A note on the irreducible characters of the Hecke algebra $H_n(q)$

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Abstract

In this note we present a formula for the irreducible characters of the Hecke algebras $H_n(q)$ of type A_{n-1}, in terms of irreducible characters of the symmetric group corresponding to classes with a single cycle of length greater than one.

1 Introduction

A number of prescriptions for calculating the irreducible characters ${}_q\chi^\lambda_\mu$ of the Hecke algebras $H_n(q)$ of type A_{n-1} are known. Thus in the literature we may find :

i) a purely combinatorial algorithm (ref. 1).

ii) a procedure derived from the relation between the Ocneanu trace on $H_n(q)$ and the Schur functions (ref. 2).

iii) a method using the properties of the fundamental invariant of $H_n(q)$ (ref. 3).

iv) an improved version of the "Murnaghan–Nakayama" rule for the characters of the Hecke algebra (ref. 4).

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The main purpose of this note is the construction of an algorithm for ${}_q\chi^\lambda_\mu$ whose components are the simple characters of S_n. Our approach relies on the A. Ram formula (ref. 5) :

$$ {}_q\chi^\lambda_\mu = \frac{1}{(q-1)^{\ell(\mu)}} \sum_\nu \prod_i \frac{(q^i-1)^{n_i}}{i^{n_i} n_i!} \chi^\lambda_\nu \phi^\mu_\nu \tag{1} $$

where $\ell(\mu)$ is the number of non-vanishing parts of the partition μ, χ^λ_μ and ϕ^μ_ν are, respectively, the simple and the induced characters of the symmetric group S_n. Hereafter, Section 2 is dedicated to recall some characteristics of the representations theory of S_n. Section 3 considers anew Ram's result. Finally, Section 4 contains our method for evaluating the characters of $H_n(q)$.

2 The Induced characters of the Symmetric Group S_n

Let n be a positive integer. The partitions of n are designated by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $\lambda = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ are positive integers with $n = \lambda_1 + \lambda_2 + \ldots + \lambda_k$; $p(n)$ is the number of partitions of n.

Let C be a class of the symmetric group S_n characterized by its cycle structure $(1^\alpha, 2^\beta, 3^\gamma, \ldots)$ This symbol denotes that the permutations in C contain α 1-cycles, β 2-cycles, γ 3-cycles, etc., where $n = \alpha + 2\beta + 3\gamma + \ldots$.

$\chi^\lambda_{(1^\alpha, 2^\beta, 3^\gamma, \ldots)}$ is an irreducible or simple character of S_n corresponding to the partition λ and to the class $(1^\alpha, 2^\beta, 3^\gamma, \ldots)$ and χ is the irreducible character table of S_n. The character tables of S_n for $n \leq 10$ may be found in ref. 6.

To each partition of n corresponds a subgroup S_λ of S_n given by the direct product $S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_k}$. Such subgroups are called canonical or Young subgroups of S_n.

For each S_{λ_i} we have :

$$ \alpha_i + 2\beta_i + 3\gamma_i + \ldots = \lambda_i. \tag{2} $$

The character induced in S_n by the identity representation of a canonical subgroup is

$$ \phi^\lambda_{(1^\alpha, 2^\beta, 3^\gamma, \ldots)} = \sum \frac{\alpha!}{\alpha_1! \alpha_2! \ldots} \cdots \frac{\beta!}{\beta_1! \beta_2! \ldots} \cdots \frac{\gamma!}{\gamma_1! \gamma_2! \ldots} \cdots \tag{3} $$

where

$$ \sum \alpha_i = \alpha, \qquad \sum \beta_i = \beta, \qquad \sum \gamma_i = \gamma, \ldots . \tag{4} $$

The sum is over all the integer solutions of the system of equations (2) and (4). ϕ is the induced character table of S_n (for $n = 4$, see ref. 7). The Tables χ and ϕ are of dimensions $p(n) \times p(n)$.

3 The Ram's formula revisited

The purpose of this section is the derivation of a special case of A. Ram's formula for the irreducible characters of $H_n(q)$. Note that in equation (1) $n_1, n_2, n_3, \ldots$ correspond to $\alpha, \beta, \gamma, \ldots$. From the algorithm for the induced characters of S_n, we deduce the expression

$$\phi^n_{1^\alpha, 2^\beta, \ldots} = 1 \quad \text{for } \lambda = n.$$

Moreover, if $n = \mu$, from eq. (1) we obtain :

$$_q\chi^\lambda_n = \frac{1}{q-1} \sum_\nu \prod_i \frac{(q^i-1)^{n_i}}{i^{n_i} n_i!} \chi^\lambda_\nu. \tag{5}$$

To a cycle of length m appearing p times we shall associate a Q-term defined by

$$Q^p_m = \begin{cases} (q-1)^{p-1} & m = 1 \\ \left(\frac{q^m-1}{q-1}\right)^p & m > 1 \end{cases} \tag{6}$$

where $\dfrac{q^m-1}{q-1} = q^{m-1} + q^{m-2} + \ldots + q + 1$.

For a class $(1^{n_1}, 2^{n_2}, 3^{n_3}, \ldots)$ the Q-terms combine according to

$$Q^r_1 Q^{n_2}_2 Q^{n_3}_3 \ldots$$

where $r = n_1 + n_2 + n_3 + \ldots$.

Hence

$$\begin{aligned} Q^r_1 Q^{n_2}_2 Q^{n_3}_3 \ldots &= \frac{(q-1)^{n_1+n_2+n_3\ldots}}{q-1} \cdot \frac{(q^2-1)^{n_2}}{(q-1)^{n_2}} \cdot \frac{(q^3-1)^{n_3}}{(q-1)^{n_3}} \cdots \\ Q^r_1 Q^{n_2}_2 Q^{n_3}_3 \ldots &= \frac{1}{q-1} \sum_i (q^i-1)^{n_i} \qquad \mu = n, \ell(n) = 1. \end{aligned} \tag{7}$$

Finally, we have

$$_q\chi^\lambda_n = \sum_\nu \frac{Q^r_1 Q^{n_2}_2 Q^{n_3}_3}{1^{n_i} n_i! 2^{n_2} n_2! \cdots} \cdots \chi^\lambda_\nu. \tag{8}$$

It must be pointed out that

- $_q\chi^\lambda_n$ is valid for the irreducible characters of $H_n(q)$ corresponding to the classes with a single cycle of length n.
- In $_q\chi^\lambda_n$ the factor involving the induced character of S_n does not appear.

Let $\mu_1 = (1^{\alpha_1}, 2^{\beta_1}, 3^{\gamma_1}, \ldots), \mu_2 = (1^{\alpha_2}, 2^{\beta_2}, 3^{\gamma_2}, \ldots), \ldots, \mu_s = (1^{\alpha_s}, 2^{\beta_s}, 3^{\gamma_s}, \ldots)$ be s classes. From the combination $\mu_1\mu_2\ldots\mu_s$ results :

$$(1^{\alpha_1+\alpha_2+\ldots+\alpha_s}, 2^{\beta_1+\beta_2+\ldots+\beta_s}, 3^{\gamma_1+\gamma_2+\ldots+\gamma_s}, \ldots)$$

We shall focus our attention on $(1^{n-\ell}, \ell)$. If $n > \ell$, each partition of ℓ written as $(1^\alpha, 2^\beta, 3^\gamma, \ldots)$ yields the combination $(1^{n-\ell})$ $(1^\alpha, 2^\beta, 3^\gamma, \ldots)$. To clarify the point, we set $n = 5$ and $\ell = 3$.

Then $(1^2)\cdot(1^3)=(1^5)$, $(1^2)\cdot(1,2)=(1^3,2)$, $(1^2)\cdot(3)=(1^2,3)$.
For a partition λ and the class $(1^{n-\ell},\ell)$ the irreducible characters are given by

$$ {}_q\chi^{\lambda}_{1^{n-\ell},\ell}=\sum\frac{Q_1^rQ_2^{n_2}Q_3^{n_3}}{1^{n_1}n_1!2^{n_2}n_2!\ldots}\chi^{\lambda}_{(1^{n-\ell}),(1^{n_1},2^{n_2},3^{n_3}\ldots)}. \tag{9}$$

The sum is over all the partitions of ℓ, $n>\ell$.

For example

$$ {}_q\chi^{\lambda}_{1^2,3}=\frac{\chi^{\lambda}_{1^5}}{6}(q-1)^2+\frac{\chi^{\lambda}_{1^3,2}}{2}(q-1)(q+1)+\frac{\chi^{\lambda}_{1^2,3}}{3}(q^2+q+1).$$

- If $n=\ell$ we recover equation (8)
- If $n\geq 1$ and $\ell=0$, ${}_q\chi^{\lambda}_{1^n}=\chi^{\lambda}_{1^n}$ (see App. A).

Table I and Table II display ${}_q\chi^{\lambda}_{1^{n-\ell},\ell}$ for $n\geq 2,3,\ldots,7$.

4 A product method for the irreducible characters of $H_n(q)$

Let λ be a partition of N and $(\lambda'),(\lambda''),(\lambda'''),\ldots$ subpartitions of λ i.e., $N=\lambda'+\lambda''+\lambda'''+\ldots$. Besides let $\mu=(1^A,2^B,3^C,\ldots)$ be a class of N, $N=A+2B+3C+\ldots$ which stems from a combination of the classes $\mu',\mu'',\mu''',\ldots$ respectively. (In all these classes a single cycle has a length >1).

Proposition

$$ {}_q\chi^{\lambda}_{\mu}=\left({}_q\chi^{\lambda'}_{\mu'}\right)\cdot\left({}_q\chi^{\lambda''}_{\mu''}\right)\cdot\left({}_q\chi^{\lambda'''}_{\mu'''}\right)\ldots$$

Each factor of the right-hand side member of the expression is evaluated by means of (8) or (9).

By way of example we shall calculate the irreducible character ${}_q\chi^{\lambda}_{3,2,1}$ of $H_6(q)$ where λ is a partition of $N=6$.

(a) : ${}_q\chi^{\lambda}_{3,2,1}={}_q\chi^{\lambda'}_{3}\cdot{}_q\chi^{\lambda''}_{2,1}={}_q\chi^{\lambda'''}_{3,1}\cdot{}_q\chi^{\lambda^{\iota\nu}}_{2}={}_q\chi^{\lambda^{\nu}}_{3}\cdot{}_q\chi^{\lambda^{\nu\iota}}_{2}\cdot{}_q\chi^{\lambda^{\nu\iota\iota}}_{1}$

$\lambda'+\lambda''=\lambda'''+\lambda^{\iota\nu}=\lambda^{\nu}+\lambda^{\nu\iota}+\lambda^{\nu\iota\iota}=6.$

(b) : Using the results of Table I, we get :

$$\begin{aligned} {}_q\chi^{\lambda}_{3,2,1} &= \frac{\chi^{\lambda}_{1^6}}{12}(q-1)^3+4\frac{\chi^{\lambda}_{2,1^4}}{12}(q-1)^2(q+1)\\ &+\frac{\chi^{\lambda}_{2^2,1^2}}{4}(q-1)(q+1)^2+\frac{\chi^{\lambda}_{3,1^3}}{6}(q-1)(q^2+q+1)\\ &+\frac{\chi^{\lambda}_{3,2,1}}{6}(q+1)(q^2+q+1).\end{aligned}$$

To particularize, let $\lambda=(5,1)$. The irreducible characters of S_6 are in such a case :

$$\chi^{5,1}_{1^5}=5;\quad \chi^{5,1}_{2,1^4}=3;\quad \chi^{5,1}_{2^2,1^2}=1;\quad \chi^{5,1}_{3,1^3}=2\quad\text{and}\quad \chi^{5,1}_{3,2,1}=0.$$

Therefore : ${}_q\chi^{5,1}_{3,2,1}=\dfrac{24q^3-24q^2}{12}=2q^3-2q^2.$

Proof : For briefness we are going to consider the case $({}_q\chi^{\lambda'}_{\mu'})\,({}_q\chi^{\lambda''}_{\mu''})$.

By means of eq. (8), we may write :

$$({}_q\chi^{\lambda'}_{\mu'})\,({}_q\chi^{\lambda''}_{\mu''}) = \sum_\nu \frac{Q_1^r Q_2^{n_2} Q_3^{n_3}\dots}{1^{n_1}n_1!2^{n_2}n_2!\dots}\chi^{\lambda'}_\nu \sum_\rho \frac{Q_1^R Q_2^{N_2} Q_3^{N_3}\dots}{1^{N_1}N_1!2^{N_2}N_2!\dots}\chi^{\lambda''}_\rho$$

$$({}_q\chi^{\lambda'}_{\mu'})\,({}_q\chi^{\lambda''}_{\mu''}) = \sum_\nu\sum_\rho \frac{Q_1^{r+R} Q_2^{m_2} Q_3^{m_3}\dots}{1^{n_1+N_1}n_1!N_1!2^{n_2+N_2}n_2!N_2!\dots}\chi^{\lambda'}_\nu\chi^{\lambda''}_\rho .$$

Let $m_i = n_i + N_i$; we have :

$$({}_q\chi^{\lambda'}_{\mu'})\,({}_q\chi^{\lambda''}_{\mu''}) = \sum\sum \frac{Q_1^{r+R} Q_2^{m_2} Q_3^{m_3}\dots}{1^{m_1}2^{m_2}\dots n_1!N_1!n_2!N_2!\dots}\chi^{\lambda'}_\nu\chi^{\lambda''}_\rho .$$

Multiplying by $\displaystyle\sum_i \frac{m_1!m_2!\dots}{(n_i+N_i)!} = 1$

$$({}_q\chi^{\lambda'}_{\mu'})\,({}_q\chi^{\lambda''}_{\mu''}) = \sum_\nu\sum_\rho \frac{Q_1^{n+R} Q_2^{m_2} Q_3^{m_3}\dots}{1^{m_1}m_1!2^{m_2}m_2!\dots}\chi^{\lambda'}_\nu\chi^{\lambda''}_\rho \sum_i \frac{m_1!m_2!\dots}{n_1!n_2!\dots N_1!N_2!\dots}.$$

If we take into account a class $\mu = u'\mu''$, the sums involving the indices ν and ρ can be expressed as a single sum over the partitions ordered from $1^{\mu'+\mu''}$ to (μ',μ'').

Remark that :

$$\sum_i \frac{m_1!m_2!}{n_i!N_i!} = \phi^{\mu',\mu''}_\mu \qquad \text{(see Sect. 2)}$$

and using expression (1), we obtain :

$$({}_q\chi^{\lambda'}_{\mu'})\,({}_q\chi^{\lambda''}_{\mu''}) = \frac{1}{(q-1)^2}\sum_\mu\prod_j \frac{(q^j-1)^{m_j}}{j^{m_j}m_j!}\chi^\lambda_\mu\phi^{(\mu',\mu'')}_\mu$$

that is :

$$({}_q\chi^{\lambda'}_{\mu'})\,({}_q\chi^{\lambda''}_{\mu''}) = {}_q\chi^\lambda_{(\mu',\mu'')}.$$

The proof of the general case follows the same pattern.

APPENDIX A

From eq. (1) we get :

$$ {}_q\chi^{\lambda}_{1^n} = \frac{1}{(q-1)^n} \sum_{\nu} \prod_i \frac{(q^i-1)^{n_i}}{i^{n_i} n_i!} \chi^{\lambda}_{\nu} \phi^{1^n}_{\nu} $$

if $\nu = 1^n$, $\phi^{1^n}_{1^n} = n!$

Hence : ${}_q\chi^{\lambda}_{1^n} = \dfrac{1}{(q-1)^n}\, n! \prod_i \dfrac{(q^i-1)^{n_i}}{i^{n_i} n_i!} \chi^{\lambda}_{1^n}$

$$ {}_q\chi^{\lambda}_{1^n} = \frac{n!}{(q-1)^n} \frac{(q-1)^n}{1^n n!} \chi^{\lambda}_{1^n} = \chi^{\lambda}_{1^n}. $$

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TABLE I : IRREDUCIBLE CHARACTERS ${}_q\chi^{\lambda}_{1^{n-\ell},\ell}$ $n \geq 2, \ldots, 5$

$${}_q\chi^{\lambda}_{1^{n-2},2} = \frac{\chi^{\lambda}_{1^n}}{2}(q-1) + \frac{\chi^{\lambda}_{1^{n-2},2}}{2}(q+1)$$	$n \geq 2$
$${}_q\chi^{\lambda}_{1^{n-3},3} = \frac{\chi^{\lambda}_{1^n}}{6}(q-1)^2 + \frac{\chi^{\lambda}_{1^{n-2},2}}{2}(q-1)(q+1) + \frac{\chi^{\lambda}_{1^{n-3},3}}{3}(q^2+q+1)$$	$n \geq 3$
$${}_q\chi^{\lambda}_{1^{n-4},4} = \frac{\chi^{\lambda}_{1^n}}{24}(q-1)^3 + \frac{\chi^{\lambda}_{1^{n-2},2}}{4}(q-1)^2(q+1) + \frac{\chi^{\lambda}_{1^{n-4},2^2}}{8}(q-1)(q+1)^2 + \frac{\chi^{\lambda}_{1^{n-3},3}}{3}(q^3-1) + \frac{\chi^{\lambda}_{1^{n-4},4}}{4}(q^3+q^2+q+1)$$	$n \geq 4$
$${}_q\chi^{\lambda}_{1^{n-5},5} = \frac{\chi^{\lambda}_{1^n}}{120}(q-1)^4 + \frac{\chi^{\lambda}_{1^{n-2},2}}{12}(q-1)^3(q+1) + \frac{\chi^{\lambda}_{1^{n-4},2^2}}{8}(q-1)^2(q+1)^2 + \frac{\chi^{\lambda}_{1^{n-3},3}}{6}(q^3-1)(q-1) + \frac{\chi^{\lambda}_{1^{n-5},2,3}}{6}(q^3-1)(q+1) + \frac{\chi^{\lambda}_{1^{n-4},4}}{4}(q^4-1) + \frac{\chi^{\lambda}_{1^{n-5},5}}{5}(q^4+q^3+q^2+q+1)$$	$n \geq 5$

TABLE II : IRREDUCIBLE CHARACTERS ${}_q\chi^{\lambda}_{1^{n-\ell},\ell}$ $n \geq 6, 7$

$$
\begin{aligned}
{}_q\chi^{\lambda}_{1^{n-6},6} = & \frac{\chi^{\lambda}_{1^n}}{720}(q-1)^5 + \frac{\chi^{\lambda}_{1^{n-2},2}}{48}(q-1)^4(q+1) + \frac{\chi^{\lambda}_{1^{n-4},2^2}}{16}(q-1)^3(q+1)^2 \\
& + \frac{\chi^{\lambda}_{1^{n-6},2^3}}{48}(q-1)^2(q+1)^3 + \frac{\chi^{\lambda}_{1^{n-3},3}}{18}(q-1)^2(q^3+1) \\
& + \frac{\chi^{\lambda}_{1^{n-5},2,3}}{6}(q^2-1)(q^3-1) + \frac{\chi^{\lambda}_{1^{n-6},3^2}}{18}(q^3-1)(q^2+q+1) \\
& + \frac{\chi^{\lambda}_{1^{n-4},4}}{8}(q-1)(q^4-1) + \frac{\chi^{\lambda}_{1^{n-6},2,4}}{8}(q+1)(q^4-1) \\
& + \frac{\chi^{\lambda}_{1^{n-5},5}}{5}(q^5-1) + \frac{\chi^{\lambda}_{1^{n-6},6}}{6}(q^5+q^4+q^3+q^2+q+1)
\end{aligned}
$$

$$
\begin{aligned}
{}_q\chi^{\lambda}_{1^{n-7},7} = & \frac{\chi^{\lambda}_{1^n}}{5040}(q-1)^6 + \frac{\chi^{\lambda}_{1^{n-2},2}}{240}(q-1)^5(q+1) + \frac{\chi^{\lambda}_{1^{n-4},2^2}}{48}(q-1)^4(q+1)^2 \\
& + \frac{\chi^{\lambda}_{1^{n-6},2^3}}{48}(q-1)^3(q+1)^3 + \frac{\chi^{\lambda}_{1^{n-3},3}}{72}(q-1)^3(q^3-1) \\
& + \frac{\chi^{\lambda}_{1^{n-5},2,3}}{12}(q^2-1)(q^3-1)(q-1) + \frac{\chi^{\lambda}_{1^{n-7},2^2,3}}{24}(q^2-1)^2(q^2+q+1) \\
& + \frac{\chi^{\lambda}_{1^{n-6},3^2}}{18}(q^3-1)^2 + \frac{\chi^{\lambda}_{1^{n-4},4}}{24}(q-1)^2(q^4-1) \\
& + \frac{\chi^{\lambda}_{1^{n-6},2,4}}{8}(q^2-1)(q^4-1) + \frac{\chi^{\lambda}_{1^{n-7},3,4}}{12}(q^3-1)(q^3+q^2+q+1) \\
& + \frac{\chi^{\lambda}_{1^{n-5},5}}{10}(q-1)(q^5-1) + \frac{\chi^{\lambda}_{1^{n-7},2,5}}{10}(q^2-1)(q^4+q^3+q^2+q+1) \\
& + \frac{\chi^{\lambda}_{1^{n-6},6}}{6}(q^6-1) + \frac{\chi^{\lambda}_{1^{n-7},7}}{7}(q^6+q^5+q^4+q^3+q^2+q+1)
\end{aligned}
$$

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