The Bundle Structure of Noncommutative Tori over UHF-Algebras*

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Abstract

The noncommutative torus $C^*(\mathbb{Z}^n, \omega)$ of rank n is realized as the C^*-algebra of sections of a locally trivial continuous C^*-algebra bundle over $\widehat{S_\omega}$ with fibres $C^*(\mathbb{Z}^n/S_\omega, \omega_1)$ for some totally skew multiplier ω_1 on $\mathbb{Z}^n/S_\omega$. It is shown that $C^*(\mathbb{Z}^n/S_\omega, \omega_1)$ is isomorphic to $A_\varphi \otimes M_k(\mathbb{C})$ for some completely irrational noncommutative torus A_φ and some positive integer k, and that $A_\omega \otimes M_{l^\infty}$ has the trivial bundle structure if and only if the set of prime factors of k is a subset of the set of prime factors of l. This is applied to understand the bundle structure of the tensor products of Cuntz algebras with noncommutative tori.

Introduction

Given a locally compact abelian group G and a multiplier ω on G, one can associate to them the twisted group C^*-algebra $C^*(G, \omega)$, which is the universal object for unitary ω-representations of G. The twisted group C^*-algebra $C^*(\mathbb{Z}^n, \omega)$ is called a *noncommutative torus of rank* n and denoted by A_ω.

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Now the multiplier ω determines a subgroup

$$S_\omega = \{g \in G : \omega(g,h) = \omega(h,g), \quad \forall h \in G\}$$

of G, called its *symmetry group*. A multiplier ω on an abelian group G is called *totally skew* if the symmetry group S_ω is trivial. A noncommutative torus A_ω is said to be a *completely irrational noncommutative torus* if $S_\omega \cong \{0\}$. Baggett and Kleppner [1] showed that if G is a locally compact abelian group and ω is a totally skew multiplier on G, then $C^*(G,\omega)$ is a simple C^*-algebra. So $C^*(\mathbb{Z}^n,\omega)$ is simple if ω is totally skew.

Baggett and Kleppner [1] also showed that even when ω is not totally skew on a locally compact abelian group G, the restriction of ω-representations from G to S_ω induces a canonical homeomorphism of $\mathrm{Prim}(C^*(G,\omega))$ with $\widehat{S_\omega}$, and that there is a totally skew multiplier ω_1 on $\mathbb{Z}^n/S_\omega$ such that ω is similar to the pull-back of ω_1. Furthermore, it is known (see [1, 7, 10, 11]) that $C^*(G,\omega)$ may be realized as the C^*-algebra $\Gamma(\zeta)$ of sections of a locally trivial continuous C^*-algebra bundle ζ over $\widehat{S_\omega} = \mathrm{Prim}(C^*(G,\omega))$ with fibres $C^*(G,\omega)/P$ for $P \in \mathrm{Prim}(C^*(G,\omega))$ and all $C^*(G,\omega)/P$ turn out to be the simple twisted group C^*-algebra $C^*(G/S_\omega,\omega_1)$.

D. Poguntke showed in [10] that any primitive quotient of the group C^*-algebra $C^*(G)$ of a locally compact two step nilpotent group G is isomorphic to a tensor product of a completely irrational noncommutative torus A_φ with the algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a separable (possibly finite-dimensional) Hilbert space $\mathcal{H}$. Since $C^*(G/S_\omega,\omega_1)$ is a primitive quotient of $C^*(G/S_\omega(\omega_1))$, $C^*(G/S_\omega,\omega_1)$ is isomorphic to $A_\varphi \otimes \mathcal{K}(\mathcal{H})$, where $G/S_\omega(\omega_1)$ is the extension group of G/S_ω by $\mathbb{T}^1$ defined by ω_1.

It was shown in [9, 11] that $C^*(\mathbb{Z}^n,\,\omega) \otimes \mathcal{K}(\mathcal{H})$ has the trivial bundle structure when $\mathcal{H}$ is infinite dimensional.

In this paper, we study the bundle structure of $A_\omega \otimes M_{l^\infty}$.

Let A_ω be a noncommutative torus which is isomorphic to the C^*-algebra of sections of a locally trivial continuous C^*-algebra bundle over $\widehat{S_\omega}$ with fibres $A_\varphi \otimes M_k(\mathbb{C})$. Here A_φ is a completely irrational noncommutative torus. We prove that $A_\omega \otimes M_{l^\infty}$ has the trivial bundle structure if and only if the set of prime factors of k is a subset of the set of prime factors of l.

Let $\mathcal{O}_u$ be a Cuntz algebra. It is shown that $A_\omega \otimes \mathcal{O}_{2u}$ has the trivial bundle structure if and only if k and $2u-1$ are relatively prime.

1 Preliminaries

Let ω be a multiplier on a locally compact abelian group G. Define a homomorphism $h_\omega : G \to \widehat{G}$ by $h_\omega(x)(y) = \omega(x,y)\omega(y,x)^{-1}$ for all $x,y \in G$.

We introduce the concept of C^*-algebra bundle over a locally compact Hausdorff space. Let $\mathrm{Prim}(C^*(G,\omega))$ be the primitive ideal space of the twisted group C^*-algebra $C^*(G,\omega)$ of a locally compact abelian group G defined by a multiplier ω.

Proposition 1.1 ([1, 7]). *Let G be a locally compact abelian group and ω a multiplier on G. Then*

(i) *there is a totally skew multiplier ω_1 on G/S_ω such that $C^*(G,\omega)/P$ is isomorphic to $C^*(G/S_\omega,\omega_1)$ for any $P \in \operatorname{Prim}(C^*(G,\omega))$ and ω is similar to the pull-back of ω_1;*

(ii) *the restriction of ω-representations from G to S_ω induces a canonical homeomorphism of $\operatorname{Prim}(C^*(G,\omega))$ with $\widehat{S_\omega}$.*

Theorem 1.2 ([8]). *Let A be a C^*-algebra and T a locally compact Hausdorff space. Then A is isomorphic to the C^*-algebra of sections of a continuous C^*-algebra bundle over T if and only if there is a continuous open surjection of $\operatorname{Prim}(A)$ onto T.*

Here the C^*-algebra bundle is not necessarily locally trivial.

But it is known (see [7, 11]) that if A is a twisted group C^*-algebra of a locally compact abelian group by a multiplier, then its C^*-algebra bundle is locally trivial continuous. In particular, $A_\omega \cong C^*(\mathbb{Z}^n,\omega)$ is realized as the C^*-algebra of sections of a locally trivial continuous C^*-algebra bundle over $\widehat{S_\omega}$ with fibres the simple twisted group C^*-algebra $C^*(\mathbb{Z}^n/S_\omega,\omega_1)$ of a finitely generated discrete abelian group $\mathbb{Z}^n/S_\omega$ defined by a totally skew multiplier ω_1 on $\mathbb{Z}^n/S_\omega$, where ω is similar to the pull-back of ω_1.

The following result of Poguntke clarifies the structure of the fibres $C^*(\mathbb{Z}^n/S_\omega,\omega_1)$ of the canonical bundle associated to a noncommutative torus A_ω.

Theorem 1.3 ([10, Theorem 1]). *Let G be a compactly generated locally compact abelian group and ω_1 a totally skew multiplier on G. Let K be the maximal compact subgroup of $E := G(\omega_1)$, and E_ρ the stabilizer of an irreducible unitary representation ρ of K restricting on $\mathbb{T}^1$ to the identity. Then*

$$C^*(G,\,\omega_1) \cong C^*(E_\rho/K, m) \otimes \mathcal{K}(\mathcal{L}^2(E/E_\rho)) \otimes M_{\dim(\rho)}(\mathbb{C}),$$

where m is the associated Mackey obstruction.

This theorem implies that the simple twisted group C^*-algebra $C^*(\mathbb{Z}^n/S_\omega,\omega_1)$ is isomorphic to $A_\varphi \otimes M_k(\mathbb{C})$, where A_φ is a completely irrational noncommutative torus and $M_k(\mathbb{C})$ is a matrix algebra. So A_ω is realized as the C^*-algebra of sections of a locally trivial continuous C^*-algebra bundle over $\widehat{S_\omega}$ with fibres $A_\varphi \otimes M_k(\mathbb{C})$.

Note that if $M_k(\mathbb{C})$ is trivial, then $\mathbb{Z}^n/S_\omega$ is torsion-free. The torsion-free group $\mathbb{Z}^n$ is isomorphic to $S_\omega \oplus F$, where $F(\cong \mathbb{Z}^n/S_\omega)$ is a torsion-free subgroup of $\mathbb{Z}^n$, and ω splits as the trivial multiplier on S_ω and a totally skew multiplier on F up to similarity. So A_ω is isomorphic to $C(\widehat{S_\omega}) \otimes A_\varphi$. Now if $M_k(\mathbb{C})$ is not trivial and A_ω is isomorphic to $C(\widehat{S_\omega}) \otimes A_\varphi \otimes M_k(\mathbb{C})$, then $M_k(\mathbb{C})$ is factored out of A_ω, which implies $[1_{A_\omega}] = k[p]$ for some $[p] \in K_0(A_\omega)$, where $[1_{A_\omega}]$ is the class of the unit 1_{A_ω} of A_ω. This contradicts the fact that $[1_{A_\omega}] \in K_0(A_\omega)$ is primitive ([5, Theorem 2.2]). So if $M_k(\mathbb{C})$ is not trivial, then A_ω is not isomorphic to $C(\widehat{S_\omega}) \otimes A_\varphi \otimes M_k(\mathbb{C})$. Hence A_ω has the trivial bundle structure if and only if $M_k(\mathbb{C})$ is trivial.

Lemma 1.4. *For $l \in \mathbb{N}$ let $\mathbb{Z}_l := \mathbb{Z}/l\mathbb{Z}$. Let ω be a multiplier on $\mathbb{Z}^n$ and α a multiplier on $\mathbb{Z}^n \oplus \mathbb{Z}_l^n$ whose restriction to $\mathbb{Z}^n$ is similar to ω. Assume that $\alpha|_{\mathbb{Z}_l^n}$ is* *trivial, and that S_α is torsion-free with $(\mathbb{Z}^n \oplus \mathbb{Z}_l^n)/S_\alpha \cong \mathbb{Z}_l^n \oplus \mathbb{Z}_l^n$. Then $C^*(\mathbb{Z}^n \oplus \mathbb{Z}_l^n, \alpha)$ is isomorphic to $A_{l^2\omega} \otimes M_{l^n}(\mathbb{C})$, where $A_{l^2\omega}$ is identified with $C^*((l\mathbb{Z})^n, \omega|_{(l\mathbb{Z})^n})$.*

Proof: By assumption, $\mathbb{Z}^n$ acts *transitively* on $\widehat{\mathbb{Z}_l^n}$. So by the Mackey machine for twisted crossed products, $C^*(\mathbb{Z}^n \oplus \mathbb{Z}_l^n, \alpha)$ is induced from a twisted crossed product of $\mathbb{C}$ by $(l\mathbb{Z})^n$, where $(l\mathbb{Z})^n$ is the stabilizer in $\mathbb{Z}^n \oplus \mathbb{Z}_l^n$ of the identity in $\widehat{\mathbb{Z}_l^n}$. Since every C^*-algebra automorphism of $\mathbb{C}$ is trivial, this twisted crossed product is of the form $C^*((l\mathbb{Z})^n, \alpha|_{(l\mathbb{Z})^n})$. So

$$C^*(\mathbb{Z}^n \oplus \mathbb{Z}_l^n, \alpha) \cong C^*((l\mathbb{Z})^n, \alpha|_{(l\mathbb{Z})^n}) \otimes M_{l^n}(\mathbb{C}) \cong A_{l^2\omega} \otimes M_{l^n}(\mathbb{C}).$$

■

2 Tensor products of noncommutative tori with UHF-algebras and Cuntz algebras

Let us apply the previous results to understand the bundle structure of the tensor products of noncommutative tori with UHF-algebras.

We can construct inductive systems for tensor products of noncommutative tori with matrix algebras to obtain inductive limit C^*-algebras $A_\omega \otimes M_{l^\infty}$ and $A_{l^2\omega} \otimes M_{l^\infty}$ for M_{l^∞} a UHF-algebra of type l^∞, and we are going to show that they are isomorphic.

The twisted group C^*-algebra of $\mathbb{Z}^n$ defined by a multiplier ω on $\mathbb{Z}^n$ is the universal C^*-algebra generated by n unitaries $u_1, u_2, \dots, u_n$ with $u_i u_j = \rho_{ij} u_j u_i$ where $\rho_{ij} = e^{2\pi i \theta_{ij}}$ is a scalar for each pair i, j. Let $A_\omega = C^*(u_1, \cdots, u_n | u_i u_j = e^{2\pi i \theta_{ij}} u_j u_i)$, and identify $A_{l^2\omega}$ with $C^*(u_1^l, \cdots, u_n^l)$. Then from Lemma 1.4, one obtains the following.

Theorem 2.1. *$A_{l^2\omega} \otimes M_{l^\infty}$ is isomorphic to $A_\omega \otimes M_{l^\infty}$.*

Proof: Let

$$g_r := \iota \otimes id_{M_{l^r}(\mathbb{C})} : A_{l^2\omega} \otimes M_{l^r}(\mathbb{C}) \to A_\omega \otimes M_{l^r}(\mathbb{C}),$$

where ι is the canonical injection of $A_{l^2\omega}$ into A_ω, and $id_{M_{l^r}(\mathbb{C})}$ is the identity map of $M_{l^r}(\mathbb{C})$.

Let D be a finitely generated discrete abelian group such that $D \cong \mathbb{Z}^n \oplus \mathbb{Z}_l^n$. Let α be a multiplier on D such that A_ω is isomorphic to $C^*(\mathbb{Z}^n, \alpha|_{\mathbb{Z}^n})$, $\alpha|_{\mathbb{Z}_l^n}$ is trivial, and S_α is torsion-free. Let μ be the isomorphism of A_ω to $C^*(\mathbb{Z}^n, \alpha|_{\mathbb{Z}^n}) \hookrightarrow C^*(D, \alpha)$. By Lemma 1.4, $C^*(D, \alpha)$ is isomorphic to $A_{l^2\omega} \otimes M_{l^n}(\mathbb{C})$. Let ν be the isomorphism of $C^*(D, \alpha)$ to $A_{l^2\omega} \otimes M_{l^n}(\mathbb{C})$. And let

$$\varphi_r := \nu \otimes id_{M_{l^r}(\mathbb{C})} : C^*(D, \alpha) \otimes M_{l^r}(\mathbb{C}) \to A_{l^2\omega} \otimes M_{l^n}(\mathbb{C}) \otimes M_{l^r}(\mathbb{C}).$$

Let

$$\begin{aligned} \psi_r : \ &= \mu \otimes id_{M_{l^r}(\mathbb{C})} : A_\omega \otimes M_{l^r}(\mathbb{C}) \to C^*(D, \alpha) \otimes M_{l^r}(\mathbb{C}), \\ \phi_r : \ &= \varphi_r \circ \psi_r : A_\omega \otimes M_{l^r}(\mathbb{C}) \to A_{l^2\omega} \otimes M_{l^n}(\mathbb{C}) \otimes M_{l^r}(\mathbb{C}). \end{aligned}$$

Now look at the sequence

$$\begin{aligned}\cdots \to A_{l^2\omega} \quad &\otimes M_{l^r}(\mathbb{C}) \xrightarrow{g_r} A_\omega \otimes M_{l^r}(\mathbb{C}) \xrightarrow{\phi_r} A_{l^2\omega} \otimes M_{l^{r+n}}(\mathbb{C}) \\ \xrightarrow{g_{r+n}} \quad &A_\omega \otimes M_{l^{r+n}}(\mathbb{C}) \xrightarrow{\phi_{r+n}} A_{l^2\omega} \otimes M_{l^{r+2n}}(\mathbb{C}) \xrightarrow{g_{r+2n}} \cdots .\end{aligned}$$

Then by definition one has an intertwining sequence. The inductive limit of the odd terms

$$\cdots \to A_{l^2\omega} \otimes M_{l^r}(\mathbb{C}) \xrightarrow{\phi_r \circ g_r} A_{l^2\omega} \otimes M_{l^{r+n}}(\mathbb{C}) \xrightarrow{\phi_{r+n} \circ g_{r+n}} \cdots$$

is $A_{l^2\omega} \otimes M_{l^\infty}$.

Next, the inductive limit of the even terms

$$\cdots \to A_\omega \otimes M_{l^r}(\mathbb{C}) \xrightarrow{g_{r+n} \circ \phi_r} A_\omega \otimes M_{l^{r+n}}(\mathbb{C}) \xrightarrow{g_{r+2n} \circ \phi_{r+n}} \cdots$$

is $A_\omega \otimes M_{l^\infty}$. Hence by the Elliott theorem [6, Theorem 2.1], $A_{l^2\omega} \otimes M_{l^\infty}$ is isomorphic to $A_\omega \otimes M_{l^\infty}$. ∎

G.A. Elliott showed in [5, Theorem 2.2] that $K_i(A_\omega) \cong \mathbb{Z}^{2^{n-1}}$, and that the class $[1_{A_\omega}]$ of the unit is a primitive element of $K_0(A_\omega)$.

¿From now on, we assume that A_ω is isomorphic to the C^*-algebra of sections of a locally trivial continuous C^*-algebra bundle over $\widehat{S_\omega}$ with fibres $A_\varphi \otimes M_k(\mathbb{C})$ for some simple noncommutative torus A_φ and some positive integer k.

THEOREM 2.2. *$A_\omega \otimes M_{l^\infty}$ is isomorphic to $C(\widehat{S_\omega}) \otimes A_\varphi \otimes M_k(\mathbb{C}) \otimes M_{l^\infty}$ if and only if the set of prime factors of k is a subset of the set of prime factors of l.*

PROOF: Assume the set of prime factors of k is a subset of the set of prime factors of l. Let $A_\omega \cong C^*(\mathbb{Z}^n, \omega)$. Then $A_{k^2\omega} \cong C^*((k\mathbb{Z})^n, \omega|_{(k\mathbb{Z})^n})$, and the matrix subalgebra $M_k(\mathbb{C})$ is a twisted group C^*-algebra of a subgroup of $(\mathbb{Z}/k\mathbb{Z})^n$ defined by a totally skew multiplier ω_2 induced from ω. So $(k\mathbb{Z})^n/S_{\omega|_{(k\mathbb{Z})^n}}$ is torsion-free, $(k\mathbb{Z})^n$ splits as $F \oplus S_{\omega|_{(k\mathbb{Z})^n}}$ where $F \cong (k\mathbb{Z})^n/S_{\omega|_{(k\mathbb{Z})^n}}$ is a torsion-free subgroup of $(k\mathbb{Z})^n$, and $\omega|_{(k\mathbb{Z})^n}$ splits up to isomorphism as the direct sum of $\omega|_F$ and the trivial multiplier $1_{S_{\omega|_{(k\mathbb{Z})^n}}}$ on $S_{\omega|_{(k\mathbb{Z})^n}}$. Thus $A_{k^2\omega} \cong C^*(F, \omega|_F) \otimes C^*(S_{\omega|_{(k\mathbb{Z})^n}})$, which has the trivial bundle structure. By Theorem 2.1, $A_{k^2\omega} \otimes M_{k^\infty} \cong A_\omega \otimes M_{k^\infty}$. So $A_\omega \otimes M_{k^\infty}$ has the trivial bundle structure. Since M_{k^∞} can be considered as a factor of M_{l^∞}, $A_\omega \otimes M_{l^\infty}$ has the trivial bundle structure. That is, $A_\omega \otimes M_{l^\infty}$ is isomorphic to $C(\widehat{S_\omega}) \otimes A_\varphi \otimes M_k(\mathbb{C}) \otimes M_{l^\infty}$.

Conversely, assume that $A_\omega \otimes M_{l^\infty} \cong C(\widehat{S_\omega}) \otimes A_\varphi \otimes M_k(\mathbb{C}) \otimes M_{l^\infty}$. Then the unit $1_{A_\omega} \otimes 1_{M_{l^\infty}}$ maps to the unit $1_{C(\widehat{S_\omega}) \otimes A_\varphi} \otimes 1_{M_{l^\infty}} \otimes I_k$. By the Künneth theorem for tensor products [2, Theorem 23.1.3],

$$K_0(A \otimes B) = K_0(A) \otimes K_0(B)$$

for C^*-algebras A and B with $K_*(A)$ torsion-free and $K_1(B) \cong \{0\}$. So

$$\begin{aligned}[1_{A_\omega} \otimes 1_{M_{l^\infty}}] &= [1_{C(\widehat{S_\omega}) \otimes A_\varphi} \otimes 1_{M_{l^\infty}} \otimes I_k] \\ [1_{A_\omega} \otimes 1_{M_{l^\infty}}] &= [1_{A_\omega}] \otimes [1_{M_{l^\infty}}] \\ [1_{C(\widehat{S_\omega}) \otimes A_\varphi} \otimes 1_{M_{l^\infty}} \otimes I_k] &= k([1_{C(\widehat{S_\omega}) \otimes A_\varphi}] \otimes [1_{M_{l^\infty}}]).\end{aligned}$$

Under the assumption that the unit $1_{A_\omega} \otimes 1_{M_{l^\infty}}$ maps to the unit $1_{C(\widehat{S_\omega})\otimes A_\varphi} \otimes 1_{M_{l^\infty}} \otimes I_k$, if there is a prime factor q of k such that $q \not| l$, then $[1_{M_{l^\infty}}] \neq q[e_\infty]$ for any element $[e_\infty] \in K_0(M_{l^\infty})$. So there is an element $[e] \in K_0(A_\omega)$ such that $[1_{A_\omega}] = q[e]$. This contradicts the fact that $[1_{A_\omega}]$ is primitive ([5, Theorem 2.2]). Thus the set of prime factors of k is a subset of the set of prime factors of l.

Therefore, $A_\omega \otimes M_{l^\infty}$ is isomorphic to $C(\widehat{S_\omega}) \otimes A_\varphi \otimes M_k(\mathbb{C}) \otimes M_{l^\infty}$ if and only if the set of prime factors of k is a subset of the set of prime factors of l. ∎

Now let us understand the bundle structure of the tensor products of noncommutative tori with (even) Cuntz algebras.

The Cuntz algebra $\mathcal{O}_u, 2 \leq u < \infty$, is the universal C^*-algebra generated by u isometries $s_1, \ldots, s_u$, i.e., $s_j^* s_j = 1$ for all j, with the relation $s_1 s_1^* + \cdots + s_u s_u^* = 1$. Cuntz [3, 4] proved that $\mathcal{O}_u$ is simple and the K-theory of $\mathcal{O}_u$ is $K_0(\mathcal{O}_u) = \mathbb{Z}/(u-1)\mathbb{Z}$ and $K_1(\mathcal{O}_u) = 0$. He proved that $K_0(\mathcal{O}_u)$ is generated by the class of the unit.

PROPOSITION 2.3. *Let u be a positive integer such that k and $u-1$ are not relatively prime. Then $\mathcal{O}_u \otimes A_\omega$ is not isomorphic to $\mathcal{O}_u \otimes C(\widehat{S_\omega}) \otimes A_\varphi \otimes M_k(\mathbb{C})$.*

PROOF: Let p be a prime such that $p \mid k$ and $p \mid u-1$. Suppose that $\mathcal{O}_u \otimes A_\omega$ is isomorphic to $\mathcal{O}_u \otimes C(\widehat{S_\omega}) \otimes A_\varphi \otimes M_k(\mathbb{C})$. Then the unit $1_{\mathcal{O}_u \otimes A_\omega}$ maps to the unit $1_{\mathcal{O}_u \otimes C(\widehat{S_\omega})\otimes A_\varphi} \otimes I_k$. So

$$[1_{\mathcal{O}_u \otimes A_\omega}] = [1_{\mathcal{O}_u \otimes C(\widehat{S_\omega})\otimes A_\varphi} \otimes I_k] = k[1_{\mathcal{O}_u \otimes C(\widehat{S_\omega})\otimes A_\varphi}].$$

Hence there is an element $[e] \in K_0(\mathcal{O}_u \otimes A_\omega)$ such that $[1_{\mathcal{O}_u \otimes A_\omega}] = k[e]$. By the Künneth theorem for tensor products given in the proof of Theorem 2.2, $[1_{\mathcal{O}_u \otimes A_\rho}] = [1_{\mathcal{O}_u}] \otimes [1_{A_\omega}]$ and $[1_{\mathcal{O}_u}]$ is a generator of $K_0(\mathcal{O}_u) \cong \mathbb{Z}/(u-1)\mathbb{Z}$ (see [4]). But $p \mid u-1$. $[1_{\mathcal{O}_u}] \neq p[e_*]$ for any element $[e_*] \in K_0(\mathcal{O}_u)$. So $[1_{A_\omega}] = p[e']$ for some element $[e'] \in K_0(A_\omega)$. This contradicts the fact that $[1_{A_\omega}]$ is primitive (see [5, Theorem 2.2]). Hence k and $u-1$ are relatively prime.

Therefore, $\mathcal{O}_u \otimes A_\omega$ is not isomorphic to $\mathcal{O}_u \otimes C(\widehat{S_\omega}) \otimes A_\varphi \otimes M_k(\mathbb{C})$ if k and $u-1$ are not relatively prime. ∎

The following result is useful to understand the bundle structure of $\mathcal{O}_u \otimes A_\omega$.

PROPOSITION 2.4 ([12, THEOREM 7.2]). *Let A and B be unital simple inductive limits of even Cuntz algebras. If $\alpha : K_0(A) \to K_0(B)$ is an isomorphism of abelian groups satisfying $\alpha([1_A]) = [1_B]$, then there is an isomorphism $\phi : A \to B$ which induces α.*

COROLLARY 2.5. *Let l be an odd integer such that l and $2u-1$ are relatively prime. Then $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{l^\infty}$. In particular, $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{(2u)^\infty}$.*

THEOREM 2.6. *$\mathcal{O}_{2u} \otimes A_\omega$ is isomorphic to $\mathcal{O}_{2u} \otimes C(\widehat{S_\omega}) \otimes A_\varphi \otimes M_k(\mathbb{C})$ if and only if k and $2u-1$ are relatively prime.*

PROOF: Assume that k and $2u-1$ are relatively prime. Let $k = l2^v$ for some odd integer l. Then l and $2u-1$ are relatively prime. Then by Corollary 2.5 $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{l^\infty}$, and $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{(2u)^\infty} \cong \mathcal{O}_{2u} \otimes M_{(2u)^\infty} \otimes M_{(2^v)^\infty} \cong \mathcal{O}_{2u} \otimes M_{(2^v)^\infty}$. So $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{l^\infty} \otimes M_{(2^v)^\infty} \cong \mathcal{O}_{2u} \otimes M_{k^\infty}$. Thus by Theorem 2.2, $\mathcal{O}_{2u} \otimes A_\omega$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{k^\infty} \otimes A_\omega$, which in turn is isomorphic to $\mathcal{O}_{2u} \otimes M_{k^\infty} \otimes C(\widehat{S_\omega}) \otimes A_\varphi \otimes M_k(\mathbb{C})$. Thus $\mathcal{O}_{2u} \otimes A_\omega$ is isomorphic to $\mathcal{O}_{2u} \otimes C(\widehat{S_\omega}) \otimes A_\varphi \otimes M_k(\mathbb{C})$.

The converse was proved in Proposition 2.3.

Therefore, $\mathcal{O}_{2u} \otimes A_\omega$ is isomorphic to $\mathcal{O}_{2u} \otimes C(\widehat{S_\omega}) \otimes A_\varphi \otimes M_k(\mathbb{C})$ if and only if k and $2u-1$ are relatively prime. ■

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