Infinite-dimensional complex projective varieties

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Abstract

Let V_i, $0 \leq i \leq s$, $s \geq 1$, be complex Banach spaces. Assume V_i infinite-dimensional for $1 \leq i \leq s$, V_0 finite-dimensional and $V_0 \neq 0$. Let $\pi : \mathbf{P}(V_0) \times \cdots \times \mathbf{P}(V_s) \to \mathbf{P}(V_0)$ be the projection. Let X be a closed analytic subset of finite codimension of $\mathbf{P}(V_0) \times \cdots \times \mathbf{P}(V_s)$. Here we prove that $\pi(X)$ is a closed analytic subset of $\mathbf{P}(V_0)$ with the following universal property. For every finite-dimensional reduced analytic space Y and every holomorphic map $f : X \to Y$ there is a unique holomorphic map $g : \pi(X) \to Y$ such that $f = g \circ (\pi|X)$.

The aim of this note is to prove the following result.

Theorem 1. *Let V_i, $0 \leq i \leq s$, $s \geq 1$, be complex Banach spaces. Assume V_i infinite-dimensional for $1 \leq i \leq s$, V_0 finite-dimensional and $V_0 \neq 0$. Let $\pi : \mathbf{P}(V_0) \times \cdots \times \mathbf{P}(V_s) \to \mathbf{P}(V_0)$ be the projection. Let X be a closed analytic subset of finite codimension of $\mathbf{P}(V_0) \times \cdots \times \mathbf{P}(V_s)$. Then $\pi(X)$ is a closed analytic subset of $\mathbf{P}(V_0)$ with the following universal property. For every finite-dimensional reduced analytic space Y and every holomorphic map $f : X \to Y$ there is a unique holomorphic map $g : \pi(X) \to Y$ such that $f = g \circ (\pi|X)$.*

We will also give a description of $\pi(X)$ (see Proposition 1). We will need the following known result.

Lemma 1. *Let $W_1, \ldots, W_m$, $m \geq 1$, be complex Banach spaces and Y a closed analytic subset of finite codimension of $\mathbf{P}(W_1) \times \cdots \times \mathbf{P}(W_m)$. Then Y is algebraic, i.e. it is the zero-locus of finitely many continuous multihomogeneous polynomials on $W_1 \times \cdots \times W_m$.*

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Proof. Let $f : W_1 \times \cdots \times W_m \backslash \{0\} \to \mathbf{P}(W_1) \times \cdots \times \mathbf{P}(W_m)$ be the quotient map. Hence $f^{-1}(Y)$ is a closed analytic subset of $W_1 \times \cdots \times W_m \backslash \{0\}$ with finite codimension. Let Z be the closure of $f^{-1}(Y)$ in $W_1 \times \cdots \times W_m$. By [1], Prop. III.2.1.3, Z is an analytic subset of finite codimension of $W_1 \times \cdots \times W_m$. Hence there is a neighborhood U of 0 in $W_1 \times \cdots \times W_m$ such that the analytic set $Z \cap U$ is defined by finitely many holomorphic functions $f_j : U \to \mathbf{C}$. The invariance of $f^{-1}(Y)$ and hence of Z for the multiplication by a different scalar on each factor W_i shows that the multihomogeneous components (with respect to the variables in $W_1, \ldots, W_m$) of each f_j vanish on U. Thus Z is defined locally around 0 by finitely many continuos multihomogeneous polynomials. The same multihomogeneous polynomials define Y as a subset of $\mathbf{P}(W_1) \times \cdots \times \mathbf{P}(W_m)$. ∎

Lemma 2. *Let V_i, $1 \le i \le s$, be infinite-dimensional complex Banach spaces and A, B finite-codimensional non-empty closed analytic subsets of $\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_s)$. Then $A \cap B \neq \emptyset$.*

Proof. Call z the maximal codimension of one of the irreducible components of $A \cup B$. Fix any finite-dimensional projective subspace M_i of $\mathbf{P}(V_i)$ with $\dim(M_i) \ge z+s+1$ and set $M := M_1 \times \cdots \times M_s$. Hence $M \cap A$ and $M \cap B$ are non-empty closed analytic subsets of M with codimension at most z. Hence $(M \cap A) \cap (M \cap B) \neq \emptyset$, proving the lemma. ∎

Now we can give a description of the defining equations of the set $\pi(X)$ appearing in the statement of Theorem 1 in terms of the defining equations of X.

Proposition 1. *Let V_i, $0 \le i \le s$, $s \ge 1$, be complex Banach spaces. Assume V_i infinite-dimensional for $1 \le i \le s$, V_0 finite-dimensional and $V_0 \neq 0$. Let $\pi : \mathbf{P}(V_0) \times \cdots \times \mathbf{P}(V_s) \to \mathbf{P}(V_0)$ be the projection. Let X be a closed analytic subset of finite codimension of $\mathbf{P}(V_0) \times \cdots \times \mathbf{P}(V_s)$. Let $\{g_\alpha\}_{\alpha \in T}$ be a finite set of multihomogeneous polynomials defining X. Set $S := \{\alpha \in T : g_\alpha$ has degree 0 with respect to each factor $V_1, \ldots, V_s\}$. Hence each g_α, $\alpha \in S$, defines a homogeneous polynomial on V_0. We have $\pi(X) = \{P \in \mathbf{P}(V_0) : g_\alpha(P) = 0$ for every $\alpha \in S\}$. In particular $\pi(X)$ is a closed algebraic subset of $\mathbf{P}(V_0)$.*

Proof. Call z the maximal codimension of an irreducible component of X. Fix any finite-dimensional subspace M_i of $\mathbf{P}(V_i)$ with $\dim(M_i) \ge z+s+1$ and set $M := M_1 \times \cdots \times M_s$. Set $Z :== \{P \in \mathbf{P}(V_0) : g_\alpha(P) = 0$ for every $\alpha \in S\}$. Obviously, we have $X \subseteq \pi^{-1}(Z)$. It is sufficient to show that if $Q \in Z$, then $\pi^{-1}(Q) \cap X \neq \emptyset$. The closed analytic subset $\pi^{-1}(Q) \cap X$ of $\pi^{-1}(Q) \cong \mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_s)$ is defined evaluating at Q the multihomogeneous polynomials $\{g_\alpha\}_{\alpha \in T}$. Since $Q \in Z$, after the evaluation at Q none of these polynomials is a non-zero constant. Hence $M \cap (\pi^{-1}(Q) \cap X) \neq \emptyset$ by Hilbert Nullstellensatz. ∎

Proof of Theorem 1. It is sufficient to check that for all $P, Q \in X$ with $\pi(Q) = \pi(P)$ we have $f(Q) = f(P)$. Set $A := f^{-1}(f(P)) \cap \pi^{-1}(P)$ and $B := f^{-1}(f(Q)) \cap \pi^{-1}(Q) = f^{-1}(f(Q)) \cap \pi^{-1}(P)$. Since Y has finite dimension, P and Q may be locally defined in Y by finitely many germs of analytic functions. Thus A and B are finite codimensional closed analytic subsets of $\pi^{-1}(P) \cong \mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_s)$;

the non-emptiness of A and B follows for instance by their description given by Proposition 1. By Lemma 2 we have $A \cap B \neq \emptyset$. Since $f^{-1}(f(Q))$ and $f^{-1}(f(P))$ are fibers of a map, this implies $f(Q) = f(P)$, concluding the proof.

References

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