The center of primitive locally pseudoconvex algebras

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Abstract

It is shown that the center of a unital primitive locally A-pseudoconvex Hausdorff algebra over $\mathbb{C}$ and of a unital topologically primitive locally pseudoconvex Fréchet algebra over $\mathbb{C}$ are topologically isomorphic to $\mathbb{C}$.

It is well known (see [9], Corollary 2.4.5; see also [5], Proposition 3, p. 127, and [4], Theorem 2.6.26 (ii), p. 255) that the center of a unital primitive Banach algebra over $\mathbb{C}$ is topologically isomorphic to $\mathbb{C}$. Similar result holds for unital primitive spectral algebras over $\mathbb{C}$ (see the proof of Theorem 6.7 in [7] or [8], Theorem 4.2.11); for unital primitive p-Banach algebras over $\mathbb{C}$ (see [3], Corollary 9.3.7); for unital primitive locally m-convex Q-algebras over $\mathbb{C}$ (see [10], Corollary 2) and for unital primitive locally A-convex algebras over $\mathbb{C}$ in which all maximal ideals are closed (see [11], Theorem 3).

In the present paper we will show that the center of any unital primitive locally A-pseudoconvex Hausdorff algebra over $\mathbb{C}$ and of any unital topologically primitive locally pseudoconvex Fréchet algebra over $\mathbb{C}$ is topologically isomorphic to $\mathbb{C}$.

Received by the editors October 2002.

Communicated by F. Bastin.

1991 *Mathematics Subject Classification :* Primary: 46H05; Secondary: 46H20.

Key words and phrases : Primitive topological algebras, Q-algebras, locally pseudoconvex Fréchet algebras, locally A-pseudoconvex algebras, locally m-pseudoconvex algebras.

1 Introduction

1. Let X be a linear topological space over $\mathbb{K}$ (over $\mathbb{R}$ or $\mathbb{C}$), A an algebra over $\mathbb{K}$ and $\mathcal{L}(X)$ the set of all endomorphisms on X. The addition of elements and the scalar multiplication in $\mathcal{L}(X)$ we define pointwise and the multiplication of elements in $\mathcal{L}(X)$ by the composition (that is, if $f, g \in \mathcal{L}(X)$ and $\lambda \in \mathbb{K}$, then $(f+g)(x) = f(x) + g(x)$, $(\lambda f)(x) = \lambda f(x)$ and $(fg)(x) = f(g(x))$ for each $x \in X$). Then $\mathcal{L}(X)$ is an algebra over $\mathbb{K}$.

Let π be a representation of A on X (that is, a homomorphism or anti-homomorphism of A into $\mathcal{L}(X)$). If we define on X a left module multiplication ${}_{\pi}\cdot$ by $a\,{}_{\pi}\!\cdot x = \pi(a)(x)$ and a right module multiplication $\cdot_{\pi}$ similarly by $x \cdot_{\pi} a = \pi(a)(x)$ for each $a \in A$ and $x \in X$, then X becomes respectively a left A-module, which we denote by ${}_{\pi}X$, and a right A-module, which we denote by X_{π}.

The left[1] A-module ${}_{\pi}X$ is *nontrivial* if $A\,{}_{\pi} \cdot X \neq \{\theta_X\}$, where θ_X is the zero element of X, and is *irreducible* if ${}_{\pi}X$ is a nontrivial left A-module in which ${}_{\pi}X$ and $\{\theta_X\}$ are the only A-submodules of ${}_{\pi}X$. A representation π of A on X is *irreducible* if ${}_{\pi}X$ is an irreducible left A-module (respectively X_{π} is an irreducible right A-module).

2. Let A be a locally pseudoconvex space over $\mathbb{K}$. Then A has a base $\{U_\alpha : \alpha \in \mathcal{A}\}$ of neighbourhoods of zero which consists of balanced (that is, $\mu a \in U_\alpha$ if $a \in U_\alpha$ and $|\mu| \leqslant 1$) and pseudoconvex (that is,

$$U_\alpha + U_\alpha \subset 2^{\frac{1}{k_\alpha}} U_\alpha$$

for a $k_\alpha \in (0, 1]$) sets U_α. It is well known (see [12], p. 4, or [3], p. 189) that the topology of A we can give by a family $\{p_\alpha : \alpha \in \mathcal{A}\}$ of k_α-homogeneous seminorms. In particular, when A is a locally pseudoconvex algebra over $\mathbb{K}$ in which for each $a \in A$ and $\alpha \in \mathcal{A}$ there are positive numbers $M(a, \alpha)$ and $N(a, \alpha)$ such that $p_\alpha(ab) \leqslant M(a, \alpha)p_\alpha(b)$ and $p_\alpha(ba) \leqslant N(a, \alpha)p_\alpha(b)$ for all $b \in A$, then A is a *locally absorbingly pseudoconvex* or *locally A-pseudoconvex* algebra. Furthermore, if all seminorms p_α satisfy the condition $p_\alpha(ab) \leqslant p_\alpha(a)p_\alpha(b)$ for all $a, b \in A$, then A is a *locally multiplicatively pseudoconvex* or *locally m-pseudoconvex* algebra. In case, when $k_\alpha = 1$ for each $\alpha \in \mathcal{A}$, then A is a *locally convex* (respectively *locally A-convex* and *locally m-convex*) algebra. Moreover, if A has a unit element and the set of all invertible elements Inv$A$ of A is open, then A is a *Q-algebra*.

3. Let now A be a locally pseudoconvex algebra over $\mathbb{K}$, the topology of which has been given by a family $\{p_\alpha : \alpha \in \mathcal{A}\}$ of k_α-homogeneous seminorms, M a closed maximal regular left (right) ideal of A, $A - M$ the quotient space of A modulo M endowed with the quotient topology and π_M the canonical homomorphism from A onto $A - M$. Similarly as in [3], p. 108-109, it is easy to show that the quotient topology on $A - M$ coincides with the topology defined on $A - M$ by the family $\{\dot{p}_\alpha : \alpha \in \mathcal{A}\}$ of k_α-homogeneous seminorms, where

$$\dot{p}_\alpha(\pi_M(a)) = \inf\{p_\alpha(a + m) : m \in M\} \tag{1}$$

for each $a \in A$ and $\alpha \in \mathcal{A}$. The algebraic operations in $A - M$ we define, as usual, by $x_1 + x_2 = \pi_M(a_1 + a_2)$ and $\mu x_1 = \pi_M(\mu a_1)$ for each $x_1, x_2 \in A - M$ and $\mu \in \mathbb{K}$,

[1]Nontriviality and irreducibility of the right A-module X_π are defined similarly.

where a_1 and a_2 are arbitrary elements of the M-cosets x_1 and x_2 respectively. Then $A - M$ is a locally pseudoconvex space.

Let now A and B be two locally pseudoconvex spaces, the topologies of which have been given by families $\{p_\alpha : \alpha \in \mathcal{A}\}$ and $\{q_\beta : \beta \in \mathcal{B}\}$ of k_α-homogeneous and r_β-homogeneous seminorms respectively. Then a linear mapping T from A into B is continuous if and only if for each $\beta \in \mathcal{B}$ there exist an index $\alpha \in \mathcal{A}$ and a constant $C_{\alpha\beta} > 0$ such that

$$q_\beta(T(a)) \leqslant C_{\alpha\beta} p_\alpha(a)^{\frac{r_\beta}{k_\alpha}} \tag{2}$$

for each $a \in A$ (see [3], p. 192).

In particular, when A is a locally pseudoconvex Fréchet algebra over $\mathbb{K}$, then the topology of A we can define by such family $\{p_v : v \in \mathbb{N}\}$ of k_v-homogeneous seminorms (here $k_v \in (0, 1]$ for each $v \in \mathbb{N}$) which satisfies the condition

$$p_v(ab) \leqslant p_{v+1}(a)^{\frac{k_v}{k_{v+1}}}\ p_{v+1}(b)^{\frac{k_v}{k_{v+1}}} \tag{3}$$

for each $a, b \in A$ and $v \in \mathbb{N}$ (see [3], p. 207). In this case $A - M$ is a locally pseudoconvex Fréchet space for each closed maximal regular left (right) ideal M of A.

4. Let A be a topological algebra over $\mathbb{K}$. For each element $a \in A$ and each maximal regular left[2] ideal M of A let L_a^M be a mapping from $A - M$ into $A - M$ defined by $L_a^M(x) = ax$ for each $x \in A - M$ and let L_M be a mapping from A into $\mathcal{L}(A - M)$ defined by $L_M(a) = L_a^M$ for each $a \in A$. Then L_M is a representation of A on $A - M$. Since $a_{L_M} \cdot x = L_a^M(x) = ax$ for each $a \in A$ and $x \in A - M$, then ${}_{L_M}(A - M)$ is an irreducible left A-module. Indeed, $A_{L_M} \cdot (A - M)$ is not trivial, because any right regular unit element for M belongs to A. If Y is a nontrivial A-submodule of ${}_{L_M}(A - M)$, then

$$K = \{a \in A : \pi_M(a) \in Y\}$$

is closed with respect to the addition and the multiplications over $\mathbb{K}$ and A. Since $M \subset K$ and $K \setminus M$ is not empty, then $K = A$. But then $\pi_M(u) \in Y$ and $A - M = A\pi_M(u) \subset Y$ (u is a right unit of A modulo M). It means that $Y = A - M$. Hence ${}_{L_M}(A - M)$ is irreducible. Thus, L_M is an irreducible representation of A on $A - M$. Herewith,

$$\ker L_M = \{a \in A : aA \subset M\}$$

if M is a maximal regular left ideal and

$$\ker L_M = \{a \in A : Aa \subset M\}$$

if M is a maximal regular right ideal of A. In both cases $\ker L_M$ is called a *primitive ideal*. A topological algebra A is a *primitive algebra* if there is a maximal regular left (or right) ideal $M \subset A$ such that $\ker L_M = \{\theta_A\}$ and is a *topologically primitive algebra* if this ideal M is closed. In these cases L_M is an isomorphism from A into $\mathcal{L}(A - M)$. Since ${}_{L_M}(A - M)$ is an irreducible left (and $(A - M)_{L_M}$ is an irreducible

[2]The case, when M is a maximal regular right ideal of A, is similar.

right) A-module for each maximal regular left (respectively right) ideal M of A, then[3]

$$\mathcal{D}_A^M = \{T \in \mathcal{L}(A - M)\colon a[T(x)] = T(ax) \text{ for each } a \in A \text{ and } x \in A - M\}$$

is a division subalgebra of $\mathcal{L}(A-M)$ for each maximal left (respectively right) ideal M of A by Schur's lemma (see, for example, [5], p. 127).

For each closed maximal regular left ideal M of A and $a_0 \notin M$ let

$$L_M(a_0) = \{a \in A : aa_0 \in M\}.$$

Then $L_M(a_0)$ is a closed (proper) left ideal of A. If J is a left ideal of A such that $L_M(a_0) \subset J$ and $J \setminus L_M(a_0)$ is not empty, then $J\pi_M(a_0)$ is a left A-submodule of $A - M$ and $J\pi_M(a_0) \neq \{\theta_{A-M}\}$. Since ${}_{L_M}(A - M)$ is irreducible, then $J\pi_M(a_0) = A - M$. Therefore there is an element $e \in J$ such that $ea_0 - a_0 \in M$. Hence from $(a - ae)a_0 = a(a_0 - ea_0) \in M$ follows that $a - ae \in L(a_0) \subset J$ for each $a \in A$, because of which $a = (a - ae) + ae \in J$ for each $a \in A$. It means that $A = J$, which is not possible. Therefore, $L_M(a_0)$ is a closed maximal regular left ideal of A.

In the same way it is easy to show that the set $\{a \in A : a_0a \in M\}$ is a closed maximal regular right ideal of A for each closed maximal regular right ideal M of A.

2 Description of $\mathcal{D}_A^M$

To describe the set $\mathcal{D}_A^M$ in case of locally pseudoconvex algebra A over $\mathbb{C}$, we need the following

Lemma 1. (see [5], p. 127) *Let A be an algebra over $\mathbb{K}$, M a maximal regular left (right) ideal of A, $a_0 \in A \setminus M$ and π_L the canonical homomorphism from A onto $A - L_M(a_0)$. For each $y \in A - L_M(a_0)$ let U be a mapping from $A - L_M(a_0)$ into $A - M$ defined by*

$$U(y) = a\pi_M(a_0) \quad (\text{respectively } U(y) = \pi_M(a_0)a),$$

where a is any element of A for which $y = \pi_L(a)$. Then U is a module isomorphism from $A - L_M(a_0)$ onto $A - M$.

Proof. If $a, b \in A$ are such that $\pi_L(a) = \pi_L(b)$, then[4] $(a - b)a_0 \in M$. Hence $a\pi_M(a_0) = b\pi_M(a_0)$. Thus the mapping U is a well defined linear mapping. If $U(\pi_L(a)) = \theta_{A-M}$, then $a \in L_M(a_0)$. Therefore $\pi_L(a) = \theta_{A-L_M(a_0)}$. It means that U is injective. For all $b \in A$ and $\pi_L(a) \in A - L_M(a_0)$ we have that

$$U(b\pi_L(a)) = U(\pi_L(ba)) = b(a\pi_M(a_0)) = b(U(\pi_L(a))).$$

So U is a module isomorphism which is surjective because ${}_{L_M}(A - M)$ is irreducible. ∎

[3] If M is a maximal regular right ideal of A, then

$$\mathcal{D}_A^M = \{T \in \mathcal{L}(A - M)\colon [T(x)]a = T(xa) \text{ for each } a \in A \text{ and } x \in A - M\}.$$

[4] Here and later we present proofs of results only for left ideals, because all the proofs for right ideals are similar.

Theorem 1. *Let A be a locally m-pseudoconvex Hausdorff algebra over $\mathbb{C}$ with a unit element or a locally pseudoconvex Fréchet algebra over $\mathbb{C}$ with a unit element. If M is a closed maximal left (right) ideal of A, then $\mathcal{D}_A^M$ is topologically isomorphic to $\mathbb{C}$.*

Proof. a) Let A be a locally m-pseudoconvex Hausdorff algebra, topology of which has been given by a saturated family $\{p_\alpha : \alpha \in \mathcal{A}\}$ of k_α-homogeneous submultiplicative seminorms (here $k_\alpha \in (0,1]$ for each $\alpha \in \mathcal{A}$), M a closed maximal left ideal of A, π_M the canonical homomorphism from A onto $A - M$, π_L the canonical homomorphism from A onto $A - L_M(a_0)$ for a fixed $a_0 \in A \setminus M$ and U the module isomorphism from $A - L_M(a_0)$ onto $A - M$ defined by Lemma 1. Then

$$U^{-1} \circ T \circ U \in \mathcal{L}(A - L_M(a_0))$$

for each $T \in \mathcal{D}_A^M$. Moreover, if $y = \pi_L(a) = \pi_L(a')$ and

$$(U^{-1} \circ T)(\pi_M(a_0)) = \pi_L(b_T),$$

then

$$(U^{-1} \circ T \circ U)(y) = U^{-1}[T(a\pi_M(a_0))] = U^{-1}[T(a'\pi_M(a_0))] = U^{-1}[a'T(\pi_M(a_0))] = a'[(U^{-1} \circ T)(\pi_M(a_0))] = a'\pi_L(b_T).$$

For each $\alpha \in \mathcal{A}$ and $a \in A$ let

$$\tilde{p}_\alpha(\pi_L(a)) = \inf\{p_\alpha(a + l) : l \in L_M(a_0)\}. \tag{4}$$

Then $\{\tilde{p}_\alpha : \alpha \in \mathcal{A}\}$ is the family of k_α-homogeneous seminorms on $A - L_M(a_0)$ which defines the quotient topology on it. Herewith, for each nonzero $x \in A - L_M(a_0)$ there is an index $\alpha_0 \in \mathcal{A}$ such that $\tilde{p}_{\alpha_0}(x) > 0$, because $A - L_M(a_0)$ is a Hausdorff space. Since

$$\tilde{p}_\alpha[(U^{-1} \circ T \circ U)(y)] = \tilde{p}_\alpha(a'\pi_L(b_T)) = \tilde{p}_\alpha(\pi_L(a'b_T)) = \inf\{p_\alpha(a'b_T + l) : l \in L_M(a_0)\} \leqslant \inf\{p_\alpha(a'(b_T + l)) : l \in L_M(a_0)\} \leqslant p_\alpha(a')\tilde{p}_\alpha(\pi_L(b_T))$$

for each $a' \in a + L_M(a_0)$, then taking the infimum over all elements of $a + L_M(a_0)$, we have

$$\tilde{p}_\alpha[(U^{-1} \circ T \circ U)(y)] \leqslant K_{\alpha,T}\, \tilde{p}_\alpha(y)$$

for each $\alpha \in \mathcal{A}$ and $y \in A - L_M(a_0)$, where $K_{\alpha,T} > \tilde{p}_\alpha(\pi_L(b_T))$ for each $\alpha \in \mathcal{A}$. Taking this into account, for each $\alpha \in \mathcal{A}$ we can define k_α-homogeneous seminorms r_α on $\mathcal{D}_A^M$ by

$$r_\alpha(T) = \sup_{\tilde{p}_\alpha(\pi_L(a)) \leqslant 1} \tilde{p}_\alpha[(U^{-1} \circ T \circ U)(\pi_L(a))]$$

for each $T \in \mathcal{D}_A^M$. Then r_α is a k_α-homogeneous seminorm on $\mathcal{D}_A^M$ for each $\alpha \in \mathcal{A}$. To show that every r_α is submultiplicative, for each $a \in A$ let $\mu_a = n$ (for any fixed $n \in \mathbb{N}$) if $\tilde{p}_\alpha(\pi_L(a)) = 0$ and $\mu_a = (\tilde{p}_\alpha(\pi_L(a)))^{-1}$ if $\tilde{p}_\alpha(\pi_L(a)) > 0$. Then

$$\tilde{p}_\alpha(\pi_L(\mu_a^{\frac{1}{k_\alpha}} a)) \leqslant 1.$$

Therefore from

$$p_\alpha[(U^{-1}\circ T\circ U)(\pi_L(\mu_a^{\frac{1}{k_\alpha}}a))]\leqslant r_\alpha(T)$$

follows that

$$p_\alpha[(U^{-1}\circ T\circ U)(\pi_L(a))]\leqslant r_\alpha(T)\ \tilde p_\alpha(\pi_L(a)) \tag{5}$$

for each $a\in A$ (if $\tilde p_\alpha(\pi_L(a))=0$, then from

$$p_\alpha[(U^{-1}\circ T\circ U)(\pi_L(a))]\leqslant \frac{1}{n}\ r_\alpha(T)$$

follows that $\tilde p_\alpha[(U^{-1}\circ T\circ U)(\pi_L(a))]=0$). Hence

$$r_\alpha(T_1\circ T_2)=\sup_{\tilde p_\alpha(\pi_L(a))\leqslant 1} p_\alpha[(U^{-1}\circ(T_1\circ T_2)\circ U)(\pi_L(a))]=$$

$$\sup_{\tilde p_\alpha(\pi_L(a))\leqslant 1} p_\alpha\{(U^{-1}\circ T_1\circ U)[(U^{-1}\circ T_2\circ U)(\pi_L(a))]\}\leqslant$$

$$r_\alpha(T_1)\sup_{\tilde p_\alpha(\pi_L(a))\leqslant 1} p_\alpha[(U^{-1}\circ T_2\circ U)(\pi_L(a))]\leqslant r_\alpha(T_1)r_\alpha(T_2)$$

for each $T_1,T_2\in\mathcal{D}_A^M$.

Let now T be a nonzero element of $\mathcal{D}_A^M$. Then $U^{-1}\circ T\circ U$ is a nonzero element of $\mathcal{L}(A-M)$. Therefore there is an element $a\in A$ such that $(U^{-1}\circ T\circ U)(\pi_L(a))$ is nonzero in $A-L_M(a_0)$. Taking this into account, there is an index $\alpha_0\in\mathcal{A}$ such that

$$0<\tilde p_{\alpha_0}[(U^{-1}\circ T\circ U)(\pi_L(a))]\leqslant r_{\alpha_0}(T)\ \tilde p_{\alpha_0}(\pi_L(a))$$

by (5). Hence $r_{\alpha_0}(T)>0$. Consequently, $\mathcal{D}_A^M$ is a locally m-pseudoconvex Hausdorff division algebra in the topology defined on $\mathcal{D}_A^M$ by the family $\{r_\alpha:\alpha\in\mathcal{A}\}$ of k_α-homogeneous submultiplicative seminorms and therefore $\mathcal{D}_A^M$ is topologically isomorphic to $\mathbb{C}$ (see [1], Corollary 1, or [2], Theorem 3.2).

b) Let now A be a locally pseudoconvex Fréchet algebra over $\mathbb{C}$ and $\{p_v:v\in\mathbb{N}\}$ a family of k_v-homogeneous seminorms on A with $k_v\in(0,1]$ for each $v\in\mathbb{N}$, which defines the topology of A. We can assume that the system $\{p_v:v\in\mathbb{N}\}$ satisfies the condition (3) for each $v\in\mathbb{N}$ and each $a,b\in A$. Let again M be a closed maximal left ideal of A, π_M the canonical homomorphism from A onto $A-M$ and π_L the canonical homomorphism from A onto $A-L_M(a_0)$ for some $a_0\in A\setminus M$. Let $\{\dot p_v:v\in\mathbb{N}\}$ and $\{\tilde p_v:v\in\mathbb{N}\}$ denote the families of k_v-homogeneous seminorms (defined by (1) and (4)), which define the quotient topologies on $A-M$ and $A-L_M(a_0)$ respectively, and let U be the module isomorphism from $A-L_M(a_0)$ onto $A-M$ defined by Lemma 1. Then $A-M$ and $A-L_M(a_0)$ are locally pseudoconvex Fréchet spaces (because M and $L_M(a_0)$ are closed linear subspaces of A), $U^{-1}\circ T\circ U\in\mathcal{L}(A-L_M(a_0))$ for each $T\in\mathcal{D}_A^M$ and

$$\dot p_v(U(\pi_L(a)))=\dot p_v(a'\pi_M(a_0))=\dot p_v(\pi_M(a'a_0))=$$

$$\inf\{p_v(a'a_0+m):m\in M\}\leqslant\inf\{p_v(a'(a_0+m)):m\in M\}\leqslant$$

$$p_{v+1}(a')^{\frac{k_v}{k_{v+1}}}\inf\{p_{v+1}(a_0+m)^{\frac{k_v}{k_{v+1}}}:m\in M\}\leqslant$$

$$p_{v+1}(a')^{\frac{k_v}{k_{v+1}}}\ \dot p_{v+1}(\pi_M(a_0))^{\frac{k_v}{k_{v+1}}}$$

for each $a' \in a + L_M(a_0)$ by (3). Hence

$$\dot{p}_v(U(\pi_L(a))) \leqslant C_{v,M}\ \tilde{p}_{v+1}(\pi_L(a))^{\frac{k_v}{k_{v+1}}}$$

for each $\pi_L(a) \in A - L_M(a_0)$ and $v \in \mathbb{N}$, where

$$C_{v,M} > \tilde{p}_{v+1}(\pi_M(a_0))^{\frac{k_v}{k_{v+1}}}$$

for all $v \in \mathbb{N}$. Therefore U is a continuous mapping by (2).

Let now $a \in A$ and $y \in A - M$. Then there is an element $b \in A$ such that $y = \pi_M(b)$. Since

$$\dot{p}_v(L_M(a)(y)) = \dot{p}_v(\pi_M(ab)) = \inf\{p_v(ab+m) : m \in M\} \leqslant$$
$$\inf\{p_v(a(b+m)) : m \in M\} \leqslant$$
$$p_{v+1}(a)^{\frac{k_v}{k_{v+1}}} \inf\{p_{v+1}(b+m)^{\frac{k_v}{k_{v+1}}} : m \in M\} \leqslant$$
$$M_v(a)\dot{p}_{v+1}(y)^{\frac{k_v}{k_{v+1}}}$$

by (3), where

$$M_v(a) > p_{v+1}(a)^{\frac{k_v}{k_{v+1}}}$$

for all $v \in \mathbb{N}$, then $L_M(a)$ is a continuous mapping for each $a \in A$ by (2).

Defining seminorms r_v on $\mathcal{D}_A^M$ similarly as above by

$$r_v(T) = \sup_{\tilde{p}_v(\pi_L(a)) \leqslant 1} \tilde{p}_v[(U^{-1} \circ T \circ U)(\pi_L(a))]$$

for each $T \in \mathcal{D}_A^M$ and $v \in \mathbb{N}$, then $\mathcal{D}_A^M$ (in the topology defined by the family $\{r_v : v \in \mathbb{N}\}$) is a metrizable locally pseudoconvex division algebra over $\mathbb{C}$. To show that $\mathcal{D}_A^M$ is complete, let (T_n) be a Cauchy sequence in $\mathcal{D}_A^M$. Then for each $v \in \mathbb{N}$ and $\varepsilon > 0$ there is a number $N = N(v, \varepsilon) \in \mathbb{N}$ such that $r_v(T_m - T_n) < \varepsilon$ whenever $m > n > N$. Hence

$$\tilde{p}_v[(U^{-1} \circ (T_m - T_n) \circ U)(\pi_L(a))] < \varepsilon \tilde{p}_v(\pi_L(a)) \tag{6}$$

whenever $m > n > N$ for each $y = \pi_L(a) \in A - L_M(a_0)$ by the inequality (5). Therefore $((U^{-1} \circ T_n \circ U)(y))$ is a Cauchy sequence in $A - L_M(a_0)$ for each $y \in A - L_M(a_0)$. Since $A - L_M(a_0)$ is complete, then the sequence $((U^{-1} \circ T_n \circ U)(y))$ converges in $A - L_M(a_0)$ for each $y \in A - L_M(a_0)$. It means that for each $a \in A$ there exists the limit

$$T(\pi_L(a)) = \lim_{n \to \infty} (U^{-1} \circ T_n \circ U)(\pi_L(a)). \tag{7}$$

It is easy to show that T is a module homomorphism from $A - L_M(a_0)$ into $A - L_M(a_0)$. Therefore $U \circ T \circ U^{-1} \in \mathcal{L}((A - M)_{L_M})$. To show that $U \circ T \circ U^{-1} \in \mathcal{D}_A^M$, let x be an arbitrary element of $A - M$. Then there is an element $b \in A$ such that $U^{-1}(x) = \pi_L(b)$. Since U is a continuous module isomorphism from $A - L_M(a_0)$ onto $A - M$ and $L_M(a)$ is continuous for each $a \in A$, then

$$a[(U \circ T \circ U^{-1})(x)] = (aU)[T(U^{-1}(x))] = (aU)[T(\pi_L(b))] =$$

$$(L_M(a)\circ U)[\lim_{n\to\infty}(U^{-1}\circ T_n\circ U)(\pi_L(b))] = \lim_{n\to\infty}(L_M(a)\circ T_n)(x) =$$
$$\lim_{n\to\infty} a[T_n(x)] = \lim_{n\to\infty} T_n(ax) = \lim_{n\to\infty} T_n[a(U(\pi_L(b)))] =$$
$$\lim_{n\to\infty} T_n[U(\pi_L(ab))] = U[\lim_{n\to\infty}(U^{-1}\circ T_n\circ U)(\pi_L(ab))] =$$
$$(U\circ T)[\pi_L(ab)] = (U\circ T)[a\pi_L(b)] = (U\circ T)[a(U^{-1}(x))] =$$
$$(U\circ T)[U^{-1}(ax)] = (U\circ T\circ U^{-1})(ax)$$

for each $a \in A$ and $x \in A - M$. Consequently, $U\circ T\circ U^{-1} \in \mathcal{D}_A^M$.

If now $m \to \infty$ in (6), then

$$\tilde{p}_v[(T - U^{-1}\circ T_n\circ U)(\pi_L(a))] \leqslant \varepsilon\tilde{p}_v(\pi_L(a)) \tag{8}$$

by (7) whenever $n > N$ for each $\pi_L(a) \in A - L_M(a_0)$. Therefore

$$r_v(U\circ T\circ U^{-1} - T_n) = \sup_{\tilde{p}_v(\pi_L(a))\leqslant 1} \tilde{p}_v[(T - (U^{-1}\circ T_n\circ U))(\pi_L(a))] \leqslant$$
$$\sup_{\tilde{p}_v(\pi_L(a))\leqslant 1} \varepsilon\tilde{p}_v(\pi_L(a)) = \varepsilon$$

by (8) whenever $n > N$. Consequently, the sequence (T_n) converges in the topology of $\mathcal{D}_A^M$. It means that $\mathcal{D}_A^M$ is complete. Thus $\mathcal{D}_A^M$ is a locally pseudoconvex Fréchet division algebra over $\mathbb{C}$ and therefore $\mathcal{D}_A^M$ is topologically isomorphic to $\mathbb{C}$ (see again [1], Corollary 1, or [2], Corollary 3.1). ■

3 The center of primitive locally pseudoconvex algebras

Let A be a primitive topological algebra over $\mathbb{C}$. When the center $Z(A)$ of A is a field? The following result gives an answer to this question in case of locally pseudoconvex algebras.

Theorem 2. *Let A be a unital primitive locally A-pseudoconvex Hausdorff algebra over $\mathbb{C}$ or a unital topologically primitive locally pseudoconvex Fréchet algebra over $\mathbb{C}$. Then the center $Z(A)$ of A is topologically isomorphic to $\mathbb{C}$.*

Proof. Let A be a unital primitive locally A-pseudoconvex Hausdorff algebra over $\mathbb{C}$. Then A has a topology τ', finer than τ (see [2], Lemma 2.2), such that (A, τ') is a locally m-pseudoconvex Hausdorff algebra over $\mathbb{C}$. Let β be a base of τ' and τ_A the topology on A which subbase is $\beta_A = \beta \cup \{\mathrm{Inv}A\}$. Then it is easy to see that (A, τ_A) is a locally m-pseudoconvex Q-algebra over $\mathbb{C}$. Because A is primitive, then there is a closed maximal left ideal M of A such that L_M is an isomorphism from A into $\mathcal{L}(A-M)$. The same statement is true in case when A is a unital topologically primitive locally pseudoconvex Fréchet algebra over $\mathbb{C}$. Since $L_M(Z(A)) \subset \mathcal{D}_A^M$ and $\mathcal{D}_A^M$ is topologically isomorphic to $\mathbb{C}$ in both cases by Theorem 1, then in both cases $Z(A)$ is also topologically isomorphic to $\mathbb{C}$ (because $Z(A)$ is a Hausdorff space). ■

References

[1] M. Abel, *Gelfand-Mazur algebras.* In "Topological vector spaces, algebras and related areas. Pitman Research Notes in Math. Series **316**, Longman Group Ltd., Harlow, 1994", 116–129.

[2] M. Abel, A. Kokk, *Locally pseudoconvex Gelfand-Mazur algebras.* Eesti NSV Tead. Akad. Toimetised, Füüs.-Mat., **37**, 1988, 377–386 (Russian).

[3] V. K. Balachandran, *Topological Algebras.* North-Holland Math. Studies **185**, Elsevier, Amsterdam, 2000.

[4] H. G. Dales, *Banach algebras and automatic continuity.* London Math. Soc. Monographs. New Series **24**, Oxford Univ. Press, New York, 2000

[5] F. F. Bonsal, J. Duncan, *Complete normed algebras.* Springer-Verlag, Berlin, 1973.

[6] A. Mallios, *Topological Algebras. Selected Topics.* North-Holland Math. Stud. **124**, Elsevier Sci. Publ., Amsterdam, 1986.

[7] Th. W. Palmer, *Spectral algebras.* Rocky Mountain J. Math., **22**, (1992), no. 1, 293– 328.

[8] Th. W. Palmer, *Banach algebras and the general theory of* **-algebras. Vol. I. Algebras and Banach algebras.* Encyclopedia of Math. and Appl. **49**, Cambridge Univ. Press, Cambridge, 1994.

[9] C. E. Rickart, *General Topology of Banach Algebras.* D. van Nostrand, Princeton, 1960.

[10] Y. Tsertos, *On primitive topological algebras.* Bull. Math. Greek Mat. Soc.,**28**, (1987), 81–92.

[11] Y. Tsertos, *On primitive A-convex algebras.* Comment. Math. Prace Mat., **41** (2001), 203–219.

[12] L. Waelbroeck, *Topological vector spaces and algebras.* Lecture Notes in Math. **230**, Springer-Verlag, Berlin, 1973.

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