EXPOSITORY PAPER

From Delaunay to the Hopf problem : on bubbles and curvature

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Abstract

Is any soap bubble necessarily a Euclidean sphere ? In this expository text, we show how this simple question has led over the years to developments in many fields of mathematics.

Introduction

A simple physical experiment – which we all tried at a very early age – seems to convince us that soap bubbles (when they are not glued together) are simply Euclidean spheres and not – say – tori.

Is this a mathematical theorem, or a reflection on the way we produce the bubbles ?

In this expository text, we sketch the history of this question and some of the mathematical developments it motivated. We assume no previous knowledge of the subject – the text should be understandable by last year university students.

We stress that this is by no means a full survey, so that neither the text nor the bibliography gives a full account of all contributions.

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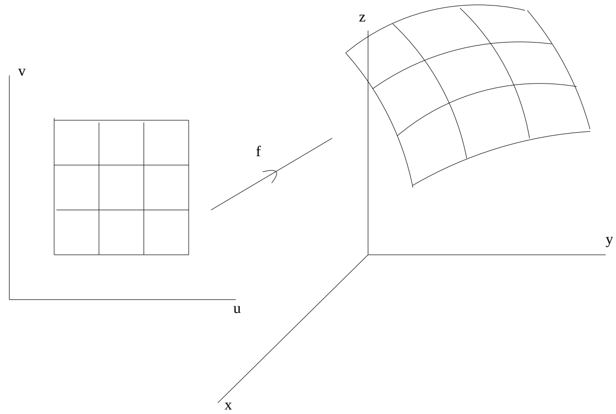
The aim is to entertain the reader and lead him through a tour of many mathematical vistas.

Now, let us begin by explaining how a surface – for instance a soap bubble – will be mathematically described.

For a (small) piece of surface, we can use a parameterization, given by three functions of two variables :

$$\begin{cases} x = f_1(u,v) \\ y = f_2(u,v) \\ z = f_3(u,v). \end{cases}$$

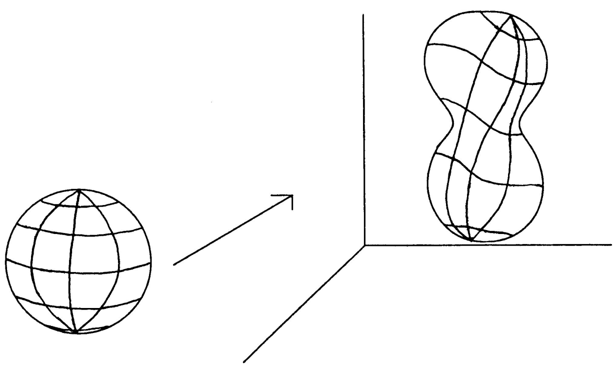
Here the point (u,v) moves in a domain of the plane $\mathbb{R}^2$, and the image of the domain is a piece of surface in $\mathbb{R}^3$.



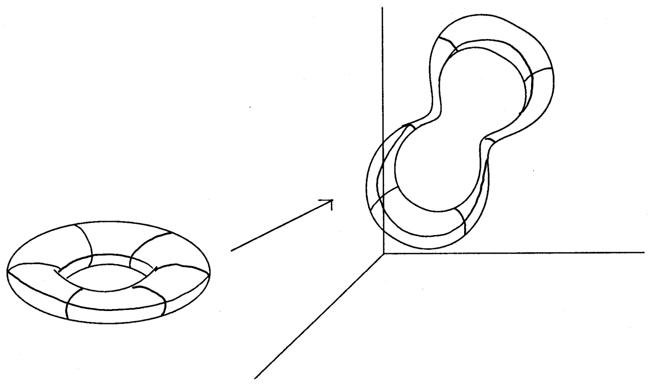
Technically, we suppose that $f = (f_1, f_2, f_3)$ is smooth (C^∞), that its differential is injective at each point and that it is a homeomorphism onto its image.

Globally, the surfaces we consider are obtained by glueing these pieces together, and could for instance be :

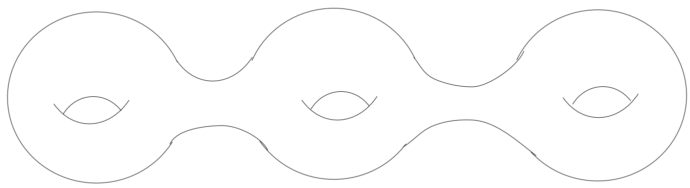
- a sphere



- a torus



- a surface of genus p ($p \geq 2$), drawn here for $p = 3$:



A classification theorem of the 19th century tells us that these are all the compact orientable surfaces.

Concerning the way these surfaces are placed in $\mathbb{R}^3$, we shall examine two different situations.

1. The surface is embedded, which means the map is injective, and the surface has no self intersection.
2. Self-intersections are allowed and we say that the surface is immersed.

Now, what is a soap bubble ?

In the first case (embedding), the surface is closed and contains an interior, in which a fixed quantity of air is locked. The soap molecules will then position themselves so as to minimise the area of the surface.

This is a typical problem of **calculus of variations** : minimise the area of a surface containing a fixed volume.

For such a surface, the first derivative of the area in the direction of any deformation that keeps the volume constant is zero, and this implies the **Euler-Lagrange equations** of the problem.

In this case, these equations form a system of elliptic second order partial differential equations in the three unknown functions $f_i(u, v)$.

If we want to consider also immersed surfaces, we leave the realm of physical properties, because a soap bubble will never cross itself without taking a different configuration. In fact, an immersed surface will not have an interior, so that the physical problem isn't there anymore.

However, a mathematical question is to find all solutions of the Euler-Lagrange system, even with self-intersections (at a crossing, each piece of surface simply ignores the presence of the other).

The equations : analysis

If we simply write locally the equations of the surface as :

$$\left[\begin{array}{l} x_1 = f_1(u,v) \\ x_2 = f_2(u,v) \\ x_3 = f_3(u,v) \end{array}\right.$$

and write the Euler-Lagrange equations of the problem (with Lagrange multiplier), we obtain a complicated system of nonlinear partial differential equations of second order (with some redeeming features : it is elliptic and linear in the second derivatives).

To give an idea, if the surface is represented locally as the graph of the function $h(x,y)$, the Euler-Lagrange equation reduces to

$$\left(1+\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2\right)\left(\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\right)$$

$$-\quad 2\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}\frac{\partial^2 h}{\partial x \partial y} = H\left(1+\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2\right)^{1/2}$$

where H is a constant.

However, the system can be simplified by choosing a conformal parameterization of the surface, that is new parameters (u,v) such that the map $f : U \to \mathbb{R}^3$ satisfies

$$\left\|\frac{\partial f}{\partial u}\right\|^2 = \left\|\frac{\partial f}{\partial v}\right\|^2 \text{ and } \langle\frac{\partial f}{\partial u},\frac{\partial f}{\partial v}\rangle = 0 \tag{1}$$

(recall that f has value in $\mathbb{R}^3$, so that $\frac{\partial f}{\partial u}(u,v)$ and $\frac{\partial f}{\partial v}(u,v)$ are vectors).

These equations mean that the differential of f at a point preserves the angle of tangent vectors and multiply their length by a factor independent of their direction.

This can be expressed by saying that the length element (squared) measured on the surface is proportional to the Euclidean one in U, i.e.

$$ds^2 = E(u,v)(du^2+dv^2) \tag{2}$$

Note that the existence of such a parameterization is not obvious : it is the theorem of existence of isothermal coordinates on a surface (and it would not be true on a higher dimensional manifold).

When f satisfies (1), the Euler-Lagrange equation of soap bubbles becomes :

$$\Delta f = 2H \cdot \left(\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} \right) \tag{3}$$

where $\times$ is the vector product in $\mathbb{R}^3$, and Δf is the Laplacian of each component of f, i.e.

$$\Delta f_i = \frac{\partial^2 f_i}{\partial u^2} + \frac{\partial^2 f_i}{\partial v^2} \qquad\qquad i = 1, 2, 3.$$

So we have an elliptic second order system in diagonal form in the second derivatives but with coupling of the first derivatives

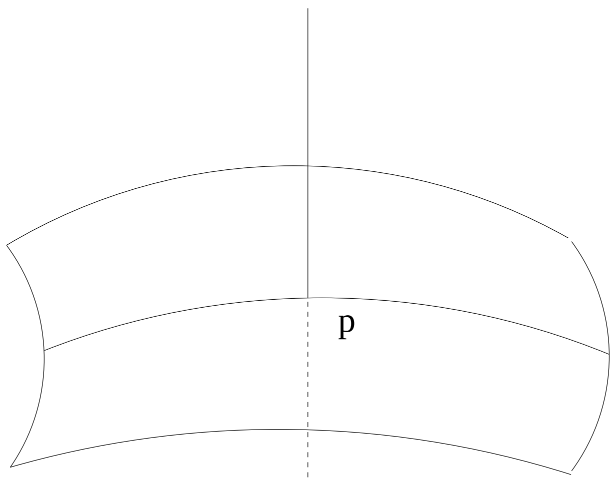
The equations : geometry

Consider first a curve in the plane $\mathbb{R}^2$, described by a parameterization $\gamma(t)$ with speed $||\gamma'(t)|| = 1$. The curvature of the curve at $\gamma(t)$ is then given by its acceleration $||\gamma"(t)||$- the greater the acceleration is, the larger the curvature.

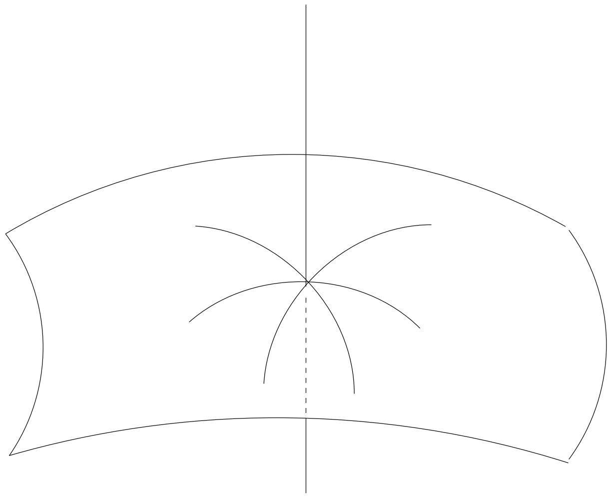
Alternatively, consider the osculating circle (i.e. the circle with best tangence) to the curve. The curvature of the curve is then one over the radius of the circle.

Now, for a surface in $\mathbb{R}^3$, we don't have a unique notion of curvature. This was clarified by Gauss, in the following way.

At a point p of the surface, consider the normal line to the surface and a plane containing that line. The intersection of the plane and the surface is a curve, which has a curvature as above (but we give it a positive or negative sign according to the direction of the acceleration).



Now, rotate the plane around the normal. You get a family of curves and in general the curvature varies with the plane.



Unless the curvature of the curves through p is constant, one can prove that it takes a maximum value k_1 and a minimal value k_2 (called the principal curvatures) for perpendicular directions of the curves of intersection (called the principal directions at p).

The classical theory of surfaces (Gauss, Codazzi, ...) then associates to the point p two notions of curvature :

Definition :

$K(p) = k_1(p) \cdot k_2(p)$ is the Gauss curvature

$H(p) = \dfrac{k_1(p) + k_2(p)}{2}$ is the normal mean curvature.

The Gauss curvature is intrinsic to the surface, in the sense that it depends only on the ds^2 above.

The mean curvature H is the number appearing as Lagrange multiplier in equation (3) above. Thus :

Theorem *The solutions of the Euler-Lagrange equations of soap bubbles are the surfaces of constant mean curvature (i.e. $H =$ constant).*

Remark : A soap film (without difference of pressure of air on either side of the surface) satisfies $H = 0$. It is referred to as a **minimal surface**, or **surface of zero mean curvature**.

Some obvious examples

The Euclidean sphere of radius r is a soap bubble, and a surface of CMC (constant mean curvature). Not only do we know this since childhood, but we see easily that at each point $k_1 = k_2 = 1/r$, so $H = 1/r$ is constant.

The Euclidean cylinder of radius r (which of course is not compact) satisfies :

$$k_1 = \frac{1}{r} \text{ and } k_2 = 0,$$

so it is of CMC with $H = \dfrac{1}{2r}$.

Note that its Gauss curvature $k_1 \cdot k_2 = 0$ - from the intrinsic point of view the cylinder cannot be distinguished locally from a Euclidean plane.

Note finally that an ellipsoid with distinct lengths of axis is not CMC. It is not surprising that at the extremities of the longest axis the mean curvature is larger than anywhere else.

The problem of Hopf

We shall mainly retrace the history of the following :

Problem

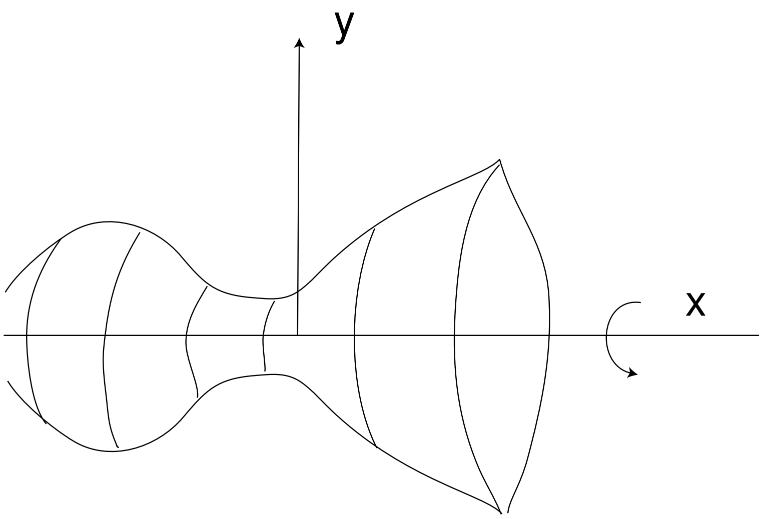
Is every compact surface of constant normal mean curvature a Euclidean sphere ?

However, to study this problem, we need to consider also some non compact surfaces.

1841

Following experiments of the Belgian physicist Joseph Plateau, Charles Delaunay published in 1841 a remarkably clear paper [4] on this question.

Starting from the statement that partial differential equations are too difficult to handle, he reduces the problem to an ordinary differential equation by considering only surfaces of revolution, say generated by rotation of the graph of a function $y(x)$



The variational problem (with Lagrange multiplier) becomes :

$$J(y) = \pi \int \left(y^2 + 2\lambda y \sqrt{1 + y'^2} \right) dx,$$

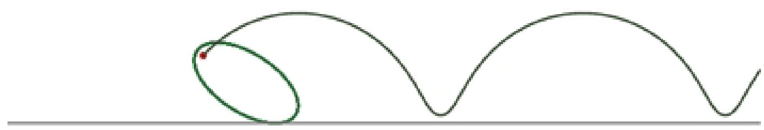
and since x does not appear explicitly in the integrand F we know that $F - y' \cdot F_{y'}$ must be constant along the solutions, i.e.

$$y^2 + \frac{2\lambda y}{\sqrt{1 + y'^2}} = b$$

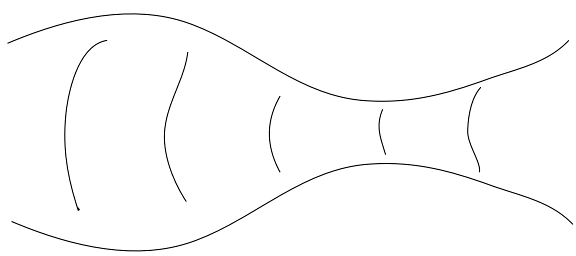
This can be integrated, in terms of elliptic integrals.

But Delaunay goes further, and identifies all the solutions with curves of geometric origin : the locus of one focus of a rolling conic.

Consider first an ellipse, which rolls (without sliding) on the axis, and follow the trace of one focus

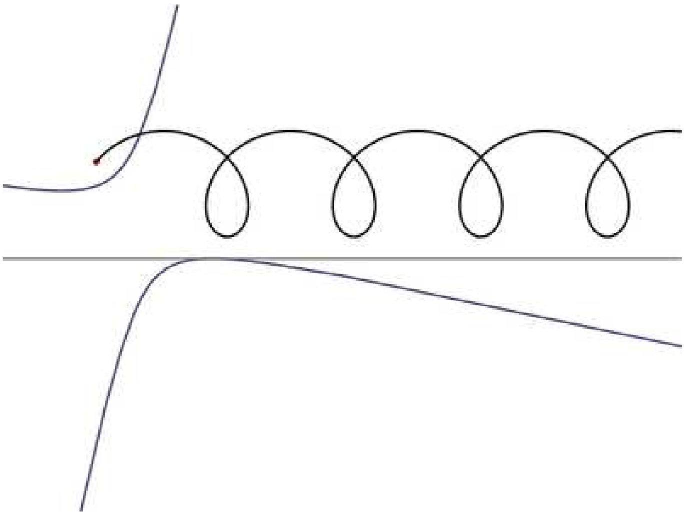


Rotation of this curve around the axis generates a surface of CMC called an unduloid.

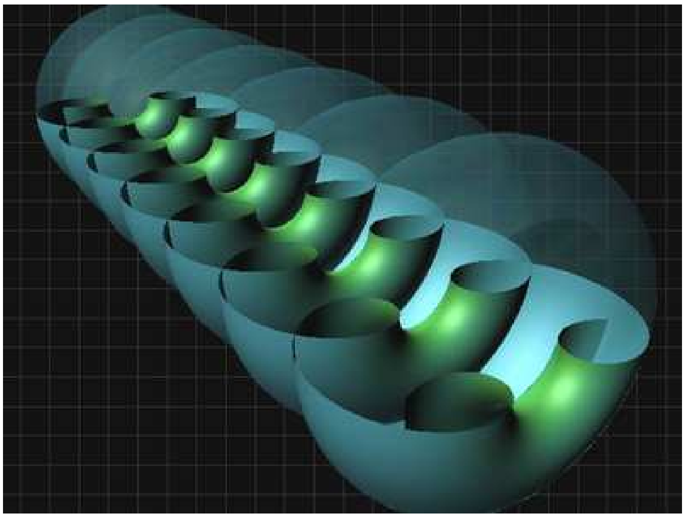


Now roll a hyperbola which means that it rolls alternatively on each branch. When the contact point tends to infinity to the right, a new contact point with the other branch appears at infinity to the left.

The curve described by one focus has points of self-intersection.



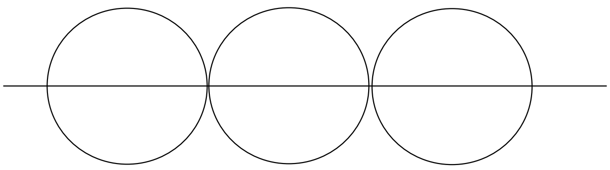
Rotating it around the axis yields a surface of CMC which is immersed and not embedded - indeed each point as above gives rise to a circle of self-interaction of the surface. It is called a nodoid.



In each of the cases above, we get a family of distinct surfaces, varying with the excentricity of the conic.

Now we could also roll a circle - the (single) focus describes a horizontal line and the surface of revolution is a cylinder.

We could also take a limit case of ellipse, and "roll" a line segment, whose extremities play the role of focus. Then the surface obtained is a string of spheres – not a smooth surface.



Last case : we can roll a parabola. Its focus will describe a catenary, and the surface of revolution is called a catenoid and is a minimal surface ($H = 0$).

This is the full list of solutions, and we can observe that the only compact surface of revolution with CMC is the sphere.

1853

In this year, J.H. Jellet [7] proved that **any star-shaped surface of CMC must be a Euclidean sphere**.

1951

Almost a century later, a major step was made by H. Hopf, [5], who proved :

If an immersed surface of CMC is topologically a sphere, then it is a Euclidean sphere.

We give some indication about the proof, to illustrate the role of the hypothesis that the surface be a sphere. (In a first reading, all such ideas of proofs can be skipped - just go from one date to the next).

Consider again a conformal parameterization of a piece of surface by a function $f : (u, v) \to \mathbb{R}^3$, and denote by ν the normal vector of the surface at a point $f(u, v)$.

A good deal of the geometry of the surface is encoded in its **second fundamental form**, defined by

$$II = L\,du^2 + 2M\,du\,dv + N\,dv^2$$

where

$$L = \langle \frac{\partial^2 f}{\partial u^2}, \nu \rangle, \; M = \langle \frac{\partial^2 f}{\partial u \partial v}, \nu \rangle$$

and

$$N = \langle \frac{\partial^2 f}{\partial v^2}, \nu \rangle.$$

We mention here that the eigenvalues of this form with respect to the ds^2 form (2) are the principal curvatures k_1 and k_2.

Hopf's construction is to use in the plane (u, v) the complex variable $w = u + iv$ and to define the "Hopf differential" $\Phi(w) \cdot (dw)^2$ where

$$\Phi(w) = \left(\frac{L - N}{2} - iM \right)$$

The classical equations of Codazzi for a surface then become :

$$\frac{\partial \Phi}{\partial \bar{w}} = E \cdot \frac{\partial H}{\partial w},$$

and since H is constant here, $\Phi(w)$ is a holomorphic function.

Now the surface is not described by a single map $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, but as a union of these coordinate charts, which overlap on open sets.

When changing from one chart to another, the function $\Phi(w)$ is not preserved - so in fact it is not well defined on the whole surface.

However, the differential $\Phi(w) \cdot (dw)^2$ is preserved by a change of chart, and thus well defined on the whole surface. With Φ holomorphic, it is called a **holomorphic quadratic differential**.

On a sphere, any such form is identically zero (this can be checked directly by writing it in two charts, one around each pole, and comparing the Taylor series of the two expressions : it follows all coefficients must vanish).

So $L = N$, $M = 0$ and II is proportional to ds^2, so that $k_1 = k_2$ everywhere.

One can show this only happens on a Euclidean sphere.

Note that on a torus a holomorphic quadratic differential is constant (but not necessarily zero), and it is known that on a surface of genus $p \geq 2$, the space of such forms has complex dimension $3p - 3$. This justifies that Hopf's result applies only to the sphere.

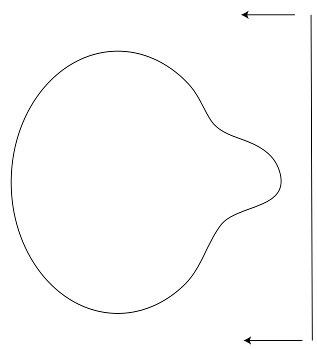
1962

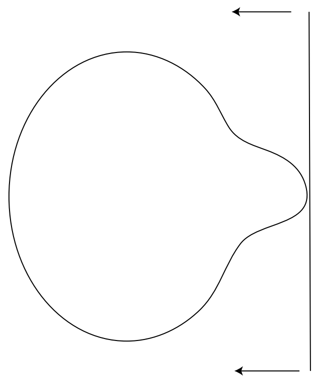
By a completely different approach, A.D. Alexandrov [2] showed that :

If an embedded surface has CMC, then it is a Euclidean sphere.

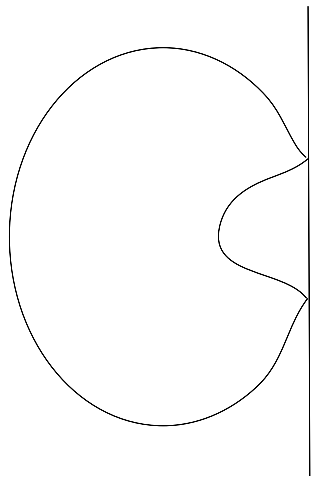
The idea of the proof introduces the method of the "moving plane".

Consider a plane on one side of the surface, and move it in a parallel way until it touches the surface.

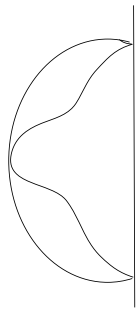




Continue the movement and use the plane as a mirror to reflect the piece of surface already crossed



Do this until the first contact between the initial surface and its reflection occurs.



At that point, the reflected surface is inside the other one. Since they have the same mean curvature H, it can be seen quite easily that they must coincide everywhere (intuitively, one piece is stuck into the other, and the equality of curvature forces it to glue to it). This means that the surface is symmetric with respect to a plane parallel to the given one.

This is true whatever plane direction is chosen, and a surface symmetric in all plane directions must be a sphere.

(In fact, more care has to be taken in the proof when the first contact point between the two pieces of surface also belongs to the plane).

1982

The preceding results gave rise to the idea that any compact CMC in $\mathbb{R}^3$ should be a Euclidean sphere (this statement was sometimes called the Hopf conjecture, eventhough he did not make such claim).

This question made no progress for 20 years, but in 1982 Wu-Yi Hsiang solved the analogous question in higher dimensions [6] :

There exist non-round CMC spheres of dimension n in $\mathbb{R}^{n+1}$, for $n \geq 3$.

This proof goes by reduction to an ordinary differential equation, the sphere being realized as a surface of revolution in a non-standard way.

1984-1986

At this stage, Henri Wente created a surprise by announcing in 1984 that :

There exists a CMC immersed torus in $\mathbb{R}^3$ [14].

His proof is a delicate study of partial differential equations. With the function $E(u,v)$ appearing in (2), and defining ω by

$$E(u,v) = \frac{1}{4H} \cdot e^{2\omega},$$

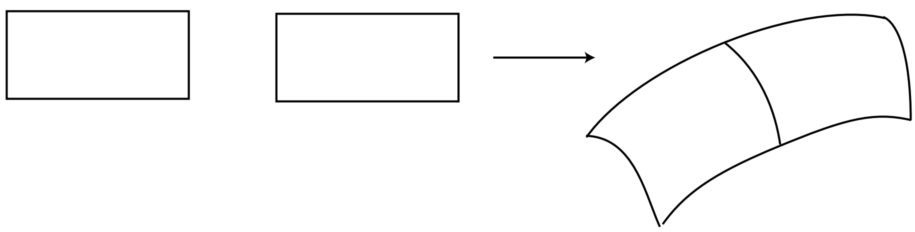
it turns out that if the surface has CMC, then ω satisfies the Sinh-Gordon equation :

$$\Delta\omega + \sinh\omega \cosh\omega = 0. \tag{4}$$

Conversely, if ω is a solution of (4), integration along paths will yield a surface of CMC on any simply connected domain of the plane.

The problem is that this solution will generally not close to form a compact surface.

Wente first shows existence of solutions of (4) on rectangles, with boundary data such that successive rectangles can be glued together.



Still, these building blocks need not close to give a torus, but Wente shows existence of a continuum of solutions, in such a way that for some of them the angle between the copies of pieces of surfaces is rational. After a sufficient number of pieces, the surface will then close.

This existence proof goes not give a description of the immersed torus, and when asked about its shape, Wente answered it might look like a bunch of grapes.

1987

Using methods of numerical analysis on a computer, Uwe Abresch obtained pictures of the tori of Wente.

In order to make the drawings understandable, he programmed the computer to draw a number of lines of principal curvature on the surface, and on some pictures made the observation that all lines of one family (the ones associated to the smallest principal curvature) were straight.

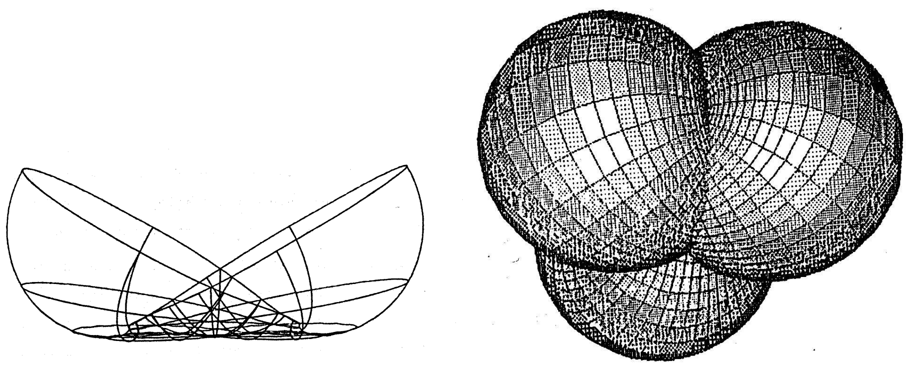
In $\mathbb{R}^3$, this means that these lines are each contained in a plane.

So he decided to study the CMC tori with the supplementary restriction that all lines of smallest principal curvature are planar.

This adds a new partial differential equation, and leads to an overdetermined system.

However, it induces a separation of variables in the Sinh-Gordon equation (4), which can then be solved explicitly using elliptic function.

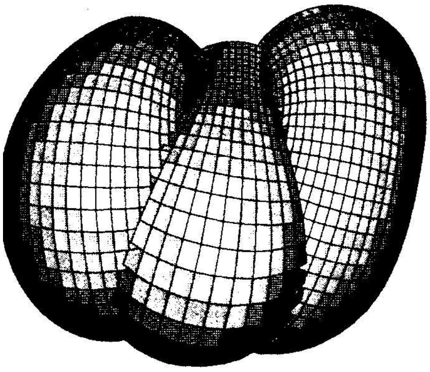
This allowed Uwe Abresch to obtain a large family of explicit CMC tori, and also to have them drawn by a computer programme [1].

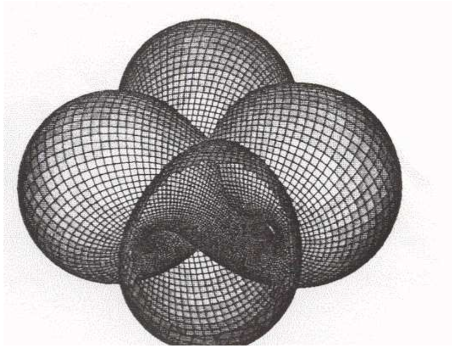


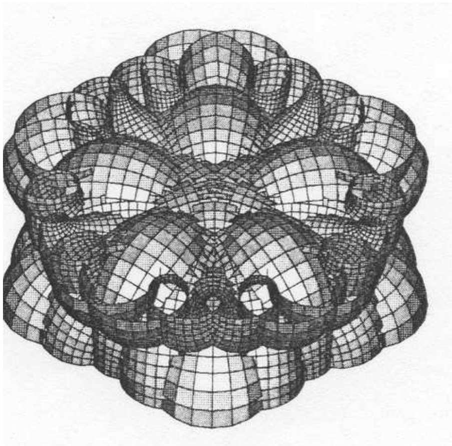
R. Walter [13] then gave more explicit formulas for these tori, in terms of theta functions.

1989

Ulrich Pinkall and Ivan Sterling extended the preceding constructions to include all CMC tori in $\mathbb{R}^3$ [12]. The family of Abresch appears as a first case, all the others being obtained by an inductive method. These constructions are very well suited to computer graphics giving again a clear geometric vision of the solutions.







1991

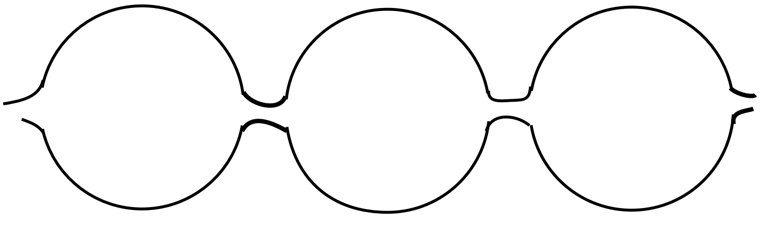
Alexander Bobenko gave an alternative construction of all CMC tori, by means of theta-functions on surfaces of higher genus associated to the different tori [3].

1987-1991

By a complete different approach, Nikolaos Kapouleas proved in [8], [9], that **for each $p \geq 3$, there exist immersed CMC surfaces of genus p in $\mathbb{R}^3$.**

His construction uses the surfaces constructed by Delaunay in 1841. Remember that the unduloids, for instance, form a family parametrized by the excentricity τ of the ellipse.

For the parameter τ tending to zero, the surface approaches a string of spheres, i.e. looks like

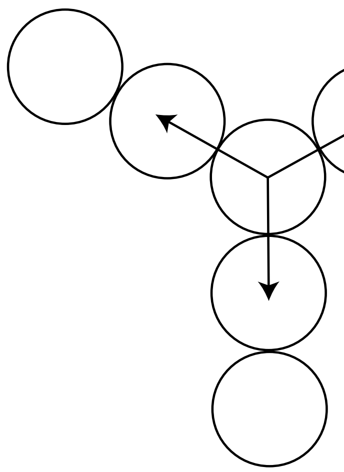


Kapoulea's idea is to glue together a number of strings of spheres each of them glued to the new ones by small bridges approaching catenoids, then to deform this configuration to a CMC surface.

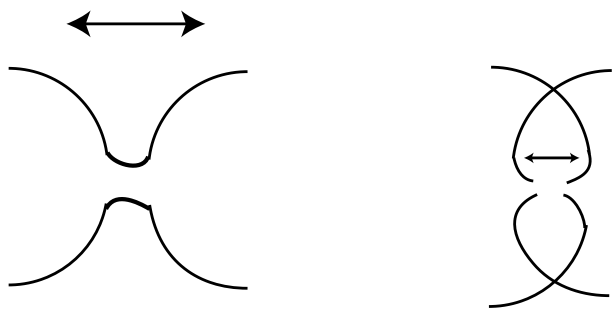
So, starting from a surface which is not CMC, he looks for a small deformation that will make it CMC.

Delicate arguments (using in particular an implicit function theorem in Banach space) reduce this problem to a linear system for the deformation, which Kapouleas manages to solve provided a "balancing condition" is satisfied.

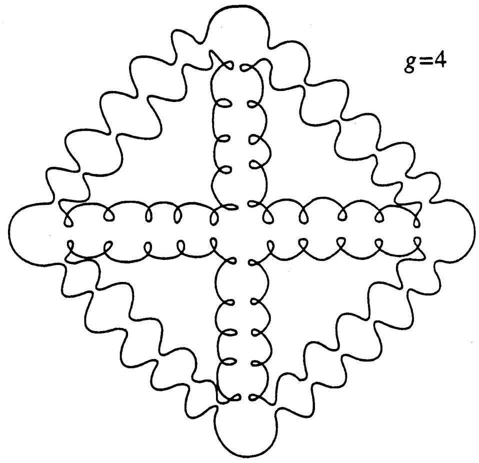
Namely, if one attempts to glue n unduloids around a sphere, if τ_i denote the excentricity and v_i a vector along the axis of the unduloid, then the condition $\sum_{i=1}^{n} \tau_i v_i = 0$ must be satisfied. In some sense, each unduloid pushes the central sphere, and these pushes have to balance.



At first sight, no compact configuration would satisfy this condition but – remarkably – for a nodoid the τ changes sign and the nodoid will "pull" and not "push". This can be loosely explained as follows. The push occurs at the junction of two spheres, and according to the configuration of unduloid or nodoid, it applies in opposite directions.



Therefore, Kapouleas obtains surfaces of genus $p \geq 3$, for instance as follows :



Note that these examples are coherent with Alexandrov's result : the balancing condition imposes to use both unduloids and nodoids, so the surface is never embedded.

1992-1995

This left open the question of existence of CMC surfaces of genus 2, which cannot be built by glueing together Delaunay surfaces (the balancing condition cannot be satisfied at each sphere).

It is again Kapouleas who proved existence, by glueing together two Wente tori [10], [11].

By way of conclusion

Over the years, a naive question (without any practical application) has motivated the development or refinement of mathematical tools which in turn have been applied to many other problems.

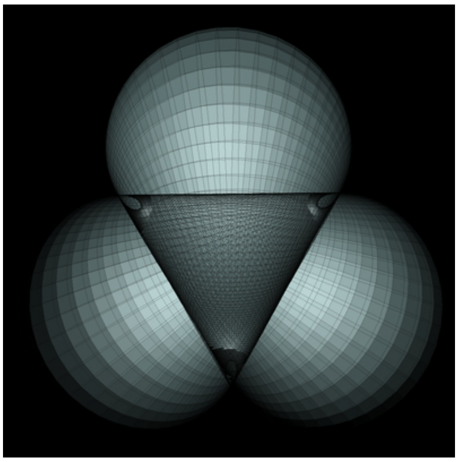
The Hopf differential has been used in other contexts, the moving plane method has become one of the most powerful tools for the study of symmetric solutions of partial differential equations, the use of computer graphics to built conjectures and later prove them has been generalized.

One must stress also that the answers above confirm the power of the unity of mathematics. The problem is a mixture of real geometry and partial differential equations, and the methods used involve complex numbers, symmetries, Riemann surfaces, theta functions, implicit function theorems and much more.

The above text gives essentially the situation today, with respect to compact CMC surfaces.

Present research is focused on complete non compact surfaces, which have a number of ends going to infinity. Progress in the understanding of these is currently very fast.

Finally, let us mention the web-site http://www.gang.umass.edu providing superb pictures of minimal and CMC surfaces, as well as software allowing for personal experiments. All the pictures of CMC surfaces reproduced here come from this site or the original research articles.



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