Geodesic laminations as geometric realizations of Arnoux–Rauzy sequences

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Abstract

We consider a family of minimal sequences on a 3-symbol alphabet with complexity $2n+1$, which satisfy a special combinatorial property. These sequences were originally defined by P. Arnoux and G. Rauzy in [2] as a generalization of the binary sturmian sequences. We prove that the dynamical system associated to each of these sequences of this family, can be realized as a dynamical system defined on a geodesic lamination on the hyperbolic disk. This is a generalization of the results shown in [17]. We also show some applications of these results.

1 Introduction

Let $\mathbf{u}$ be an one-sided infinite sequence in the finite alphabet $\mathcal{A}$. We associate to the sequence $\mathbf{u}$ a dynamical system (Ω, σ) where Ω is the closure of the orbit of $\mathbf{u}$ under the shift map σ, i.e. $\sigma(v_0 v_1 v_2 \cdots) = v_1 v_2 \cdots$. We are interested in finding a geometrical interpretation of the symbolic system, i.e. to find a dynamical system defined on a geometrical structure such that there is a semiconjugacy between these systems.

The *complexity of the sequence* $\mathbf{u}$ is the function $p(n)$ which is defined as the number of different subwords of length n of $\mathbf{u}$. Hedlund and Morse proved that $\mathbf{u}$ is

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eventually periodic if and only if $p(n) \leq n$, for some n (cf [11]). So the simplest non-trivial sequences have complexity $p(n) = n+1$, which are called *sturmian sequences*. The most well know example of sturmian sequence is the Fibonacci sequence, which is the fixed point of the substitution $1 \to 12$, $2 \to 1$, i.e. $\mathbf{u} = 12112121\cdots$. These sequences are obtained from the symbolic coding of the orbit of a point on the circle under a rotation by an irrational number, using the partition given by the continuity intervals (cf. [11, 12]). The sturmian sequences have been extensively studied, see for example [9, 8, 4, 5]. There has been defined different generalizations of sturmian sequences, one of them is due to Arnoux and Rauzy. They introduced in [2] a family of minimal sequences in the alphabet of three symbols, of complexity $p(n) = 2n+1$ which satisfy an additional combinatorial property called the $*$ condition, see definition 2.1. A well known sequence of this family is obtained by the fixed point of the tribonacci substitution, $1 \to 12$, $2 \to 13$, $3 \to 1$. It was showed that the dynamical systems associated to these sequences are realizable as interval exchange transformations. They also show that these sequences can be obtained as an infinite composition of three different substitutions. It was conjectured that these sequences come from rotations of the two dimensional torus, with a suitable partition for the coding. As it happens in the tribonacci substitution, in this case the coding of the orbit is done according to the partition obtained by the Rauzy fractal (cf. [14]). In [6] it was showed that this conjecture is not true. These sequences have also been studied in [1, 15, 5].

In this paper we will give a new geometrical realization of these symbolic systems. We construct a geodesic lamination Λ on the disk associated to the symbolic system and define a dynamical system (Λ, F) on this lamination. We shall show in Theorem 3.4 that (Λ, F) is a geometrical realization of (Ω, σ), i.e. the system (Λ, F) is semiconjugate to (Ω, σ). The dynamical system (Λ, F) helps to understand better the dynamics of the interval exchange map associated to the original sequence, since it gives a description of the points on the interval that have the same symbolic representation (Theorems 3.1 and 3.3). This is a generalization of the results of [16] and [17]. There the author showed that the dynamical system associated to the tribonacci substitution can be realized as a dynamical system on a geodesic lamination. The results of this paper can be generalized in a straight forward manner to minimal sequences in the alphabet of k symbols and complexity $p(n) = (k-1)n+1$ which satisfy the $*$ property for k symbols.

2 Preliminaries and Notation

Let $\mathcal{A} = \{1, 2, 3\}$ be the alphabet and $\mathbf{u}$ a sequence in this alphabet. A word is *allowed or admissible* in $\mathbf{u}$ if it is a finite subword of the sequence $\mathbf{u}$.

Definition 2.1. *We say that the sequence $\mathbf{u} = u_0u_1\ldots$ has the $*$* property *if for all n there are allowed subwords of length n, V_n and W_n such that V_n1, V_n2, V_n3 and $1W_n$, $2W_n$, $3W_n$ are also allowed words.*

Let us consider the following substitutions:

$$\Pi_1 : \begin{cases} 1 & \to & 1 \\ 2 & \to & 12, \\ 3 & \to & 13 \end{cases} \quad \Pi_2 : \begin{cases} 1 & \to & 21 \\ 2 & \to & 2, \\ 3 & \to & 23 \end{cases} \quad \Pi_3 : \begin{cases} 1 & \to & 31 \\ 2 & \to & 32 \\ 3 & \to & 3 \end{cases}$$

Theorem 2.1 ([2]). *Let* $\mathbf{u}$ *be a minimal sequence in the alphabet* $\{1,2,3\}$*. Then* $\mathbf{u}$ *has complexity* $p(n) = 2n+1$ *and satisfies the* $*$ *condition, if and only if there exists a sequence* $\{i_k\}_k$ *with values in* $\{1,2,3\}$ *such that each symbol appears infinitely many times and*

$$\mathbf{u} = \lim_{k\to\infty} \Pi_{i_1} \cdots \Pi_{i_k}(\mathbf{u}).$$

If the sequence $\{i_k\}_k$ is periodic then the sequence $\mathbf{u}$ is the fixed point of the substitution $\Pi_{i_1} \cdots \Pi_{i_l}$, where $\{i_k\}_k = \{i_1, \ldots, i_l, i_1, \ldots, i_l, \ldots\}$. This substitution is Pisot, which means that the incidence matrix of the substitution has one eigenvalue greater than one and all other eigenvalues are less than one in modulus. In the case that $\mathbf{u}$ is the fixed point of the tribonacci substitution, i.e. $1 \to 12$, $2 \to 13$, $3 \to 1$, the sequence $\{i_k\}_k$ of Theorem 2.1 is the periodic sequence $\{1,2,3,1,2,3,\ldots\}$. Since $\Pi_1\Pi_2\Pi_3$ is the cube of the previous substitution. A general reference for substitution dynamical systems is [13].

We will denote by $\overline{\mathbf{u}}$ the sequence $\ldots u_{-2}u_{-1}u_0$ where $u_{-j} = u_j$, it is called the *reverse sequence of* $\mathbf{u}$. Note that the dynamical system associated to $\overline{\mathbf{u}}$, using the right shift, is the same one associated to $\mathbf{u}$.

Corollary 2.1. *Let* $\mathbf{u}$ *be a sequence that satisfies the hypothesis of the previous theorem and* $\overline{\mathbf{u}}$ *its reverse sequence. Then* $\overline{\mathbf{u}}$ *can be written as*

$$\overline{\mathbf{u}} = \lim_{k\to\infty} \overline{\Pi}_{i_1} \cdots \overline{\Pi}_{i_k}(\overline{\mathbf{u}})$$

where $\{i_k\}$ *is the sequence obtained in theorem 2.1 and*

$$\overline{\Pi}_1 : \begin{cases} 1 & \to & 1 \\ 2 & \to & 21, \\ 3 & \to & 31 \end{cases} \quad \overline{\Pi}_2 : \begin{cases} 1 & \to & 12 \\ 2 & \to & 2, \\ 3 & \to & 32 \end{cases} \quad \overline{\Pi}_3 : \begin{cases} 1 & \to & 13 \\ 2 & \to & 23 \\ 3 & \to & 3. \end{cases}$$

Let

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

be the matrices associated to the substitutions (they are also called the incidence matrices of the substitutions) Π_1, $\overline{\Pi}_1$, Π_2, $\overline{\Pi}_2$ and Π_3, $\overline{\Pi}_3$ respectively.

The image of the positive cone under the infinite product $M_{i_1} \cdots M_{i_k} \cdots$ is a straight line passing through the origin. Since products of the form $M_i M_j^l M_k$ appear infinitely many times in the infinite product and they are contractions of the positive cone by a ratio smaller than 1 (cf.[2]). Let (α, β, γ) be the element of norm 1 in this line.

Let f be the interval exchange transformation (iet) defined as $f = L_I \circ L_{I_1} \circ L_{I_2} \circ L_{I_3}$ where $I = [0,1)$, $I_1 = [0,\alpha)$, $I_2 = [\alpha, \alpha+\beta)$, $I_3 = [\alpha+\beta, 1)$ and L_J denotes the rotation of order 2 on the interval $J = [a,b)$, i.e.

$$L_J(x) = \begin{cases} x + \frac{b-a}{2} & \text{if } a \le x < \frac{a+b}{2} \\ x - \frac{b-a}{2} & \text{if } \frac{a+b}{2} \le x < b \\ x & \text{otherwise.} \end{cases}$$

Let $\nu : I \to \{1,2,3\}$ the map that give the coding according to the partition given by the canonical intervals I_1, I_2, I_3, i.e. $\nu(x) = j$ if $x \in I_j$. Let $\theta : I \to \Omega$ be the map that give the coding of the forward orbit of a point under f, i.e. $\theta(x) = \{\nu(f^k(x))\}_{k\geq 0}$. This map is continuous to the right and has the property that $\theta(f(x)) = \sigma(\theta(x))$ for all $x \in I$ (cf.[2]).

Since the iet f is an invertible map, we can codify the forward as well as the backward f-orbits of the points of the circle using the canonical partition. We define

$$\widetilde{\Omega} = \overline{\{\nu(f^n(x)) \,|\, x \in [0,1),\ n \in \mathbb{Z}\}}$$

so we have the map $\tilde{\theta} : I \to \widetilde{\Omega}$ that send the point x to itinerary of its two sided infinite f-orbit. The space $\widetilde{\Omega}$ consists of the extensions of the sequences of Ω to two sided infinite sequences and $\tilde{\theta}$ is the extension of the map θ to $\widetilde{\Omega}$. The map $\tilde{\theta}$ is right-continuous.

Let $x_1 = 0$, $x_2 = \alpha$, $x_3 = \alpha + \beta$ be the extremities of the canonical intervals and $y_1 = \alpha/2$, $y_2 = \alpha + \beta/2$, and $y_3 = \alpha + \beta + \gamma/2$ be the middle points of the canonical intervals. These points are the discontinuities of the iet f.

Proposition 2.1. *The coding of the backward orbits of x_1, x_2, x_3 is given by $\overline{\mathbf{u}}$.*

Proof: Suppose that I_1 is the largest of the canonical intervals and $\alpha > 1/2$. Let $\hat{f}$ be the induced map of f on $J = f(I_1)$, its extreme points are identified in order to obtained the map defined on the circle. On J we consider the partition $J_1 = f(I_1) \cap I_1$, $J_2 = I_2$, $J_3 = I_3$. The relationship between the coding of the orbit of a point in J under $\hat{f}$ and under f is given by $\overline{\Pi}_1$, i.e. let the map $\nu'(x) = j$ if $x \in J_j$, then $\nu(f^n(x)) = \overline{\Pi}_1(\nu'(\hat{f}^n(x)))$. Observe that the boundary points of the intervals of the partition on J are the same as the partition on I. We continue this process, each time taking the induction, on the larger interval, as it is described in [2]. So the points x_1, x_2 and x_3 are always boundary points of the intervals of the corresponding partition. In the limit we get that the coding for the backward orbit of x_1, x_2 and x_3 is given by $\overline{\mathbf{u}}$. ■

Proposition 2.2. $\theta(f(y_1)) = \theta(f(y_2)) = \theta(f(y_3)) = \mathbf{u}$.

Proof: Since $f = L_I \circ L_{I_1} \circ L_{I_2} \circ L_{I_3}$ and each L_{I_i} is an involution, f is conjugate to f^{-1} by L_I, i.e. rotation by $1/2$. Observe that $L_I(x_i) = f(y_i)$ then the coding of forward orbit of $f(y_i)$ is given by $\mathbf{u}$ for $i = 1,2,3$. ■

3 Construction of the geodesic lamination

Let $\mathcal{L}$ be the set of geodesics of the hyperbolic disk, $\mathbb{D}^2$. We can think of $\mathbb{S}^1$ as the circle at infinity of $\mathbb{D}^2$ and of the interval exchange map f as acting on it. We identify $\mathbb{S}^1$ to $I = [0,1)$.

The topology of $\mathcal{L} \cup \mathbb{S}^1$ is given by the following basis of neighbourhoods:

- If λ is an element of $\mathcal{L}$ with end points a and b in $\mathbb{S}^1$, consider the collection of neighbourhoods $(a-\epsilon, a+\epsilon)$ and $(b-\epsilon, b+\epsilon)$ for $\epsilon > 0$. Then the basis elements containing λ are given by the set of geodesics with one end point in $(a-\epsilon, a+\epsilon)$ and the other in $(b-\epsilon, b+\epsilon)$.

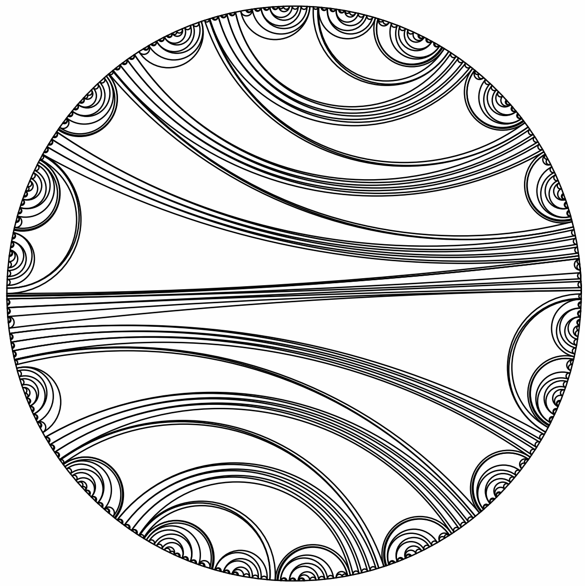

Figure 1: The geodesic lamination Λ for $\Pi_1\Pi_2^3\Pi_3$.

- If t is in $\mathbb{S}^1$, consider the collection of neighbourhoods in $\mathbb{S}^1$ given by $(t-\epsilon, t+\epsilon)$ for $\epsilon > 0$, then the basis elements containing t are given by the point t and the set of geodesics with one end point in $(t-\epsilon, t)$ and the other in $(t, t+\epsilon)$.

Let $v_0 \dots v_k$ be an allowed word and

$$[v_0 \dots v_k] = I_{v_0} \cap f^{-1}(I_{v_1}) \cap \dots \cap f^{-k}(I_{v_k}) \tag{3-1}$$

the corresponding cylinder on I. We consider the σ-algebra generated by these cylinders. Since f is an iet the cylinder is a finite collection of intervals.

The construction of the geodesic lamination Λ is as follows: we join by geodesics consecutive extreme points that belong to different components of a given cylinder. We do this for all the cylinders and then the closure of the union of all these geodesics is taken. The elements of Λ are either geodesics of the hyperbolic disk or points in $\mathbb{S}^1$. In the later case those points are called *degenerate geodesics.* A *geodesic lamination of the disk* is a non-empty closed subset of $\mathcal{L} \cup \mathbb{S}^1$ whose elements are disjoint. See [7] for geodesic laminations on surfaces.

We will give an alternative description of the set Λ. We are interested in studying when different points in $\widetilde{\Omega}$ have the same backward orbit (past) but different forward orbits (future), reciprocally when they have the same future but different pasts, and in finding the corresponding points in $[0, 1)$. Clearly this kind of behaviour is associated to the discontinuity points of the interval exchange map f.

Theorem 3.1. Λ *is the closure of the geodesics* λ *such that the image under* $\tilde{\theta}$ *of the end points of* λ *have the same past and different futures.*

Proof: Let $[v_0 \dots v_k]$ be a cylinder. It is clear that the extremities of the intervals that correspond to this cylinder are backward images under f of its discontinuity points that have the same past, i.e. x_1, x_2, x_3. So the neighbouring extremities in a

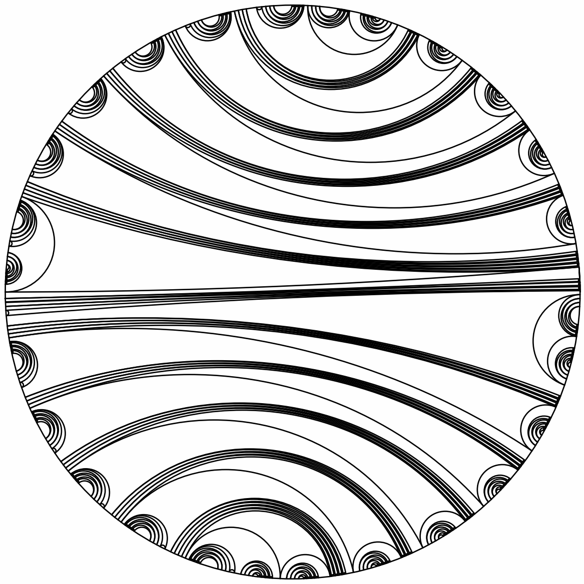

Figure 2: The geodesic lamination Λ for $\Pi_1\Pi_2^5\Pi_3$.

gap of the cylinder are of the form $f^{-l}(x_i)$ and $\lim_{t\to x_i^-} f^{-l}(t)$, which will be denoted by $f^{-l}(x_i^-)$, for some nonnegative integer $l \leq k$. These points have the same past and different futures (Proposition 2.1). Due to the density of the orbits and the construction of Λ follows the theorem. ∎

Using the facts that the neighbouring extremities in a gap of a cylinder are points of the form $f^{-l}(x_i)$ and $f^{-l}(x_i^-)$, for some positive l and $1 \leq i \leq 3$, the orbits are disjoint and the absence of periodic points. We can conclude that the cylinders of the form (3-1) consist of at most three intervals.

Theorem 3.2. Λ *is a geodesic lamination on disk.*

Proof: We have to prove that two distinct geodesics of Λ are disjoint, perhaps with the exception of the end points. By density we can restrict to geodesics joining the extremities of the cylinders. Let C_1 and C_2 be two cylinders, they are either disjoint or one is contained in the other. If C_1 is contained in C_2 then the geodesics joining points of C_1 are contained in the convex hull of C_2, which is limited by the geodesics joining the extreme points.

We shall prove that if they are disjoint then one is contained in the gap of the other. Moreover we show that we can fill the gaps between two components of any cylinder. Consider C_1, a cylinder of two intervals, $[a_1, a_2)$, $[a_3, a_4)$, a similar argument is used if it has more components. The points a_2, a_3 are extremities for other cylinder. If this new cylinder has two components, it is clear that it lies in the gap, since the cylinders are disjoint. But if it has more than two components, all of them have to lie in the gap of the original cylinder. Suppose that it is not the case, so $C_2 = [a_2, b_1) \cup [b_2, a_3) \cup [b_3, b_4)$ with $a_3 < a_4 < b_3$. Since a_2 and a_3 are neighbouring extreme points of the cylinder C_1, we have shown in the proof of Theorem 3.1, that these points are of the form $f^{-l}(x_i)$ and $f^{-l}(x_i^-)$ for some positive integer l and some i in $\{1,2,3\}$. On the other hand a_3 and b_3 are neighbouring extremities of the cylinder C_2, so they are of the form $f^{-m}(x_j)$ and $f^{-m}(x_j^-)$ for some positive integer $m \neq l$ and some j in $\{1,2,3\}$. But this is not possible due to the absence of periodic orbits of f and the disjointness of the f orbit of the x_i's. ∎

Theorem 3.3. *Λ is the closure of the geodesics λ such that the image under $\tilde{\theta}$ of the end points of λ have the same future and different pasts.*

Proof: Let $v_0 \dots v_k$ be an allowed word in $\mathbf{u}$. We associate to it the set $I_{v_0} \cap \cdots \cap f^k(I_{v_k})$. In Proposition 2.2 we showed that the maps f and f^{-1} are conjugate by L_I, the rotation by $1/2$, i.e. $f = L_I \circ f^{-1} \circ L_I$. Therefore $I_{v_0} \cap \cdots \cap f^k(I_{v_k})$ is in the σ-algebra generated by the cylinders described in (3-1). This set consists of a finite union of intervals, whose extreme points are forward images under f of its discontinuity points that have the same future, i.e. y_i for $i = 1,2,3$. Therefore the neighbouring extremities in a gap of the set are of the form $f^l(y_i)$ and $f^l(y_i^-)$, for some i and non-negative integer $l \leq k$. ∎

As we saw in the proof of Proposition 2.2 the rotation by $1/2$, maps the backward f orbits of x_i to the forward orbit of $f(y_i)$, so from Theorems 3.1 and 3.3, Λ is invariant under rotation by $1/2$.

Corollary 3.1. *The geodesic lamination Λ is invariant under the rotation by $1/2$.*

The geodesic laminations associated to quadratic Julia sets have also the same property of being invariant under rotation by $1/2$ (cf. [3, 10]).

On Λ we define the map F as follows: let λ be an element of the lamination with end point in I, $a_\lambda < b_\lambda$, then $F(\lambda)$ is the geodesic with end points $f(a_\lambda)$ and $f(b_\lambda^-)$. If λ is a degenerate geodesic, say $\lambda = a_\lambda$ in I, then $F(\lambda)$ is defined as the geodesic with end points $f(a_\lambda)$ and $f(a_\lambda^-)$. In order to verify that $F(\lambda)$ is an element of the lamination Λ. Suppose that happens an intersection between $F(\lambda)$ and an element of the lamination. For Theorem 3.1, we can assume that the intersection is with a geodesic that join points of the form $f^{-l}(x_i)$ and $f^{-l}(x_i^-)$ for some $l \geq 0$ and $1 \leq i \leq 3$. Then there will be an intersection between λ and the geodesics that joins $f^{-l-1}(x_i)$ and $f^{-l-1}(x_i^-)$, so λ could not be in Λ.

Theorem 3.4. *The dynamical system (Λ, F) is semiconjugate to (Ω, σ).*

Proof: The continuity of F has only to be checked at the geodesics that have end points at the discontinuities of f, since on the other geodesics follows from the continuity of f. At the geodesics that join pairwise the points x_1, x_2, x_3 the map F is continuous since these points the map f is right-continuous. To study the continuity of F at y_1, y_2 and y_3, we need first to show that the elements of Λ with extreme points at y_1, y_2 and y_3 are degenerate geodesics. The coding of the forward orbit of $f(y_i)$ is given by $\mathbf{u}$, so $f(y_i)$'s are joined by geodesics. Due to the invariance of Λ by F, the points y_i's are joined by geodesics or they are degenerate geodesics. But they cannot be joined by geodesics since they belong to different intervals (cylinders), whose extremities, i.e. x_i, are joined by geodesics. Therefore the points y_1, y_2, y_3 are degenerate geodesics.

Let $\epsilon > 0$. The preimage under F of the set of non-degenerate geodesics in the lamination with one end point in $(f(y_i^-) - \epsilon, f(y_i^-) + \epsilon)$ and the other in $(f(y_i) -$

$\epsilon, f(y_i) + \epsilon)$ is the neighbourhood of y_i given by the set of geodesics in $\Lambda \cap \mathcal{L}$ with one end point in $(y_i - \epsilon, y_i)$ and the other in $(y_i, y_i + \epsilon)$. So F is continuous at y_i, for $i = 1, 2, 3$.

We define a map $\Theta : \Lambda \to \Omega$ as follows: let λ_i the geodesic that joins $f(y_i)$ with $f(y_{i+1})$, here $i + 1$ is taken $\bmod 3$. So $\Theta(F^k(\lambda_i))$ is defined as $\theta(f^k(y_i))$, which is $\sigma^k(\mathbf{u})$. Let λ be a geodesic in Λ whose end points are a_λ and b_λ ($a_\lambda = b_\lambda$ if λ is degenerate). Since $\{F^k(\lambda_i) \,|\, k > 0\}$ is dense in Λ, there exists a subsequence k_n such that the limit of $f^{k_n}(y_i)$ is a_λ and the limit of $f^{k_n}(y_{i+1})$ is b_λ. So we define $\Theta(\lambda) = \lim_{n\to\infty} \sigma^{k_n}(\mathbf{u})$. By construction Θ is continuous. It is straight-forward to prove that this map is surjective. However it is not injective, consider the point $\mathbf{u}$ in Ω, there are three geodesics that are mapped to this point: the geodesics that join $f(y_i)$ for $i = 1, 2, 3$. ∎

Corollary 3.2. *The dynamical system* (Λ, F) *is semiconjugate to* $(\widetilde{\Omega}, \sigma)$.

Proof: The inverse of the map F is well defined due to theorem 3.3. Let $\widetilde{\Theta} : \Lambda \to \widetilde{\Omega}$ be the map defined as $\widetilde{\Theta}(\lambda) = \tilde{\theta}(a_\lambda)$ where λ is a geodesic in Λ and its end points are $0 \le a_\lambda \le b_\lambda < 1$. We shall show that $\tilde{\theta}(a_\lambda) = \tilde{\theta}(b_\lambda^-)$. In fact we have seen that $\theta(f^k(y_i)) = \theta(f^k(y_{i+1}))$, for all $k > 0$. Furthermore $\theta(f^k(y_{i+1})) = \theta(f^k(y_{i+1})^-) = \theta(f^k(y_i^-))$. Since f^{-1} is not discontinuous at y_i, $\tilde{\theta}(f^k(y_i)) = \tilde{\theta}(f^k(y_i^-))$ for all k. Hence $\tilde{\theta}(a_\lambda) = \tilde{\theta}(b_\lambda^-)$.

The map $\widetilde{\Theta}$ is continuous. In fact: Let λ be a non-degenerate geodesic in Λ, with end points a_λ and b_λ. The point a_λ is the extremity of two different geodesics, otherwise it will be a degenerate geodesic. So a neighbourhood of λ in Λ consists of all non-degenerate geodesics with one end point at $(a_\lambda, a_\lambda + \epsilon)$ and the other at $(b_\lambda - \epsilon, b_\lambda)$, for some $\epsilon > 0$, i.e. there are no nondegenerate geodesics in the lamination with end points in $(a_\lambda - \epsilon, a_\lambda)$ and the other at $(b_\lambda, b_\lambda + \epsilon)$. Therefore the continuity of $\widetilde{\Theta}$ at λ follows from the right continuity of $\tilde{\theta}$ at a_λ. Let λ be a degenerate geodesic, i.e. $\lambda = a_\lambda$. Let $\epsilon > 0$, then by the right continuity of $\tilde{\theta}$ there exists $\epsilon' > 0$ such that if λ' is a geodesic with end points $a_{\lambda'}$ and $b_{\lambda'}$, satisfying $a_\lambda - a_{\lambda'} < \epsilon'$ and $b_{\lambda'} - a_\lambda < \epsilon'$, then $\mathrm{dist}(\tilde{\theta}(b_{\lambda'}^-), \tilde{\theta}(a_\lambda)) < \epsilon$. Hence $\mathrm{dist}(\tilde{\theta}(a_{\lambda'}), \tilde{\theta}(a_\lambda)) < \epsilon$, where dist is the usual distance in $\widetilde{\Omega}$.

It is straight-forward to prove that this map $\widetilde{\Theta}$ is surjective. However $\widetilde{\Theta}$ is not an open map, since any open neighbourhood of a degenerate geodesic, in the topology of Λ, does not include other degenerate geodesics. ∎

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