k \in \mathbb{N}$. Denote by π_k the canonical mapping $E \to E/\mathrm{Ker}\ p_k$. The map $\overline{p_k} : E/\mathrm{Ker}\ p_k \to [0,\infty), \pi_k(x) \to p_k(x)$ is a norm on $E/\mathrm{Ker}\ p_k$. Let E_k be the completion of $(E/\mathrm{Ker}\ p_k, \overline{p_k})$ and q_k a continuous norm on E_k with $q_k(\pi_k(x)) = p_k(x), x \in E$. Put $\mathbb{N}_k = \{n \in \mathbb{N} : p_k(x_n) > 0\}$; by (2) we obtain $\{1,\dots,k\} \subset \mathbb{N}_k$. Set $x_n^k = \pi_k(x_n)$, $y_n^k = \pi_k(y_n)$, $n \in \mathbb{N}$. For all $n \in \mathbb{N}$ with $n \geq k$ we have

$$q_k(x_n^k - y_n^k) < s_k q_k(x_n^k) \text{ or } q_k(x_n^k - y_n^k) = q_k(x_n^k) = 0.$$

Hence $x_n^k = y_n^k = 0$ for $n \in (\mathbb{N} \setminus \mathbb{N}_k)$. By X_k and Y_k we denote the closure of $\mathrm{lin}\{x_n^k : n \in \mathbb{N}\}$ and $\mathrm{lin}\{y_n^k : n \in \mathbb{N}\}$, respectively. Clearly $(x_n^k : n \in \mathbb{N}_k)$ is a Schauder basis in X_k which is s_k-orthogonal with respect to q_k. Let $(f_n^k : n \in \mathbb{N}_k) \subset X_k'$ be the sequence of coefficient functionals associated with the basis $(x_n^k : n \in \mathbb{N}_k)$. Set $f_n^k(x) = 0, n \in (\mathbb{N} \setminus \mathbb{N}_k), x \in X_k$. Let $x \in X_k$. Then $x = \sum_{n=1}^\infty f_n^k(x) x_n^k$ and $q_k(f_n^k(x)x_n^k) \to 0$, as $n \to \infty$. Hence $q_k(f_n^k(x)(x_n^k - y_n^k)) \to 0$, as $n \to \infty$, and the series $\sum_{n=1}^\infty f_n^k(x)(x_n^k - y_n^k)$ is convergent in E_k. Consider the linear map

$$T_k : X_k \to E_k, x \to \sum_{n=k}^{\infty} f_n^k(x)(x_n^k - y_n^k).$$

We have $q_k(T_k x) < q_k(x)$ for $x \in (X_k \setminus \{0\})$. Indeed, for $x \in (X_k \setminus \{0\})$ we obtain

$$q_k(T_k x) \leq \max_{n \geq k} q_k(f_n^k(x)(x_n^k - y_n^k)) \leq \max_{n \geq k} s_k q_k(f_n^k(x) x_n^k) \leq$$

$$\max_{n \in \mathbb{N}} s_k q_k(f_n^k(x) x_n^k) \leq q_k(\sum_{n=1}^{\infty} f_n^k(x) x_n^k) = q_k(x),$$

and at least one of the above inequalities is strict.

Hence the linear map

$$Q_k : X_k \to E_k, x \to (x - T_k x)$$

is isometric. We have

$$Q_k(x) = \sum_{n=1}^{k-1} f_n^k(x) x_n^k + \sum_{n=k}^{\infty} f_n^k(x) y_n^k,\ x \in X_k.$$

Denote by X_k^k and Y_k^k the closure of $\mathrm{lin}\{x_n^k : n \geq k\}$ and $\mathrm{lin}\{y_n^k : n \geq k\}$, respectively. The map

$$P_k : X_k \to Y_k, x \to \sum_{n=1}^{\infty} f_n^k(x) y_n^k$$

is linear and continuous. Since $\mathrm{lin}\,\{y_n^k : n \geq k\} \subset P_k(X_k^k) = Q_k(X_k^k) \subset Y_k^k$ and Q_k is isometric, then $P_k(X_k^k) = Q_k(X_k^k) = Y_k^k$. Hence $P_k(X_k) = Y_k$.

By induction we prove that the map $P_k : X_k \to Y_k$ is a linear homeomorphism and $(y_n^k : n \in N_k)$ is a Schauder basis in Y_k. It is clear for $k = 1$ because $P_1 = Q_1$ and $P_1(x_n^1) = y_n^1, n \in \mathbb{N}_1$. Assume that it is true for $k = m$. We show that it is true for $k = m+1$. The map $P_{m+1} : X_{m+1} \to Y_{m+1}$ is one-to-one. Indeed, let $x \in X_{m+1}$ and $P_{m+1}(x) = 0$. Then

$$p_{m+1}(\sum_{n=1}^{t} f_n^{m+1}(x)y_n) = q_{m+1}(\sum_{n=1}^{t} f_n^{m+1}(x)y_n^{m+1}) \to 0\,, \mathrm{as}\, t \to \infty,$$

hence

$$q_m(\sum_{n=1}^{t} f_n^{m+1}(x)y_n^m) = p_m(\sum_{n=1}^{t} f_n^{m+1}(x)y_n) \to 0\,, \mathrm{as}\, t \to \infty.$$

Thus $\sum_{n=1}^{\infty} f_n^{m+1}(x)y_n^m = 0$ in E_m. Since $\{1, \ldots, m\} \subset \mathbb{N}_m$ and $(y_n^m : n \in \mathbb{N}_m)$ is a Schauder basis in Y_m, then $f_n^{m+1}(x) = 0$ for $1 \leq n \leq m$. Hence

$$Q_{m+1}(x) = P_{m+1}(x) + \sum_{n=1}^{m} f_n^{m+1}(x)(x_n^{m+1} - y_n^{m+1}) = 0, \text{ so } x = 0.$$

By the open mapping theorem ([6], Corollary 2.74) the map P_{m+1} is a linear homeomorphism. Since $P_{m+1}(x_n^{m+1}) = y_n^{m+1}, n \in \mathbb{N}_{m+1}$ we obtain that $(y_n^{m+1} : n \in \mathbb{N}_{m+1})$ is a Schauder basis in Y_{m+1}.

Hence for every $k \in \mathbb{N}$ the sequence $(y_n^k : n \in \mathbb{N}_k)$ is r_k-orthogonal with respect to q_k for some $r_k \in (0, 1]$, so (y_n) is (r_k)-orthogonal with respect to (p_k). Thus (y_n) is orthogonal in E. Since $y_n \neq 0, n \in \mathbb{N}$, then (y_n) is a basic sequence in E.

By X and Y we denote the closure of $\mathrm{lin}\{x_n : n \in \mathbb{N}\}$ and $\mathrm{lin}\{y_n : n \in \mathbb{N}\}$, respectively. Let $(f_n) \subset X'$ be the sequence of coefficient functionals associated with the Schauder basis (x_n) in X. It is easy to see that $p_k(y_n) = p_k(x_n)$ for all $k, n \in \mathbb{N}$ with $k \leq n$. Hence for every sequence $(\alpha_n) \subset \mathbb{K}$ the sequence $(\alpha_n x_n)$ converges to 0 in E iff the sequence $(\alpha_n y_n)$ converges to 0 in E. Hence the map

$$P : X \to Y, x \to \sum_{n=1}^{\infty} f_n(x)y_n$$

is one-to-one and surjective. Moreover P is continuous. In fact, let $k \in \mathbb{N}$ and $z \in X$. For $n \in \mathbb{N}_k$ we have $f_n^k(\pi_k(z)) = f_n(z)$. Thus $\pi_k(Pz) = P_k(\pi_k z)$, so $p_k(Pz) = q_k(P_k(\pi_k z))$. This implies the continuity of P. By the open mapping theorem the map P is a linear homeomorphism. Clearly $Px_n = y_n, n \in \mathbb{N}$. Hence (y_n) is equivalent to (x_n). ∎

Remark. For any basic sequence (x_n) in a Fréchet space E there exists a non--decreasing base (p_k) in $\mathcal{P}(E)$ such that (x_n) is (1)-orthogonal with respect to (p_k) and $p_k(x_k) > 0$ for every $k \in \mathbb{N}$.

Corollary 2. *Let (x_n) be a basic orthogonal sequence in a metrizable lcs E and (p_k) a non-decreasing base in $\mathcal{P}(E)$ such that*

(1) *(x_n) is (s_k)-orthogonal with respect to (p_k) for some $(s_k) \subset (0,1]$;*

(2) *$p_k(x_k) > 0$ for all $k \in \mathbb{N}$.*

Assume that $(y_n) \subset E$ and $p_k(x_n - y_n) < s_k p_k(x_n)$ or $p_k(x_n - y_n) = p_k(x_n) = 0$ for all $k, n \in \mathbb{N}$ with $k \leq n$. Then (y_n) is a basic orthogonal sequence in E which is equivalent to (x_n) and (r_k)-orthogonal with respect to (p_k) for some $(r_k) \subset (0,1]$.

Proof. Let F be the completion of E and q_k a continuous seminorm on F with $q_k|E = p_k, k \in \mathbb{N}$. Then (q_k) is a non-decreasing base in $\mathcal{P}(F)$. The sequence (x_n) is a basic sequence in F. Using Theorem 1 we obtain our Corollary . ■

For Schauder bases in Fréchet spaces we show the following stability theorem.

Theorem 3. *Let (x_n) be a Schauder basis in a Fréchet space E and (p_k) a non--decreasing base in $\mathcal{P}(E)$ such that*

(1) *(x_n) is (s_k)-orthogonal with respect to (p_k) for some $(s_k) \subset (0,1]$;*

(2) *$p_k(x_k) > 0$ for all $k \in \mathbb{N}$.*

Assume that $(y_n) \subset E, (t_k) \subset (0,1)$ and $p_k(x_n - y_n) \leq t_k s_k p_k(x_n)$ for all $k, n \in \mathbb{N}$ with $k \leq n$. Then (y_n) is a Schauder basis in E which is equivalent to (x_n) and (r_k)-orthogonal with respect to (p_k) for some $(r_k) \subset (0,1]$.

Proof. By Theorem 1, (y_n) is a basic sequence in E which is equivalent to (x_n) and (r_k)-orthogonal with respect to (p_k) for some $(r_k) \subset (0,1]$. It is enough to prove that the sequence (y_n) is linearly dense in E. We will use notation from the proof of Theorem 1. Let $k \in \mathbb{N}$. Then $X_k = E_k$. For $x \in E_k$ we obtain $q_k(T_k x) \leq t_k q_k(x)$. Hence $q_k(T_k^m x) \leq t_k^m q_k(x)$ for $m \in \mathbb{N}, x \in E_k$. Thus $T_k^m x \to 0$ in E_k, as $m \to \infty$, and the map

$$S_k : E_k \to E_k, x \to \sum_{m=0}^{\infty} T_k^m x$$

is continuous. Since $S_k Q_k = Q_k S_k = \mathrm{id}_{E_k}$, the map $Q_k : E_k \to E_k$ is a linear homeomorphism.

By $Q_k(X_k^k) = Y_k^k$ we obtain $\dim(E_k/Y_k^k) = \dim(E_k/X_k^k) = k-1$. Hence $Y_k = E_k$ because $(y_n^k : n \in \mathbb{N}_k)$ is a Schauder basis in Y_k and $\{1, \ldots, k\} \subset \mathbb{N}_k$. We have showed that for every $k \in \mathbb{N}$ the sequence $(\pi_k(y_n) : n \in \mathbb{N})$ is linearly dense in $(E/\mathrm{Ker}\, p_k, \overline{p_k})$. It follows that the sequence (y_n) is linearly dense in E. ■

Corollary 4. *Let (x_n) be an orthogonal basis in a metrizable lcs E and (p_k) a non--decreasing base in $\mathcal{P}(E)$ such that*

(1) *(x_n) is (s_k)-orthogonal with respect to (p_k) for some $(s_k) \subset (0,1]$;*

(2) *$p_k(x_k) > 0$ for all $k \in \mathbb{N}$.*

Assume that $(y_n) \subset E, (t_k) \subset (0,1)$ and $p_k(x_n - y_n) \leq t_k s_k p_k(x_n)$ for all $k, n \in \mathbb{N}$ with $k \leq n$. Then (y_n) is an orthogonal basis in E which is equivalent to (x_n) and (r_k)-orthogonal with respect to (p_k) for some $(r_k) \subset (0,1]$.

Now we can prove that any orthogonal basis in a metrizable lcs with a continuous norm is stable.

Proposition 5. (a) *If a lcs E has a stable orthogonal basis then there is a continuous norm on E.*

(b) *If there is a continuous norm on a metrizable lcs E then every orthogonal basis in E is stable.*

Proof. (a) Assume that (x_n) is a stable orthogonal basis in E. Let (U_n) be a sequence of neighborhoods of the zero in E such that any sequence $(y_n) \subset E$ with $y_n \in (x_n + U_n), n \in \mathbb{N}$ is an orthogonal basis in E. Let p be a non-zero continuous seminorm on E. Since the set $\{x \in E : p(x) > 0\}$ is dense in E, for every $n \in \mathbb{N}$ there exists $y_n \in (x_n + U_n)$ with $p(y_n) > 0$. Then (y_n) is an orthogonal basis in E and there exists a continuous seminorm q on E such that (y_n) is 1-orthogonal with respect to q and $q \geq p$. Let $(g_n) \subset E'$ be the sequence of coefficient functionals associated with the Schauder basis (y_n). Take $y \in E$. Then $y = \sum_{n=1}^{\infty} g_n(y)y_n$ and $q(y) = \max_n |g_n(y)|q(y_n)$. Hence q is a continuous norm on E.

(b) Let q be a continuous norm on E and $(t_k) \subset (0,1)$. Let (x_n) be an orthogonal basis in E and (p_k) a non-decreasing base in $\mathcal{P}(E)$ with $p_1 \geq q$ such that (x_n) is (1)-orthogonal with respect to (p_k). For every $n \in \mathbb{N}$ the set

$$U_n = p_1^{-1}[0, t_1 p_1(x_n)) \cap p_2^{-1}[0, t_2 p_2(x_n)) \cap \ldots \cap p_n^{-1}[0, t_n p_n(x_n))$$

is a neighborhood of the zero in E. By Corollary 4 any sequence $(y_n) \subset E$ with $y_n \in (x_n + U_n), n \in \mathbb{N}$ is an orthogonal basis in E which is equivalent to (x_n). Thus the basis (x_n) is stable. ∎

Corollary 6. *An orthogonal basis in a metrizable lcs E is stable iff there is a continuous norm on E. In particular, a Schauder basis in a Fréchet space E is stable iff there is a continuous norm on E.*

Corollary 7. *If a metrizable lcs E with a continuous norm has an orthogonal basis, then every dense subspace of E has an orthogonal basis. In particular, if a Fréchet space E with a continuous norm has a Schauder basis, then every dense subspace of E has a Schauder basis.*

For a metrizable lcs without a continuous norm we have the following.

Proposition 8. (a) *If E is a lcs without a continuous norm and F is its dense subspace with a continuous norm, then F has no orthogonal basis.*

(b) *If a metrizable lcs E without a continuous norm is not of finite type, then E contains a dense subspace without an orthogonal basis.*

(c) *If a metrizable lcs E without a continuous norm is of finite type, then every dense subspace of E has an orthogonal basis.*

Proof. (a) Assume that F has an orthogonal basis (x_n). Then (x_n) is an orthogonal Schauder basis in E. Let $(f_n) \subset E'$ be the sequence of coefficient functionals associated with the basis (x_n). There exists a continuous seminorm p on E such that (x_n) is 1-orthogonal with respect to p and $p|F$ is a norm on F. Let $x \in E$. Then $x = \sum_{n=1}^{\infty} f_n(x)x_n$ and $p(x) = \max_n |f_n(x)|p(x_n)$. Hence p is a continuous norm on E, a contradiction. This shows that F has no orthogonal basis.

(b) It is enough to consider the case when E has an orthogonal basis (x_n). Let p be a continuous seminorm on E with $\dim(E/\operatorname{Ker} p) = \infty$ such that (x_n) is 1-orthogonal with respect to p. The set $\mathbb{N}_1 = \{n \in \mathbb{N} : p(x_n) > 0\}$ is infinite and the set $\mathbb{N}_2 = (\mathbb{N} \setminus \mathbb{N}_1)$ is not empty. Let $\{M_i : i \in \mathbb{N}_2\}$ be a partition of $\mathbb{N}_1$ into infinite subsets. Take $(\alpha_n) \subset (\mathbb{K} \setminus \{0\})$ with $\alpha_n x_n \to 0$ in E. Set $z_n = \alpha_n x_n + x_i$ for $n \in M_i$ and $i \in \mathbb{N}_2$. The closure of the subspace $F = \operatorname{lin}\{z_n : n \in \mathbb{N}_1\}$ contains the set $\{x_i : i \in \mathbb{N}_2\} \cup \{x_n : n \in \mathbb{N}_1\}$, so F is a dense subspace of E. Moreover $p|F$ is a norm on F. Indeed, let $k \in \mathbb{N}, \beta_1, \ldots, \beta_k \in \mathbb{K}, n_1, \ldots, n_k \in \mathbb{N}_1, z = \sum_{i=1}^{k} \beta_i z_{n_i}$ and $p(z) = 0$. Then $p(\sum_{i=1}^{k} \beta_i \alpha_{n_i} x_{n_i}) = 0$, hence $\max_{1 \le i \le k} |\beta_i \alpha_{n_i}| p(x_{n_i}) = 0$. Thus $\beta_i = 0$ for $1 \le i \le k$, so $z = 0$. By (a), F has no orthogonal basis.

(c) Since any metrizable lcs of finite type has an orthogonal basis ([3], Theorem 3.5) and any subspace of a space of finite type is of finite type, then we have (c). ■

By Proposition 8(b) we obtain the following.

Example 9. *The Fréchet space $c_0 \times \mathbb{K}^{\mathbb{N}}$ has an orthogonal basis and contains a dense subspace without an orthogonal basis.*

For basic orthogonal sequences in a metrizable lcs we show the following.

Proposition 10. (a) *A basic orthogonal sequence in a metrizable lcs E is stable iff there is a continuous norm on its linear span.*

(b) *A metrizable lcs E has a stable basic orthogonal sequence iff E is not of finite type.*

(c) *Every basic orthogonal sequence in a metrizable lcs E is stable iff there is a continuous norm on E.*

Proof. (a) Let (x_n) be a stable basic orthogonal sequence in E and (U_n) a sequence of neighborhoods of the zero in E such that any sequence $(y_n) \subset E$ with $y_n \in (x_n + U_n), n \in \mathbb{N}$ is a basic orthogonal sequence in E. Assume that there is no continuous norm on the linear span of (x_n). Then there exists a subsequence (x_{n_k}) of (x_n) such that the linear span X of (x_{n_k}) is of finite type. Let $p \in \mathcal{P}(E)$ and $p(x_{n_1}) > 0$. For every $k \in \mathbb{N}$ there exists $y_{n_k} \in (x_{n_k} + U_{n_k} \cap X)$ with $p(y_{n_k}) > 0$. Then (y_{n_k}) is an orthogonal sequence in X. Thus there exists $q \in \mathcal{P}(E)$ with $q \ge p$ such that (y_{n_k}) is 1-orthogonal with respect to q. It follows that q is a norm on the linear span of (y_{n_k}). Hence X is not of finite type, a contradiction. Thus there is a continuous norm on the linear span of (x_n).

Now assume that there is a continuous norm q on the linear span X of a basic orthogonal sequence (x_n) in E. Let (p_k) be a non-decreasing base in $\mathcal{P}(E)$ with $(p_1|X) \ge q$ such that (x_n) is (1)-orthogonal with respect to (p_k). For every $n \in \mathbb{N}$ the set

$$U_n = p_1^{-1}[0, p_1(x_n)) \cap p_2^{-1}[0, p_2(x_n)) \cap \ldots \cap p_n^{-1}[0, p_n(x_n))$$

is a neighborhood of the zero in E. By Corollary 2 any sequence $(y_n) \subset E$ with $y_n \in (x_n + U_n), n \in \mathbb{N}$ is a basic orthogonal sequence in E which is equivalent to (x_n). Thus (x_n) is a stable basic orthogonal sequence in E.

(b) Since any infinite-dimensional metrizable lcs has a basic orthogonal sequence ([9], Theorem 2), and a lcs is not of finite type iff it contains an infinite-dimensional subspace with a continuous norm, then (a) implies (b).

(c) If there is no continuous norm on E, then E contains an infinite-dimensional subspace F of finite type ([1], Proposition 2.6). F has a basic orthogonal sequence which is not stable. Hence E has a basic orthogonal sequence which is not stable.

If there is a continuous norm on E, then every basic orthogonal sequence in E is stable. ∎

Corollary 11. (a) *A basic sequence in a Fréchet space is stable iff there is a continuous norm on its linear span.*

(b) *A Fréchet space E has a stable basic sequence iff E is not of finite type.*

(c) *Every basic sequence in a Fréchet space E is stable iff there is a continuous norm on E.*

An orthogonal basis (a basic orthogonal sequence, a Schauder basis, a basic sequence) (x_n) in a lcs E will be called *quasi stable* if there exists a sequence (U_n) of neighborhoods of the zero in E such that any sequence $(y_n) \subset E$ with $y_n \in (x_n + U_n)$, $n \in \mathbb{N}$ is an orthogonal basis (a basic orthogonal sequence, a Schauder basis, a basic sequence) in E.

By the proofs of Propositions 5 and 10(a) we obtain the following.

Proposition 12. (a) *An orthogonal basis in a metrizable lcs is quasi stable iff it is stable. In particular, a Schauder basis in a Fréchet space is quasi stable iff it is stable.*

(b) *A basic orthogonal sequence in a metrizable lcs is quasi stable iff it is stable. In particular, a basic sequence in a Fréchet space is quasi stable iff it is stable.*

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