On the stability of Appolonius' equation*

Kil-Woung Jun   Hark-Mahn Kim†

Abstract

In this paper, using an idea from the direct method of Hyers and Ulam, we investigate the situations so that the generalized Hyers-Ulam stability problem for Appolonius' equation

$$f(x-z)+f(y-z)=\frac{1}{2}f(x-y)+2f\Big(z-\frac{x+y}{2}\Big)$$

is satisfied.

1 Introduction

In 1940, S. M. Ulam [15] raised a question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is approximately a homomorphism, then there exists a true homomorphism near it with small error.

The case of approximately additive functions was solved by D. H. Hyers [5] and generalized by Th. M. Rassias [13]. During the last decades, the stability problems

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†Corresponding author:hmkim@math.cnu.ac.kr
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of several functional equations have been extensively investigated by a number of authors (see [6, 7, 8, 10, 12, 14] and references therein).

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is related to a symmetric biadditive function ([1, 6]). It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x)=B(x,x)$ for all x (see [1]). The biadditive function B is given by

$$B(x,y)=\frac{1}{4}(f(x+y)-f(x-y)). \tag{1.2}$$

Thus we call the equation (1.1) *quadratic functional equation* and every solution of the quadratic equation (1.1) is said to be a *quadratic function*. A stability problem for the quadratic functional equation (1.1) has been widely investigated by a lot of authors [2, 11]. Further, Jun and Lee [9] proved the generalized Hyers-Ulam stability of the pexiderized quadratic equation (1.1).

Now we may verify by direct calculation that the following Appolonius' identity

$$\|x-z\|^2+\|y-z\|^2=\frac{1}{2}\|x-y\|^2+2\left\|z-\frac{x+y}{2}\right\|^2$$

holds for any elements in an inner product space.

This identity may be formulated by the following Appolonius' equation,

$$f(x-z)+f(y-z)=\frac{1}{2}f(x-y)+2f\Big(z-\frac{x+y}{2}\Big), \tag{1.3}$$

of which the solution is called the Appolonius' function. In fact, it will be easily verified that Appolonius' equation (1.3) in the class of functions between real vector spaces is identically equivalent to the quadratic functional equation (1.1), and thus we establish the generalized Hyers-Ulam stability problem for the equation (1.3) in the sense of Hyers, Ulam and Rassias in Section 2.

2 Stability of (1.3)

First we seek for the general solution of (1.3) in the class of functions between real vector spaces. Putting $x=y=z=0$ in the equation (1.3) yields $f(0)=0$. Substituting $x=y=0$ in (1.3) we see that f is even. Replacing z by 0 one obtains that

$$f(x)+f(y)=\frac{1}{2}f(x-y)+2f\Big(-\frac{x+y}{2}\Big). \tag{2.1}$$

Replacing y by $-x$ in (2.1) and using the evenness of f one gets that $f(2x)=4f(x)$. This fact and the equation (2.1) imply that

$$2f(x)+2f(y)=f(x-y)+f(x+y). \tag{2.2}$$

Thus f is quadratic. Conversely, if f is quadratic, then it is obvious that f satisfies the equation (1.3).

Throughout this section X and Y will be a real vector space and a real Banach space, respectively, unless we give any specific reference. Given $f : X \to Y$, we set

$$Df(x, y, z) := f(x - z) + f(y - z) - \frac{1}{2}f(x - y) - 2f\Big(z - \frac{x + y}{2}\Big)$$

for all $x, y, z \in X$.

Let $\varphi : X \times X \times X \to \mathbb{R}_+ := [0, \infty)$ be a mapping satisfying one of the conditions $(\mathcal{A})$, $(\mathcal{B})$

$$\Phi(x, y, z) := \sum_{k=0}^{\infty} \frac{1}{4^k}\varphi(2^k x, 2^k y, 2^k z) < \infty \qquad (\mathcal{A})$$

$$\Psi(x, y, z) := \sum_{k=1}^{\infty} 4^k \varphi(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k}) < \infty \qquad (\mathcal{B})$$

for all $x, y, z \in X$.

The conditions $(\mathcal{A})$, $(\mathcal{B})$ will be needed to derive a quadratic function near an approximately Appolonius' function as in the following theorem.

Theorem 2.1. *Assume that a function $f : X \to Y$ satisfies*

$$\|Df(x, y, z))\| \le \varphi(x, y, z) \tag{2.3}$$

for all $x, y, z \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying (1.3) such that in case $(\mathcal{A})$

$$\Big\| f(x) - \frac{f(0)}{2} - Q(x) \Big\| \le \frac{1}{2}\Phi(x, -x, x) + \frac{1}{4}\Phi(x, x, -x) \tag{2.4}$$

and in case $(\mathcal{B})$

$$\|f(x) - Q(x)\| \le \frac{1}{2}\Psi(x, -x, x) + \frac{1}{4}\Psi(x, x, -x) \tag{2.5}$$

for all $x \in X$.

The function Q is given by

$$\begin{cases} Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}} & \text{if } (\mathcal{A}) \text{ holds} \\ Q(x) = \lim_{n\to\infty} 2^{2n} f(\frac{x}{2^n}) & \text{if } (\mathcal{B}) \text{ holds} \end{cases}$$

for all $x \in X$

Proof. Putting $y = z = x$ in (2.3) yields

$$\|f(0)\| \le 2\varphi(x, x, x) \tag{2.6}$$

for all $x \in X$. Replacing y, z by $x, -x$ in (2.3), respectively, yields

$$\|2f(2x) - 2f(-2x) - \frac{1}{2}f(0)\| \le \varphi(x, x, -x) \tag{2.7}$$

for all $x \in X$. Setting $-x, x$ instead of y, z in (2.3), respectively, we arrive at

$$\|f(0) + f(-2x) - \frac{1}{2}f(2x) - 2f(x)\| \le \varphi(x, -x, x) \tag{2.8}$$

for all $x \in X$. Combining the relation (2.7) and (2.8) we obtain the crucial inequality

$$\|f(2x) - 4f(x) + \frac{3}{2}f(0)\| \le \varphi(x, x, -x) + 2\varphi(x, -x, x) \tag{2.9}$$

for all $x \in X$.

Case 1. Assume that φ satisfies the condition $(\mathcal{A})$.

Dividing both sides of (2.9) by 4, we have

$$\left\|f(x) - \frac{f(0)}{2} - \frac{f(2x) - \frac{f(0)}{2}}{4}\right\| \le \frac{\varphi(x, -x, x)}{2} + \frac{\varphi(x, x, -x)}{4} \tag{2.10}$$

for all $x \in X$. Replacing x by $2^{n-1}x$ and dividing by 4^{n-1} in (2.10) we obtain

$$\begin{aligned} &\left\|\frac{f(2^{n-1}x) - \frac{f(0)}{2}}{4^{n-1}} - \frac{f(2^n x) - \frac{f(0)}{2}}{4^n}\right\| \\ &\le \frac{1}{2}\frac{\varphi(2^{n-1}x, -2^{n-1}x, 2^{n-1}x)}{4^{n-1}} + \frac{\varphi(2^n x, 2^n x, -2^n x)}{4^n} \end{aligned} \tag{2.11}$$

for all $x \in X$ and for all $n \in \mathbb{N}$.

Thus from the formula (2.10), the inequality (2.11), and the triangle inequality we prove by induction that

$$\begin{aligned} &\left\|f(x) - \frac{f(0)}{2} - \frac{f(2^n x) - \frac{f(0)}{2}}{4^n}\right\| \\ &\le \frac{1}{2}\sum_{i=0}^{n-1}\frac{\varphi(2^i x, -2^i x, 2^i x)}{4^i} + \sum_{i=0}^{n-1}\frac{\varphi(2^i x, 2^i x, -2^i x)}{4^{i+1}} \end{aligned} \tag{2.12}$$

for all $x \in X$ and for all $n \in \mathbb{N}$.

Now we claim that the sequence $\{\frac{f(2^n x) - \frac{f(0)}{2}}{4^n}\}$ is Cauchy in the Banach space Y. In fact it follows from (2.12) that

$$\begin{aligned} &\left\|\frac{f(2^m x) - \frac{f(0)}{2}}{4^m} - \frac{f(2^{n+m}x) - \frac{f(0)}{2}}{4^{n+m}}\right\| \\ &= \frac{1}{4^m}\left\|f(2^m x) - \frac{f(0)}{2} - \frac{f(2^n \cdot 2^m x) - \frac{f(0)}{2}}{4^n}\right\| \\ &\le \frac{1}{2}\sum_{i=0}^{n-1}\frac{\varphi(2^{m+i}x, -2^{m+i}x, 2^{m+i}x)}{4^{m+i}} + \sum_{i=0}^{n-1}\frac{\varphi(2^{m+i}x, 2^{m+i}x, -2^{m+i}x)}{4^{m+i+1}} \end{aligned} \tag{2.13}$$

for all $x \in X$ and for all $n, m \in \mathbb{N}$. Since the right hand side of (2.13) tends to zero as $m \to \infty$, the sequence $\{\frac{f(2^n x) - \frac{f(0)}{2}}{4^n}\}$ is Cauchy for all $x \in X$ and thus converges by the completeness of Y. Therefore we can define a function $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty}\frac{f(2^n x) - \frac{f(0)}{2}}{4^n} = \lim_{n\to\infty}\frac{f(2^n x)}{4^n}, \qquad x \in X.$$

Replacing x, y, z in (2.3) by $2^n x, 2^n y, 2^n z$, respectively, dividing both sides by 4^n, and after then taking the limit in the resulting inequality, we have

$$Q(x-z)+Q(y-z)-\frac{1}{2}Q(x-y)-2Q\Big(z-\frac{x+y}{2}\Big)=0. \tag{2.14}$$

Therefore the function Q is Appolonius and quadratic.

Taking the limit in (2.12) as $n\to\infty$, we obtain that

$$\Big\|f(x)-\frac{f(0)}{2}-Q(x)\Big\|\le\frac{1}{2}\Phi(x,-x,x)+\frac{1}{4}\Phi(x,x,-x) \tag{2.15}$$

for all $x\in X$.

To prove the uniqueness, let Q' be another Appolonius' function satisfying (2.15). Then it is obvious that $Q'(2^n x)=4^n Q'(x)$ for all $x\in X$. Thus we have

$$\begin{aligned}&\|Q(x)-Q'(x)\|=\frac{1}{4^n}\|Q(2^n x)-Q'(2^n x)\|\\ &\le\frac{1}{4^n}\Big\{\|Q(2^n x)-f(2^n x)+\frac{f(0)}{2}\|+\|f(2^n x)-\frac{f(0)}{2}-Q'(2^n x)\|\Big\}\\ &\le\frac{2}{4^n}\Big[\frac{1}{2}\Phi(2^n x,-2^n x,2^n x)+\frac{1}{4}\Phi(2^n x,2^n x,-2^n x)\Big].\end{aligned}$$

Taking the limit as $n\to\infty$, we conclude that $Q(x)=Q'(x)$ for all $x\in X$.

Case 2. Let φ satisfy the condition $(\mathcal{B})$. Then the proof is analogous to that of **Case 1**. Indeed, putting $x=y=z=0$ in (2.3) we have $f(0)=0$ since $\Psi(0,0,0)<\infty$ and so $\varphi(0,0,0)=0$.

Replacing x by $\frac{x}{2}$ in (2.9) we get

$$\|f(x)-4f(\frac{x}{2})\|\le 2\varphi(\frac{x}{2},-\frac{x}{2},\frac{x}{2})+\varphi(\frac{x}{2},\frac{x}{2},-\frac{x}{2}) \tag{2.16}$$

for all $x\in X$.

An induction argument together with (2.16) implies that

$$\begin{aligned}&\Big\|f(x)-4^n f(\frac{x}{2^n})\Big\| \\ &\le\frac{1}{2}\sum_{i=1}^{n}4^i\varphi(\frac{x}{2^i},-\frac{x}{2^i},\frac{x}{2^i})+\frac{1}{4}\sum_{i=1}^{n}4^i\varphi(\frac{x}{2^i},\frac{x}{2^i},-\frac{x}{2^i})\\ &\le\frac{1}{2}\Psi(x,-x,x)+\frac{1}{4}\Psi(x,x,-x)\end{aligned} \tag{2.17}$$

for all $x\in X$ and for all $n\in\mathbb{N}$.

Repeating the similar method to (2.13) with the aid of (2.17) one concludes that $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x\in X$ and thus converges. Therefore one can define a function $Q: X\to Y$ by

$$Q(x)=\lim_{n\to\infty}4^n f(\frac{x}{2^n}),\ x\in X.$$

Using the similar argument to that of **Case 1**, we see easily that Q is the unique quadratic mapping satisfying (1.3) such that

$$\|f(x)-Q(x)\|\le\frac{1}{2}\Psi(x,-x,x)+\frac{1}{4}\Psi(x,x,-x)$$

for all $x\in X$. This completes the proof. ■

From the main Theorem 2.1, we obtain the following corollary concerning the stability of the equation (1.3).

Corollary 2.2. *Let X and Y be a real normed space and a real Banach space, respectively. Let p, ϵ be real numbers such that $\epsilon \geq 0$, $p \neq 2$. Assume that a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y,z))\| \leq \epsilon(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.18}$$

for all $x, y, z \in X$ ($X\backslash\{0\}$ if $p < 0$). Then there exists a unique quadratic function $Q : X \to Y$ which satisfies (1.3) and the inequality

$$\|f(x) - \frac{f(0)}{2} - Q(x)\| \leq \frac{9\epsilon\|x\|^p}{|4 - 2^p|}$$

for all $x \in X$ ($X\backslash\{0\}$ if $p < 0$), where $f(0) = 0$ if $p > 0$.

Proof. Letting $\varphi(x,y,z) := \epsilon(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$ and then applying Theorem 2.1 we obtain easily the result. Here if $p > 0$, $f(0) = 0$ since one gets that $\varphi(0,0,0) = 0$ by putting $x = y = z = 0$ in (2.18). ■

The exclusion of the case $p = 2$ in Corollary 2.2 is necessary as shown by the following counterexample, which is a modification of the example of Czerwik [2]. In fact, let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\begin{cases} \varphi(x) = \frac{\epsilon}{24}x^2 & \text{if } |x| < 1 \\ \varphi(x) = \frac{\epsilon}{24} & \text{if } |x| \geq 1, \end{cases}$$

where $\epsilon > 0$, and put

$$f(x) = \sum_{i=0}^{\infty} \frac{\varphi(2^i x)}{4^i}, \qquad x \in \mathbb{R}.$$

Then φ and f are bounded by $\frac{\epsilon}{24}$ and $\frac{\epsilon}{18}$ on $\mathbb{R}$, respectively. Utilizing the similar argument to that of [2] we conclude that

$$\left| f(x-z) + f(y-z) - \frac{1}{2}f(x-y) - 2f\Big(z - \frac{x+y}{2}\Big)\right| \leq \epsilon(|x|^2 + |y|^2 + |z|^2)$$

for all x, y, z in $\mathbb{R}$, but that there is no quadratic function $g : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - g(x)| \leq Kx^2$ for all $x \in \mathbb{R}$ and for some $K > 0$. Therefore Appolonius equation is not stable in the sense of Hyers, Ulam and Rassias if $p = 2$ is assumed in the inequality (2.18).

The following corollary is an immediate consequence of Theorem 2.1.

Corollary 2.3. *Assume that for some $\theta > 0$, a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y,z))\| \leq \theta \tag{2.19}$$

for all $x, y, z \in X$. Then there exists a unique quadratic function $Q : X \to Y$ which satisfies (1.3) and the inequality

$$\|f(x) - \frac{f(0)}{2} - Q(x)\| \leq \theta$$

for all $x \in X$

Proof. Putting $\varphi(x,y,z) := \theta$, we get immediately the result. ■

Let X be a normed linear space and let $H : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\varphi_0 : \mathbb{R}_+ \to \mathbb{R}_+$ be mappings such that

$$\begin{aligned} &\varphi_0(\lambda) > 0, \quad \text{for all} \quad \lambda > 0 \\ &\varphi_0(2) \neq 4, \\ &\varphi_0(2\lambda) = \varphi_0(2)\varphi_0(\lambda), \quad \text{for all} \quad \lambda > 0 \\ &H(\lambda u, \lambda v, \lambda w) \leq \varphi_0(\lambda) H(u,v,w), \quad \text{for all} \quad u,v,w \in \mathbb{R}_+, \quad \lambda > 0. \end{aligned} \tag{2.20}$$

Recall that a function $H : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is homogeneous of degree $p > 0$ if it satisfies

$$H(tu, tv, tw) = t^p H(u,v,w) \quad \text{for all} \quad t,u,v,w \in \mathbb{R}_+.$$

Thus in particular, if for $0 < p \neq 2$ we consider $\varphi_0(t) = t^p$, the homogeneous function of degree p is a special case of the preceding properties (2.20).

Now, we consider in the next corollary

$$\varphi(x,y,z) := H(\|x\|, \|y\|, \|z\|).$$

Then one obtains that

$$\begin{aligned} \varphi(2^k x, 2^k y, 2^k z) &= H(2^k\|x\|, 2^k\|y\|, 2^k\|z\|) \\ &\leq \varphi_0(2^k) H(\|x\|, \|y\|, \|z\|) \\ &= \varphi_0(2)^k H(\|x\|, \|y\|, \|z\|), \end{aligned}$$

and

$$\begin{aligned} \varphi\Big(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k}\Big) &= H\Big(\frac{\|x\|}{2^k}, \frac{\|y\|}{2^k}, \frac{\|z\|}{2^k}\Big) \\ &\leq \varphi_0\Big(\frac{1}{2^k}\Big) H(\|x\|, \|y\|, \|z\|) \\ &= \frac{1}{\varphi_0(2)^k} H(\|x\|, \|y\|, \|z\|), \end{aligned} \tag{2.21}$$

since we remark that $1 = \varphi_0(1) = \varphi_0(2)\varphi_0(\frac{1}{2})$ and $\frac{1}{\varphi_0(2)^k} = \varphi_0(\frac{1}{2^k})$ by induction.

Therefore in case $\varphi_0(2) < 4$ we have

$$\begin{aligned}\Phi(x,y,z) &\leq \sum_{k=0}^{\infty} \frac{\varphi_0(2)^k H(\|x\|,\|y\|,\|z\|)}{4^k} \\ &= \frac{4H(\|x\|,\|y\|,\|z\|)}{4-\varphi_0(2)}.\end{aligned}$$

On the other hand, in case $\varphi_0(2) > 4$ it is noted by (2.21) that $H(0,0,0) = 0$ and

$$\begin{aligned}\Psi(x,y,z) &\leq \sum_{k=1}^{\infty} \frac{4^k H(\|x\|,\|y\|,\|z\|)}{\varphi_0(2)^k} \\ &= \frac{4H(\|x\|,\|y\|,\|z\|)}{\varphi_0(2)-4}.\end{aligned}$$

Hence, we see that the following corollary holds by Theorem 2.1.

Corollary 2.4. *Assume that a function $f : X \to Y$ satisfies*

$$\|Df(x,y,z)\| \leq H(\|x\|,\|y\|,\|z\|)$$

for all $x,y,z \in X$. Then there exists a unique quadratic function $Q : X \to Y$ which satisfies (1.3) and either the inequality

$$\|f(x) - \frac{f(0)}{2} - Q(x)\| \leq \frac{3H(\|x\|,\|x\|,\|x\|)}{4-\varphi_0(2)}$$

or

$$\|f(x) - Q(x)\| \leq \frac{3H(\|x\|,\|x\|,\|x\|)}{\varphi_0(2)-4}$$

for all $x \in X$.

In the last part of this paper, let B be a unital Banach algebra with norm $|\cdot|$, and let ${}_B\mathbb{B}_1$ and ${}_B\mathbb{B}_2$ be left Banach B-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively. A quadratic function $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is called B-quadratic if

$$T(ax) = a^2 T(x), \quad \forall a \in B, \forall x \in {}_B\mathbb{B}_1.$$

In the following theorem, we are going to prove the generalized Hyers-Ulam stability problem for Appolonius' equation in Banach modules over a unital Banach algebra.

Theorem 2.5. *Let φ be defined as in Theorem 2.1. Assume that a function $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfies*

$$\|f(\alpha x - \alpha z) + f(\alpha y - \alpha z) - \frac{\alpha^2}{2} f(x-y) - 2\alpha^2 f(z - \frac{x+y}{2})\| \leq \varphi(x,y,z)$$

for all $\alpha \in B$ ($|\alpha| = 1$) and for all $x,y,z \in {}_B\mathbb{B}_1$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, then there exists a unique B-quadratic function $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$, defined as in Theorem 2.1, which satisfies the equation (1.3) and the inequality (2.4) (or (2.5), respectively) for all $x \in {}_B\mathbb{B}_1$.

Proof. By Theorem 2.1, it follows from the inequality of the statement for $\alpha = 1$ that there exists a unique quadratic function $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfying the equation (1.3) and inequality (2.4) (or (2.5), respectively) for all $x \in {}_B\mathbb{B}_1$. Under the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, by the same reasoning as the proof of [3], the quadratic function $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfies

$$T(tx) = t^2 T(x), \quad \forall x \in {}_B\mathbb{B}_1, \forall t \in \mathbb{R}.$$

That is, T is $\mathbb{R}$-quadratic. For each fixed $\alpha \in B$ ($|\alpha| = 1$), replacing f by T and setting $y = -x, z = 0$ in (1.3), we have $T(\alpha x) = \alpha^2 T(x)$ for all $x \in {}_B\mathbb{B}_1$. The last relation is also true for $\alpha = 0$. For each element $a \in B$ ($a \neq 0$), $a = |a| \cdot \frac{a}{|a|}$. Since T is $\mathbb{R}$-quadratic and $T(\alpha x) = \alpha^2 T(x)$ for each element $\alpha \in B(|\alpha| = 1)$,

$$\begin{aligned} T(ax) = T(|a| \cdot \frac{a}{|a|}x) &= |a|^2 \cdot T(\frac{a}{|a|}x) = |a|^2 \cdot \frac{a^2}{|a|^2} \cdot T(x) \\ &= a^2 T(x), \quad \forall a \in B(a \neq 0), \forall x \in {}_B\mathbb{B}_1. \end{aligned}$$

So the unique $\mathbb{R}$-quadratic function $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is also B-quadratic, as desired. This completes the proof. ■

Since $\mathbb{C}$ is a unital Banach algebra, the Banach spaces E_1 and E_2 are considered as Banach modules over $\mathbb{C}$. Thus we have the following corollary.

Corollary 2.6. *Let E_1 and E_2 be Banach spaces over the complex field $\mathbb{C}$. Suppose that a function $f : E_1 \to E_2$ satisfies*

$$\|f(\alpha x - \alpha z) + f(\alpha y - \alpha z) - \frac{\alpha^2}{2} f(x - y) - 2\alpha^2 f(z - \frac{x+y}{2})\| \leq \varphi(x, y, z)$$

for all $\alpha \in \mathbb{C}$ ($|\alpha| = 1$) and for all $x, y \in E_1$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then there exists a unique $\mathbb{C}$-quadratic function $T : E_1 \to E_2$ which satisfies the equation (1.3) and the inequality (2.4) (or (2.5), respectively) for all $x \in E_1$.

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Department of Mathematics
Chungnam National University
220 Yuseong-Gu, Daejeon, 305-764
Republic of Korea
email: kwjun@math.cnu.ac.kr, hmkim@math.cnu.ac.kr