Lagrangian submanifolds of constant sectional curvature and their Ribaucour transformation

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Abstract

The Ribaucour transformation is applied to the family of Lagrangian submanifolds of dimension n and nonzero constant sectional curvature c of complex space forms of complex dimension n and constant holomorphic sectional curvature $4c$. As a consequence, a process is obtained to generate a new family of such submanifolds starting from a given one. In particular, explicit parametrizations in terms of elementary functions of examples with arbitrary dimension and curvature are provided. A permutability formula is derived which provides a simple algebraic procedure to construct further examples once two Ribaucour transforms of a given submanifold are known. The analytical counterparts of the above results are also discussed.

1 Introduction

An isometric immersion $f\colon M^n \to \widetilde{M}^m$ of an n-dimensional Riemannian manifold into a Kaehler manifold of complex dimension m is said to be *totally real* if the almost complex structure of $\widetilde{M}^m$ carries each tangent space of M^n into its corresponding normal space. If in addition $n = m$ then f is said to be *Lagrangian*.

The simplest examples of Lagrangian submanifolds of complex space forms $\widetilde{M}^n(4c)$ of constant holomorphic sectional curvature $4c$ are the totally geodesic real space forms $M^n(c)$ of constant sectional curvature c. The family of non-totally geodesic

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Lagrangian isometric immersions $f\colon M^n(c) \to \widetilde{M}^n(4c)$ has been recently investigated in [CDVV] and [DT_3], the latter being devoted only to the flat case. Two general problems form the core of the investigation. First, to get a satisfactory description of the family, with an eye towards some sort of classification. Second, to construct explicit examples.

The main result in [CDVV] establishes a correspondence between elements of the family and certain types of twisted product decompositions of simply-connected Riemannian manifolds of constant sectional curvature c. This is then used to attack the second problem. For a special type of such twisted product decompositions, namely, the conformally flat ones, the associated Lagrangian submanifolds are explicitly determined. However, although some interesting submanifolds are produced with this method, no examples with constant sectional curvature $c \geq 0$ and dimension $n > 2$ are obtained besides the flat Clifford tori.

A different approach is used in [DT_3]. First, the general correspondence obtained in [DT_1] between flat n–dimensional submanifolds with flat normal bundle in complex flat space $\mathbb{C}^n$ and solutions of a certain system of PDE's is considered. Then, the solutions associated to flat Lagrangian submanifolds are characterized. This paves the way for the *Ribaucour transformation*, extended from surface theory to higher dimensions in [DT_2], to be applied. It is shown that the set of Ribaucour transforms of a given flat Lagrangian n–dimensional submanifold of $\mathbb{C}^n$ contains an $(n+1)$–parameter family of submanifolds in the same class, which admit explicit parametrizations in terms of solutions of a completely integrable linear first order system of PDE's. In particular, non-trivial examples are produced in any dimension.

In this paper we show that a similar program can be carried out to the non-flat case. Our main achievement is a process to generate parametrizations of a family of Lagrangian isometric immersions $f\colon M^n(c) \to \widetilde{M}^n(4c)$ starting from a given one, which are given in terms of solutions of a linear first order system of PDE's. In particular, parametrizations in terms of elementary functions of examples with arbitrary dimension and curvature are provided. An outline of the paper is given below.

Totally real submanifolds of complex space forms have been shown by Reckziegel ([Re_1]) to be precisely the ones that admit *horizontal lifts* into the bundle space of the Hopf fibrations. A detailed study was made in [Re_2] of horizontal isometric immersions into the bundle space of the canonical fibration of a general Sasakian manifold, in particular the Hopf fibration onto complex projective space. In §2 we show how some of these results can be adapted to the Hopf fibration of anti-de-Sitter space time onto complex hyperbolic space and provide a brief and unified account of both cases. As a result, the investigation of the aforementioned problems for Lagrangian submanifolds is reduced to the investigation of similar questions for horizontal n-dimensional submanifolds with constant sectional curvature c of either the Euclidean sphere or the anti-de-Sitter space time of dimension $2n+1$ and the same curvature c, according to $c > 0$ or $c < 0$, respectively. These submanifolds can be shown to have flat normal bundle (cf. Corollary 2 below), thus their study fits into the general theory of constant curvature submanifolds with flat normal bundle of pseudo-Riemannian space forms. We then make use of the fact that submanifolds in this last class are in correspondence with solutions of certain systems of PDE's (cf. [DT_1]– [DT_3] and Theorem 4 below) and characterize in §3 those solutions which are associated to horizontal isometric immersions.

At this point the Ribaucour transformation comes into play. We show in §4 that the set of Ribaucour transforms of a given n–dimensional horizontal submanifold of constant sectional curvature c of either the Euclidean sphere or the anti-de-Sitter space time of dimension $2n+1$ and the same curvature c contains an $(n+1)$–parameter family of submanifolds in the same class, which can be parametrized in terms of solutions of a completely integrable linear first order system of PDE's. The analytical counterpart of this result is a process to generate a family of new solutions of the associated PDE's from a given one.

In §5 we derive a permutability formula which provides a simple algebraic procedure to construct further examples once two Ribaucour transforms of a given submanifold are known. This has also an analytical interpretation in terms of the associated PDE's.

In the last section, we apply our method to construct, as far as we know, the first explicit examples of non-totally geodesic Lagrangian isometric immersions $f\colon M^n(c) \to \widetilde{M}^n(4c)$ with $c > 0$ and $n > 2$, as well as similar examples for $c < 0$.

2 The Hopf fibrations and horizontal isometric immersions

Let $\mathbb{C}^{n+1}_\epsilon$ denote the complex number $(n+1)$–space endowed with the pseudo-Euclidean metric

$$g_\epsilon = \epsilon dz_1 d\bar{z}_1 + \sum_{j=2}^{n+1} dz_j d\bar{z}_j, \quad \epsilon = \pm 1,$$

and let

$$\mathbb{S}^{2n+1}(c) = \{z \in \mathbb{C}^{n+1}_\epsilon \ : \ g_\epsilon(z,z) = \frac{1}{c}, \ \epsilon c > 0\}$$

stand for either the standard Euclidean sphere or the *anti–de–Sitter space time* of dimension $(2n{+}1)$ and constant sectional curvature c, according to $\epsilon = 1$ or $\epsilon = -1$, respectively. The complex numbers act on $\mathbb{C}^{n+1}_\epsilon$ by

$$z = (z_1, \ldots, z_{n+1}) \ \longrightarrow \ \lambda z = (\lambda z_1, \ldots, \lambda z_{n+1}).$$

The quotient space $\widetilde{M}^n(4c)$ of $\mathbb{S}^{2n+1}(c)$ under the identification induced by this action is the complex projective space $\mathbb{CP}^n(4c)$ or the complex hyperbolic space $\mathbb{CH}^n(4c)$ of complex dimension n and constant holomorphic sectional curvature $4c$, according to $c > 0$ or $c < 0$, respectively. Let $\pi\colon \mathbb{S}^{2n+1}(c) \to \widetilde{M}^n(4c)$ denote the quotient map, $\widetilde{J}$ the complex structure on $\mathbb{C}^{n+1}_\epsilon$ defined by multiplication by i and ϕ its projection onto the tangent bundle of $\mathbb{S}^{2n+1}(c)$. Then, the complex structure J on $\widetilde{M}^n(4c)$ is given by

$$J \circ \pi_* = \pi_* \circ \phi.$$

Let $\overline{\nabla}$ be the connection on $\mathbb{S}^{2n+1}(c)$. Then, it is easily checked that the following properties are satisfied by the tensor ϕ and the unit *structure vector field* $\xi = \widetilde{J}\eta$, where $\eta/\sqrt{|c|}$ is the position vector on $\mathbb{S}^{2n+1}(c)$:

$$\phi\xi = 0, \quad \langle \phi, \xi \rangle = 0, \quad \overline{\nabla}\xi = \sqrt{|c|}\,\phi. \tag{1}$$

$$\phi^2 X = -X + \epsilon\langle X, \xi\rangle \xi. \tag{2}$$

$$\langle \phi X, \phi Y\rangle = \langle X, Y\rangle + \epsilon\langle X, \xi\rangle\langle Y, \xi\rangle. \tag{3}$$

$$\overline{\nabla}_X \phi Y = \phi\overline{\nabla}_X Y - \epsilon\sqrt{|c|}(\langle X, Y\rangle\xi - \langle Y, \xi\rangle X). \tag{4}$$

An isometric immersion $f\colon M \to \mathbb{S}^{2n+1}(c)$ of a Riemannian manifold is said to be *horizontal* (or *C-totally-real*, or an *integral submanifold*) if ξ is everywhere normal to TM along f. In Theorem 1 below we put together all the properties of horizontal isometric immersions $f\colon M \to \mathbb{S}^{2n+1}(c)$ that will be needed in the sequel. Most of them are stated and proved in [Re$_2$] for the case $\epsilon = 1 = c$, but the proofs carry over with slight modifications to the general case. They are included here for the convenience of the reader.

Theorem 1. *Let $f\colon M \to \mathbb{S}^{2n+1}(c)$ be a horizontal isometric immersion. Then the following holds:*
i) f is anti-invariant *with respect to ϕ, that is, ϕ carries each tangent space of M into its corresponding normal space.*
ii) The second fundamental form $\alpha_f\colon TM \times TM \to TM^\perp$ of f takes its values in the subbundle orthogonal to ξ and satisfies

$$\phi\alpha_f(X, Y) = -A_{\phi Y}X, \quad \text{for all } \ X, Y \in TM, \tag{5}$$

where A_ζ stands for the shape operator in the normal direction ζ.
iii) The normal connection and normal curvature tensor of f satisfy

$$\begin{aligned} \nabla^\perp_X \phi Y &= \phi\nabla_X Y - \epsilon\sqrt{|c|}\langle X, Y\rangle\xi \\ R^\perp(X, Y)\xi &= 0 \\ \langle R^\perp(X, Y)\phi Z, \phi W\rangle &= \langle R(X, Y)Z, W\rangle - c(\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle) \end{aligned} \tag{6}$$

Proof: It follows easily from (1)–(3) and the Gauss equation that

$$\langle \alpha_f(X, Y), \xi\rangle = \sqrt{|c|}\ \langle X, \phi Y\rangle$$

for all tangent vectors X, Y. Since the term on the left-hand-side is symmetric and that on the right-hand-side is anti-symmetric, we conclude that both terms vanish. This proves i) and the first half of ii). By comparing the tangent and normal components of (4) we get (5) and the first of formulas (6). The second follows from the Ricci equation and $A_\xi = 0$. Finally, the last of equations (6) follows by a straightforward computation using the first one together with equations (1) and (3). ■

Part *iii*) of Theorem 1 has the following immediate consequence.

Corollary 2. *A horizontal isometric immersion $f\colon M^n \to \mathbb{S}^{2n+1}(c)$ has flat normal bundle if and only if M^n has constant sectional curvature c.*

The next result shows that studying Lagrangian isometric immersions $g\colon M \to \widetilde{M}^n(4c)$ is equivalent to doing the same for horizontal isometric immersions $f\colon M \to \mathbb{S}^{2n+1}(c)$. We refer to [Re$_1$] or [Re$_2$] for a proof.

Theorem 3. [Re$_1$] *If $f\colon M \to \mathbb{S}^{2n+1}(c)$ is horizontal then $g = \pi \circ f$ is Lagrangian. Conversely, let $g\colon M \to \widetilde{M}^n(4c)$ be a Lagrangian isometric immersion and let $(x_0, y_0) \in M \times \mathbb{S}^{2n+1}(c)$ be some initial data with $g(x_0) = \pi(y_0)$. Then, there exist a Riemannian manifold $\hat{M}$, an isometric covering map $\tau\colon \hat{M} \to M$, a horizontal isometric immersion $\hat{f}\colon \hat{M} \to \mathbb{S}^{2n+1}(c)$ and a point $\hat{x} \in \hat{M}$ such that $\pi \circ \hat{f} = g \circ \tau$, $\tau(\hat{x}) = x_0$ and $f(\hat{x}) = y_0$.*

3 Horizontal submanifolds of constant sectional curvature

Given an isometric immersion $f\colon M^n \to \mathbb{Q}_s^N(c)$ of a Riemannian manifold into a pseudo–Riemannian space form of constant sectional curvature c and index s, we denote by $N_1^f(x)$ the *first normal space* of f at $x \in M^n$, which is the subspace of $T_xM^\perp$ spanned by the image of the second fundamental form α_f at x. We say that $N_1^f(x)$ is *nondegenerate* if $N_1^f(x) \cap N_1^f(x)^\perp = \{0\}$. We also denote by $\nu_f(x)$ the *index of relative nullity* of f at x, defined as the dimension of the kernel of α_f at x. From now on we deal for simplicity only with submanifolds whose index of relative nullity is everywhere vanishing.

The following result was proved in [DT$_1$] for Lorentzian space forms as ambient spaces (cf. Proposition 4, Lemma 5 and Theorem 7). We take the opportunity to point out that the statement of Proposition 4 contains an unnecessary extra assumption.

Theorem 4. *Assume that $M^n(c)$ is simply connected and let $f\colon M^n(c) \to \mathbb{Q}_s^N(c)$ be an isometric immersion with flat normal bundle and $\nu_f \equiv 0$. If $s \geq 1$, suppose further that N_1^f is nondegenerate everywhere. Then $N \geq 2n$ and there exist a global principal coordinate system $(u_1, \ldots, u_n)$ on $M^n(c)$, an orthonormal normal frame $\xi_1, \ldots, \xi_{N-n}$ and smooth functions $v_1, \ldots, v_n > 0$ and h_{ir}, $1 \leq i \leq n$, $n+1 \leq r \leq N-n$, such that*

$$ds^2 = \sum_i v_i^2\, du_i^2, \quad \alpha_f(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}) = v_i \delta_{ij} \xi_i \tag{7}$$

and

$$\nabla_{\frac{\partial}{\partial u_i}} X_j = h_{ji} X_i, \quad \nabla^\perp_{\frac{\partial}{\partial u_i}} \xi_s = h_{is} \xi_i, \quad 1 \leq i \neq j \leq n, \ \ 1 \leq s \neq i \leq N-n, \tag{8}$$

where $X_i = (1/v_i)\partial/\partial u_i$ and $h_{ij} = (1/v_i)\partial v_j/\partial u_i$ for $i \neq j$. Moreover, the pair (v, h), where $v = (v_1, \ldots, v_n)$ and $h = (h_{is})$, satisfies the completely integrable system of PDEs

$$(I) \quad \begin{cases} i)\ \dfrac{\partial v_i}{\partial u_j} = h_{ji} v_j, \quad ii)\ \dfrac{\partial h_{ij}}{\partial u_i} + \dfrac{\partial h_{ji}}{\partial u_j} + \displaystyle\sum_k h_{ki} h_{kj} + c v_i v_j = 0, \\[2ex] iii)\ \dfrac{\partial h_{is}}{\partial u_j} = h_{ij} h_{js}, \quad iv)\ \epsilon_j \dfrac{\partial h_{ij}}{\partial u_j} + \epsilon_i \dfrac{\partial h_{ji}}{\partial u_i} + \displaystyle\sum_s \epsilon_s h_{is} h_{js} = 0, \end{cases}$$

where always $i \neq j$, $\{k, s\} \cap \{i, j\} = \emptyset$ and $\epsilon_s = \langle \xi_s, \xi_s \rangle$.

Conversely, let (v, h) be a solution of (I) on an open simply connected subset $U \subset \mathbb{R}^n$ such that $v_i \neq 0$ everywhere. Then there exists an immersion $f\colon U \to \mathbb{Q}_s^N(c)$ with flat normal bundle, $\nu_f \equiv 0$, nondegenerate first normal bundle of rank n and induced metric $ds^2 = \sum_i v_i^2\, du_i^2$ of constant sectional curvature c.

Proof: Let $X_1, \ldots, X_n$ be an orthonormal principal frame for f. By the Gauss equations and the assumption on nondegeneracy of N_1^f when $s \geq 1$, the vectors $\eta_i = \alpha_f(X_i, X_i)$, $1 \leq i \leq n$, are pairwise orthogonal and $\|\eta_i\| \neq 0$ for $1 \leq i \leq n$. Set $\eta_i = v_i^{-1}\xi_i$, where $v_i > 0$ and $\xi_1, \ldots, \xi_n$ are orthonormal. The Codazzi equations yield

$$\nabla_{X_i} X_j = v_i^{-1} X_j(v_i) X_i \quad \text{and} \quad \nabla^\perp_{X_i} \xi_j = v_i^{-1} X_i(v_j)\xi_i, \;\; i \neq j. \tag{9}$$

It follows from the first equation that $[v_i X_i, v_j X_j] = 0$ for $i \neq j$, hence there exists a coordinate system $(u_1, \ldots, u_n)$ on $M^n(c)$ with $\partial/\partial u_i = v_i X_i$ for $1 \leq i \leq n$. Then, (9) gives the first equations in (8) and also the second ones for $1 \leq s \neq i \leq n$. On the other hand, an easy calculation using the second equations in (9) shows that the normal connection of f induces a flat connection on $N_1^{f\perp}$. The second equation in (8) for $n+1 \leq s \neq i \leq N-n$ follows by choosing $\xi_{n+1}, \ldots, \xi_{N-n}$ to be a parallel orthonormal frame of $N_1^{f\perp}$ with respect to this connection. Using (8) to express that $ds^2 = \sum_i v_i^2\, du_i^2$ has constant sectional curvature c and that f has flat normal bundle yields $ii)$, $iii)$ and $iv)$ of system (I).

For the converse, we consider on U the metric $ds^2 = \sum_i v_i^2\, du_i^2$, and verify from $i)$, $ii)$ and $iii)$ that it has constant sectional curvature c. Set $M^n(c) = \{U, ds^2\}$. To conclude the proof from the Fundamental Theorem of Submanifolds, consider the trivial vector bundle $E = M^n(c) \times \mathbb{R}^{N-n}$, where $\mathbb{R}^{N-n} = \mathrm{span}\{e_1, \ldots, e_{N-n}\}$ is endowed with the inner product

$$\langle e_s, e_{s'} \rangle = \epsilon_s \delta_{ss'}.$$

The compatible vector bundle connection ∇' defined by

$$\nabla'_{\partial/\partial u_i} e_s = h_{is} e_i, \quad i \neq s,$$

is flat from equations $iii)$ and $iv)$. Define $\alpha \in C^\infty(Hom(TM \times TM, E))$ by

$$\alpha\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right) = v_i \delta_{ij} e_i. \tag{10}$$

Clearly, α satisfies the Gauss equations. The Codazzi equations follow from $i)$ and the Ricci equations are satisfied because ∇' is flat and α is orthogonally diagonalizable. ■

If $f\colon M^n(c) \to \mathbb{S}^{2n+1}(c)$ is an isometric immersion with flat normal bundle, $\nu_f \equiv 0$ and nondegenerate first normal spaces everywhere when $c < 0$, then N_1^f is a vector subbundle of rank n of $T_fM^\perp$ which is everywhere either Riemannian or Lorentzian. For the isometric immersions we are most interested in, namely, horizontal isometric immersions, the first possibility always holds. In fact, by part $ii)$ of Theorem 1, in this case N_1^f is precisely the vector subbundle of $T_fM^\perp$ orthogonal to the structure vector field ξ. It will be convenient to have Theorem 4 explicitly restated for this particular case.

Corollary 5. *Assume that $M^n(c)$ is simply connected and let $f\colon M^n(c) \to \mathbb{S}^{2n+1}(c)$ be an isometric immersion with flat normal bundle, $\nu_f \equiv 0$, and Riemannian first normal bundle when $c < 0$. Then there exist a global principal coordinate system $(u_1, \ldots, u_n)$ on $M^n(c)$, a smooth orthonormal normal frame $\xi_1, \ldots, \xi_{n+1}$ and smooth functions $v_1, \ldots, v_n > 0$, $\rho_1, \ldots, \rho_n$ such that*

$$ds^2 = \sum_i v_i^2\, du_i^2, \quad \alpha_f\Big(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\Big) = v_i \delta_{ij} \xi_i$$

and

$$\nabla_{\frac{\partial}{\partial u_i}} X_j = h_{ji} X_i, \quad \nabla^\perp_{\frac{\partial}{\partial u_i}} \xi_j = h_{ij} \xi_i, \ \ i \neq j, \ \ \nabla^\perp_{\frac{\partial}{\partial u_i}} \xi_{n+1} = \rho_i \xi_i, \tag{11}$$

where $X_i = (1/v_i)\partial/\partial u_i$ and $h_{ij} = (1/v_i)\partial v_j/\partial u_i$ for $i \neq j$. Moreover, the triple (v, h, ρ), where $v = (v_1, \ldots, v_n)$, $h = (h_{ij})$ and $\rho = (\rho_1, \ldots, \rho_n)$, satisfies the completely integrable system of PDEs

$$(II)\ \begin{cases} i)\ \dfrac{\partial v_i}{\partial u_j} = h_{ji} v_j, \quad ii)\ \dfrac{\partial h_{ik}}{\partial u_j} = h_{ij} h_{jk}, \quad iii)\ \dfrac{\partial \rho_i}{\partial u_j} = h_{ij} \rho_j, \\ iv)\ \dfrac{\partial h_{ij}}{\partial u_i} + \dfrac{\partial h_{ji}}{\partial u_j} + \displaystyle\sum_k h_{ki} h_{kj} + c v_i v_j = 0, \\ v)\ \dfrac{\partial h_{ij}}{\partial u_j} + \dfrac{\partial h_{ji}}{\partial u_i} + \displaystyle\sum_k h_{ik} h_{jk} + \epsilon \rho_i \rho_j = 0, \ \ \epsilon = c/|c|, \ \ i \neq j \neq k \neq i. \end{cases}$$

Conversely, let (v, h, ρ) be a solution of (II) on an open simply connected subset $U \subset \mathbb{R}^n$ such that $v_i \neq 0$ everywhere. Then there exists an immersion $f\colon U \to \mathbb{S}^{2n+1}(c)$ with flat normal bundle, $\nu_f \equiv 0$, Riemannian first normal bundle of rank n and induced metric $ds^2 = \sum_i v_i^2\, du_i^2$ of constant sectional curvature c.

We call (v, h, ρ) the *associated triple* to $f\colon M^n(c) \to \mathbb{S}^{2n+1}(c)$. Our next result characterizes the triples associated to horizontal isometric immersions.

Theorem 6. *The isometric immersion $f\colon M^n(c) \to \mathbb{S}^{2n+1}(c)$ is horizontal if and only if its associated triple (v, h, ρ) satisfies*

$$h_{ij} = h_{ji} \quad \textit{and} \quad \rho_i = \sqrt{|c|}\, v_i. \tag{12}$$

Proof: Assume first that $f\colon M^n(c) \to \mathbb{S}^{2n+1}(c)$ is horizontal. Let $X_1, \ldots, X_n$ be an orthonormal tangent frame of principal directions and $\xi_1, \ldots, \xi_{n+1}$ an orthonormal normal frame as in Corollary 5. We easily obtain from part *ii)* of Theorem 1 that, up to signs, $\phi X_i = \xi_i$ and $\xi_{n+1} = \xi \circ f$. It now follows from (11) and the first of formulas (6) that

$$h_{ij} = \langle \nabla^\perp_{\frac{\partial}{\partial u_i}} \xi_j, \xi_i \rangle = \langle \nabla^\perp_{\frac{\partial}{\partial u_i}} \phi X_j, \phi X_i \rangle = \langle \nabla_{\frac{\partial}{\partial u_i}} X_j, X_i \rangle = h_{ji}.$$

and

$$\rho_i = \langle \nabla^\perp_{\frac{\partial}{\partial u_i}} \xi_{n+1}, \xi_i \rangle = \langle \nabla^\perp_{\frac{\partial}{\partial u_i}} \xi, \phi X_i \rangle = -\langle \nabla^\perp_{\frac{\partial}{\partial u_i}} \phi X_i, \xi \rangle = \sqrt{|c|}\, v_i.$$

Conversely, assume that the solution (v,h,ρ) of system (II) associated to $f\colon M^n(c)\to \mathbb{S}^{2n+1}(c)$ satisfies (12). Let $F=i\circ f$ be the composition of f with the umbilical inclusion i of $\mathbb{S}^{2n+1}(c)$ into the underlying flat space $\mathbb{C}^{n+1}_\epsilon$. Define a complex structure $\tilde{J}$ on $TM\oplus T_FM^\perp$ by setting

$$\tilde{J}X_i=\xi_i,\quad \tilde{J}(\sqrt{|c|}F)=\xi_{n+1}.$$

Denote by $\widetilde{\nabla}$ the derivative in $\mathbb{C}^{n+1}_\epsilon$. Then, using the symmetry of h and (11), it is easy to verify that

$$\widetilde{\nabla}_{X_i}\tilde{J}X_j=\tilde{J}\widetilde{\nabla}_{X_i}X_j\quad\text{and}\quad \widetilde{\nabla}_{X_i}\tilde{J}\xi_j=\tilde{J}\widetilde{\nabla}_{X_i}\xi_j.$$

On the other hand, using that $\rho_i=\sqrt{|c|}v_i$ we get from (11) that

$$\widetilde{\nabla}_{X_i}\tilde{J}(\sqrt{|c|}F)=\widetilde{\nabla}_{X_i}\xi_{n+1}=\sqrt{|c|}\xi_i=\tilde{J}(\sqrt{|c|}X_i)=\tilde{J}\widetilde{\nabla}_{X_i}\sqrt{|c|}F.$$

Therefore, $\tilde{J}$ is parallel with respect to $\widetilde{\nabla}$ along F, hence it is the restriction to $TM\oplus T_FM^\perp$ of an almost complex structure in $\mathbb{C}^{n+1}_\epsilon$. Clearly, $f\colon M^n(c)\to\mathbb{S}^{2n+1}(c)$ is horizontal with respect to its projection onto the tangent bundle of $\mathbb{S}^{2n+1}(c)$. ∎

Corollary 7. *Let $M^n(c)$ be simply connected and let $f\colon M^n(c)\to\mathbb{S}^{2n+1}(c)$ be a horizontal isometric immersion with $\nu_f\equiv 0$. Then there exists a global principal coordinate system $(u_1,\ldots,u_n)$ on $M^n(c)$ with*

$$ds^2=\sum_i v_i^2\,du_i^2,\ v_i>0\quad\textit{and}\quad \alpha_f(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j})=\delta_{ij}\phi\frac{\partial}{\partial u_i},$$

where $v=(v_1,\ldots,v_n)$ and $h=(h_{ij})$ satisfy the completely integrable system of PDEs

$$(III)\quad\begin{cases} i)\ \dfrac{\partial v_i}{\partial u_j}=h_{ji}v_j, & ii)\ \left(\displaystyle\sum_k\frac{\partial}{\partial u_k}\right)h_{ij}+cv_iv_j=0,\\[2ex] iii)\ \dfrac{\partial h_{ij}}{\partial u_k}=h_{ik}h_{jk}, & h_{ij}=h_{ji},\quad i\neq j\neq k\neq i.\end{cases}$$

Conversely, let (v,h) be a solution of (III) on an open simply connected subset $U\subset\mathbb{R}^n$ such that $v_i\neq 0$ everywhere. Then there exists a horizontal immersion $f\colon U\to\mathbb{S}^{2n+1}(c)$ with $\nu_f\equiv 0$ and induced metric $ds^2=\sum_i v_i^2\,du_i^2$ of constant sectional curvature c.

Remarks 8. 1) The main result in [CDVV] mentioned in the introduction can be derived from Corollary 7. In fact, let $U={}_{v_1}I_1\times\cdots\times_{v_n}I_n$ be a twisted product of intervals and let $\omega=\sum_i v_idu_i$ be the associated *twistor* one–form defined in [CDVV]. Then, ω being closed is equivalent to the symmetry of $h=(h_{ij})$. Moreover, under this assumption U has constant sectional curvature c if and only if the pair (v,h) satisfies system (III).

2) Orthogonal coordinate systems in euclidean space whose associated pairs (v,h) satisfy system (III) with $c=0$ were named E-systems by Bianchi ([Bi]), after

Egorov who first studied them. It was shown in [DT$_3$] that E-systems are precisely the principal coordinate systems of Lagrangian isometric immersions $f\colon\ M^n(0) \to \mathbb{C}^n$ with $\nu_f \equiv 0$. On the other hand, it follows from Corollary 7 that orthogonal coordinate systems whose associated pairs (v, h) satisfy (III) for an arbitrary $c \in \mathbb{R}$ are precisely the principal coordinate systems of horizontal isometric immersions $f\colon\ M^n(c) \to \mathbb{S}^{2n+1}(c)$ with $\nu_f \equiv 0$.

Let (v, h) be a solution of system (III) on an open simply connected subset $U \subset \mathbb{R}^n$ with $v_i \neq 0$ everywhere. In order to determine the corresponding horizontal isometric immersion $f\colon\ M^n(c) \to \mathbb{S}^{2n+1}(c)$ or, equivalently, to determine $F = i \circ f\colon\ M^n(c) \to \mathbb{C}^{n+1}_\epsilon$, one has to integrate the system of PDE's

$$(IV)\quad \begin{cases} i)\ \dfrac{\partial F}{\partial u_i} = v_i X_i, \qquad ii)\ \dfrac{\partial X_i}{\partial u_j} = h_{ij} X_j, \ \ i \neq j, \\ iii)\ \dfrac{\partial X_i}{\partial u_i} = -\displaystyle\sum_{k \neq i} h_{ki} X_k + i X_i - c v_i F, \end{cases}$$

with initial conditions $(F(u_0), X_1(u_0), \ldots, X_n(u_0))$ at some point $u_0 \in U$ chosen so that

$$\langle X_i(u_0), X_j(u_0)\rangle = \langle iX_i(u_0), X_j(u_0)\rangle = 0, \ \ i \neq j, \quad \langle X_i(u_0), X_i(u_0)\rangle = 1,$$

$$\langle F(u_0), X_i(u_0)\rangle = \langle iF(u_0), X_i(u_0)\rangle = 0 \quad \text{and} \quad \langle F(u_0), F(u_0)\rangle = \frac{1}{c}.$$

It is in general a difficult task both to find a solution of the nonlinear system (III) and to integrate the corresponding system (IV). Nevertheless, the difficulties involved in both steps were overcome in [CDVV] for solutions of system (III) satisfying $v_1 = \cdots = v_n$. However, it turns out that for $n \geq 3$ no such solutions exists if $c > 0$ and the only one with $c = 0$ is the trivial solution $v_i = \textit{constant}$ for $1 \leq i \leq n$. In the latter case, the associated submanifolds are Clifford tori. In the next section, we develop an alternative way of producing examples, by making use of the Ribaucour transformation.

4 The Ribaucour transformation

Classically, two surfaces in $\mathbb{R}^3$ are said to correspond by a Ribaucour transformation when they are related by a diffeomorphism preserving lines of curvature such that the normals at corresponding points intersect at a point equidistant to them. The surfaces can then be viewed as the focal surfaces of the 2–parameter congruence of spheres with centers at the intersecting points and with the common distances to corresponding points as radii.

This notion was extended in [DT$_2$] for isometric immersions $f\colon\ M^n \to \mathbb{Q}^N_s(c)$ as follows. First, two isometric immersions $f\colon\ M^n \to \mathbb{R}^N_s := \mathbb{Q}^N_s(0)$ and $\tilde{f}\colon\ \widetilde{M}^n \to \mathbb{R}^N_s$ are said to be related by a *Ribaucour transformation* (or $\tilde{f}$ is a *Ribaucour transform* of f) when there exist a vector bundle isometry $\mathcal{P}\colon\ f^*\mathrm{T}\mathbb{R}^N_s \to \tilde{f}^*\mathrm{T}\mathbb{R}^N_s$ covering a diffeomorphism $\Psi\colon\ M^n \to\ \widetilde{M}^n$, a smooth section $\omega \in \Gamma((f^*\mathrm{T}\mathbb{R}^N_s)^*)$ and a symmetric tensor D on M^n such that $\|f - \tilde{f} \circ \Psi\| \neq 0$ everywhere,

(a) $\mathcal{P}(Z) - Z = \omega(Z)(f - \tilde{f} \circ \Psi)$, for all $Z \in \Gamma(f^*\mathrm{T}\mathbb{R}^N_s)$,
and
(b) $\mathcal{P} \circ f_* \circ D = \tilde{f}_* \circ \Psi_*$.

When $c \neq 0$, let $i\colon\ \mathbb{Q}^N_s(c) \to \mathbb{R}^{N+1}_{s+\epsilon_0}$ be an umbilical inclusion, where $\epsilon_0 = 1$ or 0, according to $c < 0$ or $c > 0$, respectively. Set $F = i \circ f$ and $\tilde{F} = i \circ \tilde{f}$. Then $\tilde{f}$ is said to be a Ribaucour transform of f if $\tilde{F}$ is a Ribaucour transform of F determined by a 4-tuple $(\Psi, \mathcal{P}, D, \omega)$ such that $\mathcal{P}(F) = \tilde{F} \circ \Psi$ and $\omega(F) = -1$. Geometrically, it is easy to verify that for any $Z \in \mathrm{T}_{f(x)}\mathbb{Q}^N_s(c)$ the geodesics in $\mathbb{Q}^N_s(c)$ through $f(x)$ and $\tilde{f}(\Psi(x))$ tangent to Z and $\mathcal{P}(Z)$, respectively, intersect at a point which is at a common distance to $f(x)$ and $\tilde{f}(\Psi(x))$.

The following result was proved in [DT$_2$].

Theorem 9. *Let $f\colon\ M^n \to \mathbb{Q}^N_s(\tilde{c})$ be an isometric immersion with a Ribaucour transform $\tilde{f}\colon\ \widetilde{M}^n \to \mathbb{Q}^N_s(\tilde{c})$. Then there exist $\varphi \in C^\infty(M)$ and $\beta \in T^\perp_f M$ satisfying*

$$\alpha_f(\nabla\varphi, X) + \nabla^\perp_X \beta = 0 \ \textit{ for all } X \in TM, \tag{13}$$

such that

$$\tilde{F} \circ \Psi = F - 2\varphi\nu(F_*\nabla\varphi + \beta + \tilde{c}\varphi F), \quad \nu^{-1} = \|\nabla\varphi\|^2 + \langle\beta, \beta\rangle + \tilde{c}\varphi^2. \tag{14}$$

Conversely, for (φ, β) satisfying (13), let $U \subset M^n$ be an open subset where $D = I - 2\varphi\nu(Hess\,\varphi - A^f_\beta + \tilde{c}\varphi I)$ is invertible and let $\tilde{F}$ be defined on U by (14) with $\Psi = id$. Then $\tilde{F} = i \circ \tilde{f}$, where $\tilde{f}$ is a Ribaucour transform of $f|_U$.

Moreover, suppose that M^n has constant sectional curvature c. If also $\widetilde{M}^n$ has constant sectional curvature c and $n \geq 3$, then there exists $C \in \mathbb{R}$ such that

$$Hess\,\varphi - (1 - C)A^f_\beta + (\tilde{c} + C(c - \tilde{c}))\varphi I = 0 \tag{15}$$

and

$$\nu^{-1} - C(\langle\beta, \beta\rangle - (c - \tilde{c})\varphi^2) = 0. \tag{16}$$

Conversely, if (φ, β) satisfies (15) then the left hand side of (16) is a constant $K \in \mathbb{R}$. If initial conditions in (15) are chosen so that $K = 0$ then also $\widetilde{M}^n$ has constant sectional curvature c.

Theorem 9 yields the following for isometric immersions $f\colon\ M^n(c) \to \mathbb{Q}^N_s(c)$ as in Theorem 4. For simplicity of notation, we agree that the indexes i, j always range on $\{1, \ldots, n\}$ and s on $\{1, \ldots, N - n\}$.

Theorem 10. *Any Ribaucour transform $\tilde{f}\colon\ \widetilde{M}^n(c) \to \mathbb{Q}^N_s(c)$ of f is given by*

$$\tilde{F} \circ \Psi := i \circ \tilde{f} \circ \Psi = F - 2\varphi\nu(\sum_i \gamma_i F_* X_i + \sum_s \beta_s \xi_s + c\varphi F), \tag{17}$$

where $(\varphi,\gamma,\beta) := (\varphi,\gamma_1,\ldots,\gamma_n,\beta_1,\ldots,\beta_{N-n})$ *is a solution of the completely integrable linear system of first order*

$$\mathcal{R}_0 \;=\; \begin{cases} i)\ \dfrac{\partial\varphi}{\partial u_i} = v_i\gamma_i, \quad ii)\ \dfrac{\partial\gamma_j}{\partial u_i} = h_{ji}\gamma_i,\ \ i\neq j, \\[2ex] iii)\ \dfrac{\partial\gamma_i}{\partial u_i} = (1-C)\beta_i - \displaystyle\sum_{j\neq i} h_{ji}\gamma_j - cv_i\varphi, \ \ C\in\mathbb{R}, \\[2ex] iv)\ \epsilon_s\dfrac{\partial\beta_s}{\partial u_i} = \epsilon_i h_{is}\beta_i,\ \ s\neq i, \quad v)\ \dfrac{\partial\beta_i}{\partial u_i} = -\gamma_i - \displaystyle\sum_{s\neq i} h_{is}\beta_s, \end{cases}$$

and $\nu^{-1} := \sum_i \gamma_i^2 + \sum_s \epsilon_s\beta_s^2 + c\varphi^2$ *satisfies*

$$\nu^{-1} - C\sum_s \epsilon_s\beta_s^2 = 0. \tag{18}$$

Furthemore, the pair $(\tilde v,\tilde h)$ *associated to* $\widetilde f$ *is given by*

$$\tilde v_i = v_i + \frac{2\varphi\epsilon_i\beta_i}{\sum_s \epsilon_s\beta_s^2}, \qquad \tilde h_{is} = h_{is} + \frac{2\epsilon_s\gamma_i\beta_s}{\sum_s \epsilon_s\beta_s^2}. \tag{19}$$

Conversely, for any solution (φ,γ,β) *of* $\mathcal{R}_0$ *the left hand side of (18) is a constant* $K\in\mathbb{R}$*. Assume that initial conditions in* $\mathcal{R}_0$ *have been chosen so that* $K=0$*, let* U *be an open subset where* $\tilde v_i \neq 0$*,* $1\le i\le n$*, and let* $\widetilde F$ *be defined by (17) with* $\Psi = id$*. Then* $\widetilde F = i\circ\widetilde f$*, where* $\widetilde f$ *is a Ribaucour transform of* $f|_U$ *whose induced metric has constant sectional curvature* c*.*

Proof: By Theorem 9, we have that $\widetilde F = i\circ\widetilde f$ is given by

$$\widetilde F\circ\Psi = F - 2\varphi\nu(F_*\nabla\varphi + \beta + c\varphi F), \quad \nu^{-1} = \|\nabla\varphi\|^2 + \langle\beta,\beta\rangle + c\varphi^2, \tag{20}$$

where (φ,β) satisfies (13),

$$\Phi := \operatorname{Hess}\varphi - A^f_\beta + c\varphi I = -CA^f_\beta \tag{21}$$

and

$$\nu^{-1} - C\langle\beta,\beta\rangle = 0. \tag{22}$$

Set $\nabla\varphi = \sum_i \gamma_i X_i$ and $\beta = \sum_s \beta_s\xi_s$. Then (20) takes the form (17) with $\nu^{-1} = \sum_i\gamma_i^2 + \sum_s\epsilon_s\beta_s^2 + c\varphi^2$ and (22) reduces to (18). Equation $i)$ of $\mathcal{R}_0$ merely expresses the definition of the γ_i's in coordinates, whereas (13) reduces to $iv)$ and $v)$. On the other hand, (13) is equivalent to $\mathcal{F}_* = F_*\Phi$, where $\mathcal{F} = F_*\nabla\varphi + \beta + c\varphi F$. Thus, $\omega = F_*\Phi$ is a closed one–form on $M^n(c)$ with values in $\mathbb{R}^{N+1}_{s+\epsilon_0}$. Since

$$d\omega(X,Y) = F_*(\nabla_X\Phi Y - \nabla_Y\Phi X - \Phi[X,Y]) + \alpha_F(X,\Phi Y) - \alpha_F(\Phi X, Y),$$

it follows that Φ is a Codazzi tensor on $M^n(c)$ such that

$$\alpha_f(X,\Phi Y) = \alpha_f(\Phi X, Y) \quad \text{for all } X,Y\in TM.$$

Thus, $\partial/\partial u_1, \ldots, \partial/\partial u_n$ diagonalize Φ. An easy computation shows that

$$\Phi\left(\frac{\partial}{\partial u_i}\right) = \left(\frac{\partial \gamma_i}{\partial u_i} - \beta_i + \sum_{j\neq i} h_{ji}\gamma_j + cv_i\varphi\right) X_i + \sum_{j\neq i}\left(\frac{\partial \gamma_j}{\partial u_i} - h_{ji}\gamma_i\right) X_j, \tag{23}$$

hence equation (21) reduces to ii) and iii). This completes the proof of the first assertion.

It was shown in [DT$_2$] that the second fundamental forms of $\tilde{F}$ and F are related by

$$\alpha_{\tilde{F}}(\Psi_* X, \Psi_* Y) = \mathcal{P}\left(\alpha_F(DX, Y) + 2\nu\langle \Phi X, DY\rangle(\beta + c\varphi F)\right), \tag{24}$$

where $\mathcal{P} = I - 2\nu\mathcal{F}\mathcal{F}^*$ is the vector bundle isometry between the pulled–back bundles and $D = I - 2\varphi\nu\Phi$. Hence, $u_1, \ldots, u_n$ are also principal coordinates for $\tilde{f}$ and the coordinate vector fields of f and $\tilde{f}$ are related by

$$\Psi_*(\partial/\partial u_i) = (\tilde{\partial}/\partial u_i) \circ \Psi, \quad 1 \le i \le n.$$

By (21), we have that

$$\Phi(\partial/\partial u_i) = -C\epsilon_i\beta_i. \tag{25}$$

Using (22), we obtain that

$$D(\partial/\partial u_i) = \tilde{v}_i X_i, \tag{26}$$

hence the first of formulas (19) follows from

$$\tilde{F}_*((\tilde{\partial}/\partial u_i) \circ \Psi) = \tilde{F}_*\Psi_*(\partial/\partial u_i) = \mathcal{P}F_* D\partial/\partial u_i = \tilde{v}_i\mathcal{P}f_* X_i.$$

Let $\tilde{\xi}_1, \ldots, \tilde{\xi}_{N-n}$ be the orthonormal normal frame for $\tilde{f}$ given by Theorem 4. We claim that

$$\tilde{\xi}_s = \mathcal{P}(\xi_s - 2\langle\beta, \beta\rangle^{-1}\epsilon_s\beta_s\beta). \tag{27}$$

Using (7), (24) and $\mathcal{P}(F) = \tilde{F}$, we have for $1 \le i \le n$ that

$$\begin{aligned} \tilde{v}_i\tilde{\xi}_i &= \alpha_{\tilde{f}}\left(\frac{\tilde{\partial}}{\partial u_i}, \frac{\tilde{\partial}}{\partial u_i}\right) = \alpha_{\tilde{F}}\left(\frac{\tilde{\partial}}{\partial u_i}, \frac{\tilde{\partial}}{\partial u_i}\right) + c\langle\frac{\tilde{\partial}}{\partial u_i}, \frac{\tilde{\partial}}{\partial u_i}\rangle\tilde{F} \\ &= \mathcal{P}\left(\alpha_F\left(D\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i}\right) + 2\nu\langle\Phi\frac{\partial}{\partial u_i}, D\frac{\partial}{\partial u_i}\rangle(\beta + c\varphi F) + c\tilde{v}_i^2 F\right), \end{aligned} \tag{28}$$

We conclude from (25), (26) and (28) that (27) holds for $1 \le i \le n$.

On the other hand, by part i) of Corollary 27 in [DT$_2$] we have that $\nabla^\perp_X\mathcal{P}\xi = \mathcal{P}\nabla^\perp_X\xi$ for all $X \in TM$ and $\xi \in T_fM^\perp$. Moreover, $\nabla^\perp_{\partial/\partial u_i}\beta = -\gamma_i\xi_i$ by (13) and (7). Using also equation iv) of $\mathcal{R}_0$ and (8), we easily get that

$$\nabla^\perp_{\partial/\partial u_i}(\xi_s - 2\langle\beta, \beta\rangle^{-1}\epsilon_s\beta_s\beta) = (h_{is} + 2\langle\beta, \beta\rangle^{-1}\epsilon_s\gamma_i\beta_s)(\xi_i - 2\langle\beta, \beta\rangle^{-1}\epsilon_i\beta_i\beta),$$

which proves our claim and shows that $\tilde{h}_{is}$ is given by (19) for any $1 \le i \le n$ and $1 \le s \le N - n$, $s \neq i$. Finally, the converse follows from Theorem 9 and the fact that D is invertible wherever $\tilde{v}_i \neq 0$, $1 \le i \le n$. ∎

Corollary 11. *Let (v, h) and (φ, γ, β) be solutions of (I) and $\mathcal{R}_0$, respectively. Then $(\tilde{v}, \tilde{h})$ given by (19) is a new solution of (I).*

Proof: Assume first that (v, h) and (φ, γ, β) are defined on a simply connected open subset U where $v_i, \tilde{v}_i$ are nowhere vanishing for $1 \leq i \leq n$. By Theorem 4 there exists an immersion $f\colon U \to \mathbb{Q}_s^N(c)$ with (v, h) as associated pair. By Theorem 10, (φ, γ, β) gives rise to a Ribaucour transform $\tilde{f}\colon U \to \mathbb{Q}_s^N(c)$ of f whose associated pair is $(\tilde{v}, \tilde{h})$. Then $(\tilde{v}, \tilde{h})$ is a new solution of (I) by Theorem 4. The general case can be verified by a direct computation. ∎

Corollary 12. *Let $f\colon M^n(c) \to \mathbb{S}^{2n+1}(c)$ be as in Corollary 5. Then any Ribaucour transform $\tilde{f}\colon \widetilde{M}^n(c) \to \mathbb{S}^{2n+1}(c)$ of f is given by*

$$\widetilde{F} \circ \Psi = i \circ \tilde{f} \circ \Psi = F - 2\varphi\nu\Big(\sum_i(\gamma_i F_* X_i + \beta_i \xi_i) + \psi\xi_{n+1} + c\varphi F\Big), \tag{29}$$

where $(\varphi, \psi, \gamma, \beta) := (\varphi, \psi, \gamma_1, \ldots, \gamma_n, \beta_1, \ldots, \beta_n)$ is a solution of the completely integrable linear system of first order

$$\mathcal{R}_1 \;= \begin{cases} i)\ \dfrac{\partial\varphi}{\partial u_i} = v_i\gamma_i, \quad ii)\ \epsilon\dfrac{\partial\psi}{\partial u_i} = \rho_i\beta_i, \quad iii)\ \dfrac{\partial\gamma_j}{\partial u_i} = h_{ji}\gamma_i, \ \ i \neq j, \\[2ex] iv)\ \dfrac{\partial\gamma_i}{\partial u_i} = (1 - C)\beta_i - \displaystyle\sum_{j\neq i} h_{ji}\gamma_j - cv_i\varphi, \quad C \in \mathbb{R}, \\[2ex] v)\ \dfrac{\partial\beta_j}{\partial u_i} = h_{ij}\beta_i, \ \ i \neq j, \quad vi)\ \dfrac{\partial\beta_i}{\partial u_i} = -\gamma_i - \displaystyle\sum_{j\neq i} h_{ij}\beta_j - \rho_i\psi, \end{cases}$$

and $\nu^{-1} := \sum_i(\gamma_i^2 + \beta_i^2) + \epsilon\psi^2 + c\varphi^2$ satisfies

$$\nu^{-1} - C\Big(\sum_i \beta_i^2 + \epsilon\psi^2\Big) = 0. \tag{30}$$

Furthemore, the triple $(\tilde{v}, \tilde{h}, \tilde{\rho})$ associated to $\tilde{f}$ is given by

$$\tilde{v}_i = v_i + \frac{2\varphi\beta_i}{\sum_i \beta_i^2 + \epsilon\psi^2}, \quad \tilde{h}_{ij} = h_{ij} + \frac{2\gamma_i\beta_j}{\sum_i \beta_i^2 + \epsilon\psi^2}, \quad \tilde{\rho}_i = \rho_i + \frac{2\epsilon\gamma_i\psi}{\sum_i \beta_i^2 + \epsilon\psi^2}. \tag{31}$$

Conversely, for any solution (φ, γ, β) of $\mathcal{R}_1$ the left hand side of (30) is a constant $K \in \mathbb{R}$. Assume that initial conditions in $\mathcal{R}_1$ have been chosen so that $K = 0$, let U be an open subset where $\tilde{v}_i \neq 0$, $1 \leq i \leq n$, and let $\widetilde{F}$ be defined by (29). Then $\widetilde{F} = i \circ \tilde{f}$, where $\tilde{f}$ is a Ribaucour transform of $f|_U$ whose induced metric has constant sectional curvature c.

Corollary 13. *Let (v, h, ρ) and $(\varphi, \psi, \gamma, \beta)$ be solutions of (II) and $\mathcal{R}_1$, respectively. Then $(\tilde{v}, \tilde{h}, \tilde{\rho})$ given by (31) is a new solution of (II).*

By putting Theorem 6 and Corollary 12 together we get a Ribaucour transformation for horizontal isometric immersions $f\colon M^n(c) \to \mathbb{S}^{2n+1}(c)$, which is the main result of this paper.

Theorem 14. *Let $f\colon M^n(c) \to \mathbb{S}^{2n+1}(c)$ be a horizontal isometric immersion with $\nu_f \equiv 0$. Then any horizontal Ribaucour transform $\tilde{f}\colon \widetilde{M}^n(c) \to \mathbb{S}^{2n+1}(c)$ of f is given by*

$$\tilde{F} \circ \Psi = i \circ \tilde{f} \circ \Psi = F - \frac{2D(D+i)\varphi}{(D^2+1)(\sum_i \gamma_i^2 + c\varphi^2)} \left(\sum_i \gamma_i F_* X_i + c\varphi F \right), \qquad (32)$$

where $(\varphi, \gamma) := (\varphi, \gamma_1, \ldots, \gamma_n)$ is a solution of the completely integrable linear system of first order

$$\mathcal{R}_2 \quad = \begin{cases} i)\ \dfrac{\partial \varphi}{\partial u_i} = v_i \gamma_i, \quad ii)\ \dfrac{\partial \gamma_j}{\partial u_i} = h_{ji}\gamma_i, \quad i \neq j, \\ iii)\ \dfrac{\partial \gamma_i}{\partial u_i} = -D\gamma_i - \sum_{j \neq i} h_{ji}\gamma_j - c v_i \varphi, \quad D \neq 0. \end{cases}$$

Moreover, the pair $(\tilde{v}, \tilde{h})$ associated to $\tilde{f}$ is given by

$$\tilde{v}_i = v_i + \frac{2D\varphi}{\sum_i \gamma_i^2 + c\varphi^2}\gamma_i, \quad \tilde{h}_{ij} = h_{ij} + \frac{2D}{\sum_i \gamma_i^2 + c\varphi^2}\gamma_i\gamma_j. \qquad (33)$$

Conversely, given a solution (φ, γ) of $\mathcal{R}_2$, let U be an open subset where $\tilde{v}_i \neq 0$ for $1 \leq i \leq n$ and let $\tilde{F}$ be defined on U by (32). Then $\tilde{F} = i \circ \tilde{f}$, where $\tilde{f}$ is a horizontal Ribaucour transform of $f|_U$ whose induced metric has constant sectional curvature c.

Proof: It follows from (31) and Theorem 6 that the Ribaucour transform $\tilde{f}$ of f is also horizontal if and only if $\gamma_i\beta_j = \gamma_j\beta_i$ and $\epsilon\gamma_i\psi = \sqrt{|c|}\varphi\beta_i$ for all $i \neq j$. Hence, there must exist a function μ such that $\gamma_j = \mu\beta_j$ for all j and $\mu\psi = \epsilon\sqrt{|c|}\varphi$. We claim that $\mu = D = constant$. In fact, from equations $iii)$ and $iv)$ of system $\mathcal{R}_1$ and symmetry of h, we get

$$h_{ji}\gamma_i = \frac{\partial \gamma_j}{\partial u_i} = \frac{\partial \mu}{\partial u_i}\beta_j + \mu h_{ij}\beta_i = \frac{\partial \mu}{\partial u_i}\beta_j + h_{ji}\gamma_i,$$

and our claim follows. We conclude from (30) that $C > 1$ and $D^2 = C - 1$. The remaining of the proof follows from Corollary 12. ∎

Corollary 15. *Let (v, h, ρ) be a solution of (III) and let (φ, γ) be a solution of $\mathcal{R}_2$. Then $(\tilde{v}, \tilde{h})$ given by (33) is a new solution of (III).*

5 The permutability formulas

Let $f\colon M^n(c) \to \mathbb{Q}_s^N(c)$ be an isometric immersion as in Theorem 4 with (v, h) as associated pair and let $(\varphi_k, \gamma^k, \beta^k)$, $1 \leq k \leq 2$, be solutions of $\mathcal{R}_0$ with constants C_k. The correspondent Ribaucour transforms $\tilde{f}_k\colon \widetilde{M}_k^n(c) \to \mathbb{Q}_s^N(c)$ of f are called R_{C_k}-*transforms* of f. Denote by $\tilde{f}_{(c_1,c_2)}\colon \widetilde{M}^n_{(c_1,c_2)} \to \mathbb{Q}_s^N(c)$ the Ribaucour transform of f determined by

$$(\varphi, \gamma, \beta) = c_1(\varphi_1, \gamma_1, \beta_1) + c_2(\varphi_2, \gamma_2, \beta_2), \quad c_1, c_2 \in \mathbb{R},$$

where $\widetilde{M}^n_{(c_1,c_2)}$ stands for M^n with the metric induced by $\tilde{f}_{(c_1,c_2)}$. It follows from (17) that $\tilde{f}_{(c_1,c_2)} = \tilde{f}_{(b_1,b_2)}$ whenever $(c_1,c_2) = \lambda(b_1,b_2)$ for some $\lambda \neq 0$. Thus $\mathcal{H} = \{\tilde{f}_{(c_1,c_2)}, c_1, c_2 \in \mathbb{R}\}$ defines a *one*-parameter family of Ribaucour transforms of f, called the *associated family* to the solutions $(\varphi_k, \gamma^k, \beta^k)$.

The following permutability theorem is an immediate consequence of Theorem 35 in [DT$_2$], where it was proved for flat ambient spaces.

Theorem 16. *There exists another one–parameter family $\widetilde{\mathcal{H}}$ of immersions (called the* conjugate family *to $\mathcal{H}$), all of whose elements is a Ribaucour transform of any element of $\mathcal{H}$. Moreover, if $C_1 \neq C_2$ then $\widetilde{\mathcal{H}}$ contains exactly one element $\bar{f}\colon \bar{M}^n(c) \to \mathbb{Q}^N_s(c)$ that is a R_{C_j}-transform of $\tilde{f}_i$, $i \neq j$, which is explicitly given by*

$$\bar{F} = i \circ \bar{f} = F - \Gamma(\vartheta_2\varphi_1 + \tau_1\varphi_2)\mathcal{F}_1 - \Gamma(\vartheta_1\varphi_2 + \tau_2\varphi_1)\mathcal{F}_2, \tag{34}$$

where $\mathcal{F}_k = \sum_i \gamma_i^k F_* X_i + \sum_s \beta_s^k \xi_s + c\varphi_k F$, $\vartheta_k = C_k \sum_s \epsilon_s (\beta_s^k)^2$, $\tau_\ell = 2C_j(C_\ell - C_j)^{-1}(\sum_i \gamma_i^1\gamma_i^2 + (1 - C_\ell)\sum_s \epsilon_s\beta_s^1\beta_s^2 + c\varphi_1\varphi_2)$, $\ell \neq j$ and $\Gamma = 2/(\vartheta_1\vartheta_2 - \tau_1\tau_2)$.

Analytically, this yields the following for solutions of the associated system (I).

Corollary 17. *Let (v,h) be a solution of (I) and let $(\varphi_k, \gamma^k, \beta^k)$, $1 \leq k \leq 2$, be solutions of $\mathcal{R}_0$ with $C_1 \neq C_2$. Then $(\bar{v}, \bar{h})$ defined by*

$$\bar{v}_i = v_i + C_1\beta_i^1\Gamma(\vartheta_2\varphi_1 + \tau_1\varphi_2) + C_2\beta_i^2\Gamma(\vartheta_1\varphi_2 + \tau_2\varphi_1),$$

$$\bar{h}_{is} = h_{is} + C_1\beta_s^1\Gamma(\vartheta_2\gamma_i^1 + \tau_2\gamma_i^2) + C_2\beta_s^2\Gamma(\vartheta_1\gamma_i^2 + \tau_1\gamma_i^1),$$

is a new solution of (I).

Theorem 16 yields a permutability theorem for horizontal isometric immersions $f\colon M^n(c) \to \mathbb{S}^{2n+1}(c)$. A Ribaucour transform of f associated to a solution of $\mathcal{R}_2$ with a constant $D \neq 0$ is called a R_D-transform of f.

Theorem 18. *Let $\mathcal{H}$ be the associated family to the R_{D_k} – transforms $\tilde{f}_k\colon \widetilde{M}^n_k(c) \to \mathbb{S}^{2n+1}(c)$ of f, $1 \leq k \leq 2$, which are determined by solutions (φ_k, γ^k) of $\mathcal{R}_2$ with $D_1 \neq -D_2$. Then the conjugate family $\widetilde{\mathcal{H}}$ contains exactly one element $\bar{f}\colon \bar{M}^n(c) \to \mathbb{S}^{2n+1}(c)$ which is also horizontal and a R_{D_j}-transform of $\tilde{f}_i$, $i \neq j$. Moreover, $\bar{f}$ is explicitly given by (34), where now $\mathcal{F}_k = (D_k + i)D_k^{-1}(\sum_i \gamma_i^k f_* X_i + c\varphi_k F)$, $\vartheta_k = (D_k^2+1)D_k^{-2}(\sum_i(\gamma_i^k)^2 + c\varphi_k^2)$ and $\tau_\ell = -2(D_j^2+1)D_j^{-1}(D_1+D_2)^{-1}(\sum_i \gamma_i^1\gamma_i^2 + c\varphi_1\varphi_2)$, $1 \leq \ell \neq j \leq 2$.*

The analytical interpretation of the above result is given next.

Corollary 19. *Let (v,h) be a solution of (III) and let (φ_k, γ^k), $1 \leq k \leq 2$, be solutions of $\mathcal{R}_2$ with $D_1 \neq -D_2$. Then $(\bar{v}, \bar{h})$ defined by*

$$\bar{v}_i = v_i + \bar{D}_1\gamma_i^1\Gamma(\vartheta_2\varphi_1 + \tau_1\varphi_2) + \bar{D}_2\gamma_i^2\Gamma(\vartheta_1\varphi_2 + \tau_2\varphi_1),$$

$$\bar{h}_{ij} = h_{ij} + \bar{D}_1\gamma_i^1\Gamma(\vartheta_2\gamma_i^1 + \tau_2\gamma_i^2) + \bar{D}_2\gamma_i^2\Gamma(\vartheta_1\gamma_i^2 + \tau_1\gamma_i^1),$$

where $\bar{D}_k = (1 + D_k^2)/D_k$, is a new solution of (III).

6 The examples

We compute in this section explicit examples of horizontal isometric immersions $F\colon M^n(c) \to \mathbb{S}^{2n+1}(c) \subset \mathbb{C}^{n+1}_\epsilon$ by applying the Ribaucour transformation to the trivial solution of system (III) given by

$$v_1 = B \neq 0, \quad v_i = 0, \;\; 2 \le i \le n, \quad \text{and} \quad h = 0. \tag{35}$$

Although (35) is not associated to any isometric immersion, it does can be used to generate non–trivial examples. First, notice that for the solution (35) system (IV) reduces to

$$(V) \quad \begin{cases} i)\ \dfrac{\partial F}{\partial u_1} = BX_1, \quad ii)\ \dfrac{\partial F}{\partial u_i} = 0, \quad 2 \le i \le n, \quad iii)\ \dfrac{\partial X_i}{\partial u_j} = 0, \;\; i \neq j, \\ iv)\ \dfrac{\partial X_i}{\partial u_i} = iX_i, \quad 2 \le i \le n, \quad v)\ \dfrac{\partial X_1}{\partial u_1} = iX_1 - cBF. \end{cases}$$

Therefore $F = F(u_1)$ satisfies the linear second–order differential equation

$$F'' - iF' + cB^2F = 0.$$

Set

$$\Delta = 1 + 4cB^2.$$

We will distinguish three cases, according to $\Delta > 0$, $\Delta = 0$ or $\Delta < 0$. Notice that only the first case can occur when $c > 0$. In each case, a straightforward computation shows that, for a convenient choice of initial conditions, the solution of (V) is given as follows. We denote by $(E_1, \ldots, E_{n+1})$ the canonical basis of $\mathbb{C}^{n+1}$ over $\mathbb{C}$.

I) $\Delta > 0$:

$$\begin{cases} F = F(u_1) = C_1 e^{ia_1u_1} E_1 + C_2 e^{ia_2u_1} E_2, \\ X_1 = X_1(u_1) = \dfrac{i}{B}\Big(a_1C_1e^{ia_1u_1}E_1 + a_2C_2e^{ia_2u_1}E_2\Big), \end{cases}$$

where

$$a_1 = \frac{1-\sqrt{\Delta}}{2}, \;\; a_2 = \frac{1+\sqrt{\Delta}}{2}, \;\; \epsilon C_1^2 = \frac{\sqrt{\Delta}+1}{2c\sqrt{\Delta}} \;\text{ and }\; C_2^2 = \frac{\sqrt{\Delta}-1}{2c\sqrt{\Delta}}.$$

II) $\Delta = 0$:

$$\begin{cases} F = F(u_1) = \dfrac{e^{\frac{1}{2}iu_1}}{2\sqrt{-c}}\left[(2i + u_1)E_1 + u_1E_2\right], \\ X_1 = X_1(u_1) = \dfrac{e^{\frac{1}{2}iu_1}}{4B\sqrt{-c}}\left[iu_1E_1 + (2 + iu_1)E_2\right]. \end{cases}$$

III) $\Delta < 0$:

$$\begin{cases} F = F(u_1) = e^{\frac{1}{2}iu_1}(e^{au_1}V_1 + e^{-au_1}V_2), \quad a = \dfrac{1}{2}\sqrt{-\Delta} \\ X_1 = X_1(u_1) = \dfrac{e^{\frac{1}{2}iu_1}}{B}\Big[(a + \dfrac{i}{2})e^{au_1}V_1 + (-a + \dfrac{i}{2})e^{-au_1}V_2\Big], \end{cases}$$

where

$$V_j = \frac{1}{2\sqrt{-c}}\left(1 + i\frac{(-1)^j}{\sqrt{-\Delta}}\right)E_1 - \frac{(-1)^j B}{\sqrt{-\Delta}}E_2 \quad 1 \le j \le 2.$$

In all three cases,

$$X_i = X_i(u_i) = e^{iu_i}E_{i+1}, \quad 2 \le i \le n.$$

Now, system $\mathcal{R}_2$ for the Ribaucour transformation becomes

$$\mathcal{R}_3 \;=\begin{cases} i)\ \dfrac{\partial\varphi}{\partial u_1} = B\gamma_1, \quad ii)\ \dfrac{\partial\varphi}{\partial u_i} = 0,\ 2 \le i \le n, \quad iii)\ \dfrac{\partial\gamma_j}{\partial u_i} = 0, \quad i \ne j, \\ iv)\ \dfrac{\partial\gamma_i}{\partial u_i} = -D\gamma_i,\ 2 \le i \le n, \quad v)\ \dfrac{\partial\gamma_1}{\partial u_1} = -D\gamma_1 - cB\varphi, \quad D \ne 0. \end{cases}$$

Hence $\varphi = \varphi(u_1)$ satisfies the linear second–order differential equation

$$\varphi'' + D\varphi' + cB^2\varphi = 0.$$

Set

$$\tilde\Delta = D^2 - 4cB^2.$$

There are again three cases, according to $\tilde\Delta > 0$, $\tilde\Delta = 0$ or $\tilde\Delta < 0$. Clearly, the two last cases can occur only for $c > 0$.

a) $\tilde\Delta > 0$:

$$\begin{cases} \varphi(u_1) = A_1e^{\lambda_1u_1} + A_2e^{\lambda_2u_1}, \quad \lambda_i = \dfrac{-D \pm \sqrt{\tilde\Delta}}{2}, \\ \gamma_1(u_1) = \dfrac{1}{B}(A_1\lambda_1e^{\lambda_1u_1} + A_2\lambda_2e^{\lambda_2u_1}), \qquad A_1, A_2 \in \mathbb{R}. \end{cases}$$

b) $\tilde\Delta = 0$:

$$\begin{cases} \varphi(u_1) = e^{-\frac{1}{2}Du_1}(A_1 + A_2u_1), \quad A_1, A_2 \in \mathbb{R}, \\ \gamma_1(u_1) = \dfrac{e^{-\frac{1}{2}Du_1}}{2B}(2A_2 - DA_1 - DA_2u_1). \end{cases}$$

c) $\tilde\Delta < 0$:

$$\begin{cases} \varphi(u_1) = e^{-\frac{1}{2}Du_1}[A_1\cos(ku_1) + A_2\sin(ku_1)], \quad k = \dfrac{1}{2}\sqrt{-\tilde\Delta}, \quad A_1, A_2 \in \mathbb{R}, \\ \gamma_1(u_1) = \dfrac{e^{-\frac{1}{2}Du_1}}{2B}[(-DA_1 + 2kA_2)\cos(ku_1) - (DA_2 + 2kA_1)\sin(ku_1)]. \end{cases}$$

In all cases,

$$\gamma_i = \gamma_i(u_i) = B_ie^{-Du_i}, \quad B_i \in \mathbb{R}, \quad 2 \le i \le n.$$

For F, X_i, φ and γ_i above, formula (32) provides a parametrization of non–trivial examples of horizontal isometric immersions $\widetilde{F}\colon\ M^n(c) \to \mathbb{S}^{2n+1}(c) \subset \mathbb{C}^{n+1}_\epsilon$. Moreover, the pair $(\tilde v, \tilde h)$ associated to $\widetilde{F}$ is given by (33), thus $(\tilde v, \tilde h)$ is a solution of system (*III*) defined by elementary functions.

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