Reflexivity and AK-property of certain vector sequence spaces

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Abstract

Let E be a Banach space and Λ a Banach perfect sequence space. Denote by $\Lambda(E)$ the space of all Λ-summable sequences from E. In this note it is proved that $\Lambda(E)$ is reflexive if and only if Λ and E are reflexive and each member of $\Lambda(E)$ is the limit of its finite sections.

1 Introduction and preliminaries

Let Λ be a normal sequence space, with Köthe dual Λ^* and let E be a locally convex space. A sequence $(x_n)_n \subset E$ is said to be Λ-*summable* if the series $\sum \alpha_n x_n$ converges in E for all $(\alpha_n)_n$ in Λ^*. Denote by $\Lambda(E)$ the linear space of all Λ-summable sequences from E. These spaces were introduced by A. Pietsch in [6] and some of their properties were studied in [5] when Λ is endowed with the normal topology. The general case was studied by M. Florencio and P. J. Paúl in [2] and [3]. At this point, we refer the reader to [5] for details concerning Köthe theory of sequence spaces.

In this note, we give a partial solution to the following general question asked to the author by Prof. Paúl: When is the subspace $\Lambda(E)_r$ of $\Lambda(E)$, consisting of those sequences which are the limit of their sections, reflexive? Since $\Lambda(E)_r$ is isometrically isomorphic to the complete injective tensor product $\Lambda\widetilde{\otimes}_\varepsilon E$ of Λ and E (Prop. 2 in [2]), the answer would give conditions for $\Lambda\widetilde{\otimes}_\varepsilon E$ to be reflexive.

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In the sequel E stands for a Banach space with norm $||\cdot||_E$ and E^* denotes its topological dual, and Λ stands for a Banach perfect sequence space whose norm $||\cdot||_\Lambda$ satisfies:

(1) For all α and β in Λ, if $\alpha \leq \beta$, then $||\alpha||_\Lambda \leq ||\beta||_\Lambda$, and

(2) $(\Lambda, ||\cdot||_\Lambda)$ is an AK-space, i.e., every $\alpha = (\alpha_n)_n \in \Lambda$ is the $||\cdot||_\Lambda$-limit of its finite sections $(\alpha_1, \ldots, \alpha_n, 0, \ldots), n \in \mathbb{N}$. It is well-known that this condition holds if, and only if, the topological dual of Λ coincides with its Köthe dual Λ^*. In particular, Λ is reflexive if, and only if, $(\Lambda^*, ||\cdot||_{\Lambda^*})$ is also an AK-space.

The following proposition, whose proof is similar to that of Prop. 1 in [2], defines a natural norm on $\Lambda(E)$:

Proposition 1: *For every $x = (x_n)_n \in \Lambda(E)$ and $x^* \in E^*$, $(\langle x^*, x_n\rangle)_n$ belongs to Λ and*

$$||x||_{\Lambda(E)} =: \sup\{||(\langle x^*, x_n\rangle)_n||_\Lambda, x^* \in E^* : ||x^*||_{E^*} \leq 1\}$$

defines a (complete, whenever E is complete) norm on $\Lambda(E)$. Moreover, Λ and E are closed topological subspaces of $(\Lambda(E), ||\cdot||)$.

2 Reflexivity of $\Lambda(E)$

We start with the following result.

Proposition 2: *$\Lambda(E)$ is isometrically isomorphic to a closed subspace of the space $L(\Lambda^*, E)$ of all bounded linear operators from Λ^* to E. If, in addition, Λ^* is an AK-space, this embedding is even onto.*

Proof: For every $x = (x_n)_n \in \Lambda(E)$, let $T_x : \Lambda^* \to E$ be defined by $T_x(\alpha) = \sum_{n=1}^{\infty} \alpha_n x_n$. Then

$$\begin{aligned} ||T_x(\alpha)||_E &= \left\|\sum_{n=1}^{\infty} \alpha_n x_n\right\|_E \\ &= \sup\left\{\left|\sum_{n=1}^{\infty} \alpha_n \langle x^*, x_n\rangle\right| : x^* \in E^*, ||x^*||_{E^*} \leq 1\right\} \\ &\leq ||\alpha||_{\Lambda^*} ||x||_{\Lambda(E)}. \end{aligned}$$

Therefore, T_x is bounded. Moreover,

$$\begin{aligned} ||T_x|| &= \sup\left\{\left\|\sum_{n=1}^{\infty} \alpha_n x_n\right\|_E : ||\alpha||_{\Lambda^*} \leq 1\right\} \\ &= \sup\left\{\sum_{n=1}^{\infty} |\alpha_n \langle x^*, x_n\rangle| : ||x^*||_{E^*} \leq 1, ||\alpha||_{\Lambda^*} \leq 1\right\} \\ &= \sup\{\|\langle x^*, x_n\rangle)_n\|_\Lambda : ||x^*||_{E^*} \leq 1\} = ||x||_{\Lambda(E)}. \end{aligned}$$

Hence the linear mapping $T : \Lambda(E) \to L(\Lambda^*, E)$, $x \mapsto T_x$ is an isometry. It remains to show that the range of T is closed in $L(\Lambda^*, E)$. Given $u \in \overline{T(\Lambda(E))}^{L(\Lambda^*,E)}$ and $\varepsilon > 0$, take $a \in \Lambda(E)$ such that $||T_a - u||_{L(\Lambda^*,E)} \leq \varepsilon/2$. We shall prove that, if e_n is the n-th unit vector of Λ^* and $x = (u(e_n))_n$, then $x \in \Lambda(E)$. Indeed, for every

$\alpha = (\alpha_n)_n \in \Lambda^*$ (we may and do assume that $||\alpha||_{\Lambda^*} \leq 1$), there exists $n_0 \in \mathbb{N}$ such that, for every $n > m > n_0$, we have $||T_a(\sum_{p=m}^n \alpha_p e_p)|| \leq \varepsilon/2$. Therefore,

$$\begin{aligned}\left\|\sum_{p=m}^{n} \alpha_p u(e_p)\right\| &= \left\|u(\sum_{p=m}^{n} \alpha_p e_p)\right\| \\ &\leq \left\|(u - T_a)(\sum_{p=m}^{n} \alpha_p e_p)\right\| + \left\|T_a(\sum_{p=m}^{n} \alpha_p e_p)\right\| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This shows that $\sum \alpha_p u(e_p)$ is a Cauchy, hence convergent, series in E, that is, $x \in \Lambda(E)$. Since $u = T_x$, this shows that $\Lambda(E)$ is closed.

Finally, assume that Λ^* is an AK-space and take $v \in L(\Lambda^*, E)$. Since for every $\alpha = (\alpha_n)_n \in \Lambda^*$ the series $\sum \alpha_n e_n$ converges in E, we have that $v(\alpha) = \sum \alpha_n v(e_n)$. This shows that $y = (v(e_n))_n$ belongs to $\Lambda(E)$ and that $u = T_y$, so that T is onto. ■

Proposition 3: *Let x be in $\Lambda(E)$. Then T_x is compact if, and only if, x is the limit of its finite sections. Moreover, if Λ^* is an AK-space, then $\Lambda(E)_r$ is isometrically isomorphic to the subspace $K(\Lambda^*, E)$ of $L(\Lambda^*, E)$ of all compact operators from Λ^* to E.*

Proof: We introduce the following notation. If $z = (z_n)$ is a sequence, then $z^{\langle k\rangle}$ stands for the difference between z and its k-th finite section, that is, $z^{\langle k\rangle} = (0, 0, \ldots, 0, z_{k+1}, z_{k+2}, \ldots)$.

Now for the proof, let $x = (x_n)_n \in \Lambda(E)$ be the limit of its finite sections. Since $T_k : \Lambda^* \to E$, $T_k(\alpha) = \sum_{n=1}^k \alpha_n x_n$ is of finite rank for every $k \in \mathbb{N}$ and, by the preceding proposition,

$$||T_x - T_k||_{L(\Lambda^*,E)} = ||T_{x^{\langle k\rangle}}||_{L(\Lambda^*,E)}||(0, 0, \ldots, 0, x_{k+1}, x_{k+2}, \ldots)||_{\Lambda(E)},$$

hence T_x is compact. Conversely, assume that T_x is compact but x is not the limit of its finite sections. Then there exist $\varepsilon > 0$ and a subsequence $(u_j = T_{x^{\langle k_j\rangle}})_j$ of $(T_{x^{\langle k\rangle}})_k$ such that $||u_j|| > \varepsilon$, for every $j \in \mathbb{N}$ and we have

$$\sup\{||u_j^*(x^*)||\colon ||x^*|| \leq 1\} = ||u_j^*|| = ||u_j|| > \varepsilon.$$

where u_j^* denotes the adjoint of u_j. Now choose a sequence $(x_j^*)_j$ in the unit ball of E^* such that for every $j \in \mathbb{N}$ we have $||u_j^*(x_j^*)|| > \varepsilon$. But, on the other hand, for every $x^* \in E^*$ and $\alpha = (\alpha_n) \in \Lambda^*$, we have

$$\langle T_x^*(x^*), \alpha\rangle = x^* T_x(\alpha) = x^* \left(\sum_{n=1}^{\infty} \alpha_n x_n\right) = \langle(\langle x^*, x_n\rangle)_n, \alpha\rangle.$$

Since $(\langle x^*, x_n\rangle)_n$ belongs to Λ, it follows that T_x^* takes its values in Λ. Therefore, by Schauder theorem, T_x^* is a compact operator from E^* to Λ and, passing to a subsequence if necessary, we may assume that $(T_x^* x_j^*)$ is a convergent sequence with limit $\beta \in \Lambda$. Take $j_0 \in \mathbb{N}$ such that $||T_x^* x_j^* - \beta||_\Lambda \leq \varepsilon/2$ for every $j \geq j_0$, then for all $k \in \mathbb{N}$ we have

$$\left|||(T_x^* x_j^*)^{\langle k\rangle}||_\Lambda - ||\beta^{\langle k\rangle}||_\Lambda\right| \leq ||(T_x^* x_j^*)^{\langle k\rangle} - \beta^{\langle k\rangle}||_\Lambda \leq ||T_x^* x_j^* - \beta||_\Lambda \leq \varepsilon/2.$$

A straightforward computation shows that

$$u_j^*(x_j^*) = T_{x^{\langle k_j \rangle}}(x_j^*) = (T_x^* x_j^*)^{\langle k_j \rangle},$$

whence

$$\varepsilon \le ||u_j^*(x_j^*)|| = ||(T_x^* x_j^*)^{\langle k_j \rangle}||_\Lambda \le \varepsilon/2 + ||\beta^{\langle k_j \rangle}||_\Lambda$$

but this is a contradiction with the fact that $(\beta^{\langle j \rangle})$ is a null sequence because Λ is an AK-space. This finishes the proof that T_x is compact if, and only if, x is the limit of its finite sections.

Now, if Λ^* is also an AK-space, then, as we proved in the preceding proposition, $x := (u(e_n))$ belongs to $\Lambda(E)$ for all $u \in L(\Lambda^*, E)$ and, moreover, $T_x = u$. The conclusion follows from the first part of this proposition. ∎

We now give the promised characterization of the reflexivity of $\Lambda(E)$.

Main Theorem: *The following assertions are equivalent: (a)* $\Lambda(E)$ *is reflexive.*

(b) Λ *and* E *are reflexive and* $\Lambda(E)$ *is an AK-space.*

(c) $\Lambda \tilde{\otimes}_\varepsilon E$ *is reflexive.*

Proof: Assume that $\Lambda(E)$ is reflexive. Then, by Proposition 1, Λ and E are reflexive and, in particular, according to the remark made in condition (2) preceding Proposition 1, Λ^* is an AK-space so that both Λ and Λ^* have the approximation property. On the other hand, by using Proposition 2 we have that $L(\Lambda^*, E) = \Lambda(E)$ is reflexive hence, by Theorem 2 of [4], $L(\Lambda^*, E) = K(\Lambda^*, E)$. Using Proposition 3, we get $\Lambda(E) = \Lambda(E)_r$ and this proves that (a) implies (b).

Next, if (b) holds, an application of Propositions 2 and 3, and Theorem 44.2.(6) of [5] or Proposition 2 in [2], gives

$$L(\Lambda^*, E) = \Lambda(E) = \Lambda(E)_r = K(\Lambda^*, E) = \Lambda \tilde{\otimes}_\varepsilon E.$$

Again, by Theorem 2 of [4], $L(\Lambda^*, E) = \Lambda \tilde{\otimes}_\varepsilon E$ is reflexive, and this shows that (c) holds.

Finally suppose that $\Lambda \tilde{\otimes}_\varepsilon E$ is reflexive. Then Λ and E, which can be seen as closed subspaces of $\Lambda \tilde{\otimes}_\varepsilon E = \Lambda(E)_r$, are reflexive. It is well-known then (see, e.g., [1], p. 247) that

$$(\Lambda \tilde{\otimes}_\varepsilon E)^{**} = (\Lambda^* \tilde{\otimes}_\pi E^*)^* = L(\Lambda^*, E^{**}) = L(\Lambda^*, E),$$

and Proposition 2 yields the reflexivity of $\Lambda(E)$. ∎

Final remarks. There are examples of Banach spaces Λ and E for which $\Lambda(E)$ is reflexive. Such are given by $\ell_p(\ell_q)$ where $p, q \in (1; \infty)$ and $pq < p + q$, that is, $q < p^*$, the latter being the conjugate of p. The reflexivity follows from Proposition 2 and the fact that $L(\ell_{p^*}, \ell_q)$ is reflexive if $q < p^*$. (See e.g [1] p. 248). Actually, this condition turns out to be necessary. In particular, the spaces $\ell_p(\ell_q)$ are never reflexive if $2 \le p \le q$, but they are always reflexive whenever $1 < p \le q < 2$. In particular, for every separable Hilbert space H, $\ell_p(H)$ is reflexive if and only if $1 < p < 2$.

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