On global solvability and asymptotic behaviour of a mixed problem for a nonlinear degenerate Kirchhoff model in moving domains

M. M. Cavalcanti V. N. Domingos Cavalcanti J. Ferreira
R. Benabidallah

Abstract

In this paper we prove the existence, uniqueness and asymptotic behaviour of global regular solutions of the mixed problem for the Kirchhoff nonlinear model given by the hyperbolic-parabolic equation

$$(\rho_1 u_t)_t + \rho_2 u_t - M\Big(t, \int_{\alpha(t)}^{\beta(t)} |u_x|^2 dx\Big) u_{xx} = f \quad \text{in } \hat{Q},$$

where $\hat{Q} = \big\{(x,t) \in \mathbb{R}^2 \big| \ \alpha(t) < x < \beta(t), \ 0 < t < \infty\big\}$ is a noncylindrical domain of $\mathbb{R}^2$ and $\beta(\cdot)$, $\alpha(\cdot)$ are positive functions such that

$$\lim_{t\to\infty} (\beta(t) - \alpha(t)) = +\infty.$$

The real function $M(\cdot,\cdot)$ is such that $M(t,\lambda) \geq m_0 > 0 \, \forall (t,\lambda) \in [0,\infty[\times[0,\infty[$, while $\rho_1(\cdot)$, $\rho_2(\cdot)$ are given functions which satisfy some appropriate conditions.

Received by the editors April 2002.
Communicated by P. Godin.
1991 *Mathematics Subject Classification* : 35F30, 35K55.
Key words and phrases : Kirchhoff model, moving boundary, asymptotic behaviour.

1 Introduction

In this paper, we study existence, uniqueness and asymptotic behaviour of global regular solutions for the mixed problem of the Kirchhoff nonlinear model for the hyperbolic-parabolic equation

$$(\rho_1 u_t)_t + \rho_2 u_t - M\Big(t, \int_{\alpha(t)}^{\beta(t)} |u_x|^2 dx\Big) u_{xx} = f \tag{1.1}$$

in the noncylindrical domain $\hat{Q}$ of $\mathbb{R}^2$ defined by

$$\hat{Q} = \Big\{(x,t) \in \mathbb{R}^2 \Big| \quad \alpha(t) < x < \beta(t), \quad 0 < t < \infty\Big\},$$

where $\alpha(\cdot)$ et $\beta(\cdot)$ are two functions such that

$$\lim_{t\to\infty} (\beta(t) - \alpha(t)) = +\infty.$$

The lateral boundary $\hat{\Sigma}$ of $\hat{Q}$ is given by

$$\hat{\Sigma} = \bigcup_{0<t<\infty} (\alpha(t) \times \{t\}) \cup (\beta(t) \times \{t\}).$$

Moreover the solution must satisfy the boundary and initial conditions

$$u(t, \alpha(t)) = u(t, \beta(t)) = 0 \quad (0 < t < \infty), \tag{1.2}$$

$$u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1 \quad \text{in }]\alpha(0), \beta(0)[. \tag{1.3}$$

In the equation (1.1), we assume that the functions ρ_1, ρ_2 and $M(\cdot,\cdot)$ satisfy the following conditions

$$\begin{cases} M \in C^1([0,\infty[\times[0,\infty[), \\ M(t,\lambda) \geq m_0 > 0 \quad \forall (t,\lambda) \in [0,\infty[\times[0,\infty[, \\ M_t(t,\lambda) \leq 0 \quad \forall (t,\lambda) \in [0,\infty[\times[0,\infty[, \end{cases} \tag{1.4}$$

$$\begin{cases} \rho_1 \in W^{2,\infty}(0,\infty), \ \rho_2 \in W^{1,\infty}(0,\infty), \\ \rho_1(t) \geq 0, \ \forall t \in [0,\infty[, \ \rho_1(0) > 0, \\ \rho_2(t) - \frac{1}{2}|\rho_1'(t)| \geq \delta_0 > 0 \quad \forall t \in [0,\infty[, \end{cases} \tag{1.5}$$

where δ_0 and m_0 are given positive numbers.

Since $\rho_1(t) \geq 0$, the equation degenerates into a parabolic case on the subset of $[0,\infty[$ where $\rho_1(t) = 0$. Therefore, the equation (1.1) is an equation of hyperbolic-parabolic type.

The linear and nonlinear equations of hyperbolic-parabolic type has been studied by several authors see for example Lar'kin [12], Bensoussan-Lions- Papanicolau [3] and Ferreira-Lar'kin [9].

In [4], Bisognin proved the existence of local solution of (1.1) in a bounded or unbounded domain of $\mathbb{R}^n$.

When $\rho_1 = 1$ and $\rho_2 = 0$, there is a large number of papers connected with the Kirchhoff-Carrier operator

$$Lu = \partial_t^2 u - \Big(1 + M\Big(\int_\Omega |\nabla u|^2) dx\Big)\Big)\Delta u.$$

The existence of global solutions to problem (1.1) for analytic initial data, was firstly investigated by Pohozaev [18] and Arosio-Spagnolo [1]. More recently, Cavalcanti-Domingos Cavalcanti-Soriano [6] proved the existence of global solutions for regular solutions in the usual Sobolev's space without any smallness on the initial data. But until now the existence of a weak global solution, for arbitrary initial data taken in the usual Sobolev's spaces, has not been proved.

For $\Omega = \{x \in \mathbb{R}|\ x > 0\}$ we have the work of Dickey [7] and his result was generalized to $\Omega = \mathbb{R}^n$ by Menzala [15] and Menzala-Pereira [16].

In order to obtain a global solution for $Lu = f$ several authors, see for example Nishihara [18], have considered damping terms as $-\Delta u_t$ or $\Delta^2 u$, which give strong estimates resulting in the convergences of the nonlinear term showing in this way the global existence result, no matter how large the derivative of M is. Another class of dissipative mechanism was considered by Ikehata and Okazawa [10], where the author of this paper studied the equation of a stretched string with "frictional" damping

$$u_{tt} - \Big(1 + M\Big(\int_\Omega |\nabla u(x,t)|^2 dx\Big)\Big)\Delta u + \mu u_t = 0$$

and showed the existence of global strong solutions, provided μ (a parameter depending on the initial data) is large enough. The behaviour, as $t \to \infty$, of solutions to problem (1.1) in a cylinder with some modifications was studied by Nishihara [17], among other.

About local solutions of related problems with model on the operator Lu, we can mention, among others, the works of Matos [14] and Ebihara-Medeiros-Miranda [8]. A great numbers of works deal with weak or regular solutions in nondegenerate case associated the equation (1.1). The degeneracy of nonlinear hyperbolic equations brings essential difficulties in the case of noncylindrical domains, because the geometry of the domain influences directly in the corresponding problem (see Lar'kin [13]) when a domain is characteristic.

The global existence, uniqueness and asymptotic behavior of regular solutions to problem (1.1)-(1.3) was studied by Benabidallah-Ferreira [2] in the case where the length $L(t) = \beta(t) - \alpha(t)$ of the interval $]\alpha(t), \beta(t)[$ is finite. The purpose of the present paper is to improve considerably the previous results in connection with the Kirchhoff equation in moving domains, namely, to solve globally the initial value problem (1.1)-(1.3) and to study the asymptotic behaviour in the case $\lim_{t\to\infty} L(t) = \infty$. According to our best knowledge, this is the first result in the prior literature concerning equation (1.1).

The method used to prove the existence, uniqueness and decay energy result consists of transforming this problem into another initial boundary-value problem in a cylindrical domain whose sections are not time-dependent. This is done by means of suitable change of variable. Next, this new initial value problem is treated using Galerkin's approximation. We conclude returning to $\hat{Q}$ using the inverse of the change of variable.

At first we will study our problem in a cylinder $Q =]0,1[\times]0,\infty[$. We observe that the domains Q and $\hat{Q}$ are related by the diffeomorphism $\tau : \hat{Q} \longrightarrow Q$ defined by

$$\tau(x,t) = (y,t) = \left(\frac{x-\alpha}{L}, t\right) \quad \text{for } (x,t) \in \hat{Q}, \tag{1.6}$$

where

$$L = L(t) = \beta(t) - \alpha(t)$$

and $\tau^{-1} : Q \longrightarrow \hat{Q}$ is defined by

$$\tau^{-1}(y,t) = (x,t) = (\alpha + yL, t).$$

If we put

$$v(y,t) = u \circ \tau^{-1}(y,t) = u(\alpha + yL, t) \tag{1.7}$$

and we suppose that the functions $\alpha(\cdot)$ and $\beta(\cdot)$ belong to $W^{2,\infty}([0,\infty[)$ then the equation (1.1) and the conditions (1.2)-(1.3) become, if we denote by D^k ($k \in \mathbb{N}$) the differential operator $\frac{\partial^k}{\partial z^k}$ (z is a generic one spatial dimensional variable),

$$\begin{aligned}(\rho_1 v_t)_t + \rho_2 v_t - L^{-2}\tilde{M}\Big(t, L^{-1}\int_0^1 |Dv|^2 dy\Big)\Big) D^2 v - D(aDv) + \\ + a_1 D v_t + a_2 D v = g \quad \text{in } Q,\end{aligned} \tag{1.8}$$

$$v(t,0) = v(t,1) = 0 \quad \text{on }]0,\infty[, \tag{1.9}$$

$$v|_{t=0} = v_0, \quad v|_{t=0} = v_1 \quad \text{in }]0,1[, \tag{1.10}$$

where

$$\begin{cases} a(y,t) = \frac{m_0}{2L^2} - \rho_1\left(\frac{\alpha' + L'y}{L}\right)^2, \quad a_1(y,t) = -2\rho_1\left(\frac{\alpha'+L'y}{L}\right), \\ a_2(y,t) = -\left(\rho_1 \frac{\alpha'' + L''y}{L} + (\rho_1' + \rho_2)\frac{\alpha'+L'y}{L}\right), \\ \tilde{M}(t,\lambda) = M(t,\lambda) - \frac{m_0}{2}. \end{cases} \tag{1.11}$$

We assume that the functions α and β satisfy the conditions

$$\begin{cases} \alpha, \beta \in W^{3,\infty}_{\text{loc}}([0,\infty[), \\ \alpha'(t) < 0, \quad \beta'(t) > 0 \ \forall t \in [0,\infty[, \\ \rho_1(t)(\alpha'(t) + L'(t)y)^2 < \frac{m_0}{2} \quad \forall (t,y) \in [0,\infty[\times[0,1]. \end{cases} \tag{1.12}$$

Note that the assumptions $\alpha'(t) < 0$ and $\beta'(t) > 0$ mean that $\hat{Q}$ is increasing in the sense that if $t_1 > t_2$ then the projection of $[\alpha(t_1), \beta(t_1)]$ on the subspace $t = 0$ contains the projection of $[\alpha(t_2), \beta(t_2)]$ on the same subspace.

We observe that the second condition of (1.4) and the last one of (1.12) imply that

$$a(t,y) > 0 \quad \forall (t,y) \in Q, \tag{1.13}$$

$$\tilde{M}(t,\sigma) \geq \frac{m_0}{2} \quad \forall (t,\sigma) \in [0,\infty[\times[0,\infty[. \tag{1.14}$$

Moreover, since $\alpha(t) < \beta(t)$ for all $t \in [0,T]$, the second condition of (1.12) implies that

$$Da_1 = -\rho_1 \frac{L'}{L} < 0. \tag{1.15}$$

2 Existence and Uniqueness of Local Solution

Let us denote by A the operator

$$A = -D^2, \quad D(A) = H_0^1(0,1) \cap H^2(0,1).$$

It is well known that A is a positive self adjoint operator in the Hilbert space $L^2(0,1)$ for which there exist sequences $\{w_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ of eigenfunctions and eigenvalues of A such that $\{w_n\}_{n\in\mathbb{N}}$ is dense in $D(A)$ and $\lambda_1 < \lambda_2 \le \dots \lambda_n \to \infty$ as $n \to \infty$. Let us denote by

$$v_{0,m} = \sum_{i=1}^{m} (v_0, w_i) w_i, \quad v_{1,m} = \sum_{i=1}^{m} (v_1, w_i) w_i,$$

if $(v_0, v_1) \in D(A) \times H_0^1(0,1)$. So, we have $v_{0,m} \to v_0$ strongly in $D(A)$ and $v_{1,m} \to v_1$ strongly in $H_0^1(0,1)$.

Finally by V_m we will denote the space generated by $w_1, \dots, w_m$. The standard results on ordinary differential equations imply the existence of a local solution $v_{(m\varepsilon)}$ of the form

$$v_{m\varepsilon}(t) = \sum_{i=1}^{m} g_{i,m\varepsilon}(t) w_i,$$

for

$$\begin{aligned} \Big(\Big(\rho_{1\varepsilon} v'_{m\varepsilon}\Big)', w_j\Big) + \rho_2(v'_{m\varepsilon}, w_j) + \tfrac{1}{L^2}\tilde{M}\Big(t, \tfrac{1}{L}\|Dv_{m\varepsilon}\|_{L^2}^2\Big) \times \\ \times (Dv_{m\varepsilon}, Dw_j) + (aDv_{m\varepsilon}, Dw_j) + (a_1 Dv'_{m\varepsilon}, w_j) + \\ + (a_2 Dv_{m\varepsilon}, w_j) = (g, w_j) \quad (j = 1, \dots, m), \end{aligned} \tag{2.1}$$

$$v_{m\varepsilon}(x,0) = v_{0,m\varepsilon}, \quad v'_{m\varepsilon}(x,0) = v_{1,m\varepsilon} \tag{2.2}$$

where $\rho_{1\varepsilon} = \rho_1 + \varepsilon$ $(0 < \varepsilon \le 1)$ and $(\cdot,\cdot)$ denote the scalar product in $L^2(0,1)$, while v'_m denote, for simplicity, the partial derivative with respect t. In our following calculations we will omit the index ε.

To obtain the corresponding estimates we will define the following energy function associated to equation (1.8)

$$E(t,v) = \rho_1(t)\|v'(t)\|_{H^1}^2 + \|Dv\|_{H^1}^2.$$

We will denote by

$$E_m(t) = E(t, v_m).$$

In order to show the estimates result, we will prove some inequalities that the above energy function satisfies.

In fact, first we multiply equation (2.1) by $g'_{im}(t)$. Next we replace w_j by $-\lambda_j^{-1} D^2 w_j$ in (2.1) and multiply it by $g'_{jm}(t)$. Now if we sum up with respect to j $(j = 1, \dots, m)$, we obtain, after some calculations and taking into account the conditions (1.4)-(1.5), (1.12)-(1.13) and (1.15), the estimate

$$E_m(t) + \int_0^t \|v'_m\|_{H^1}^2 ds \le Const \quad \forall t \in [0, T^*] \tag{2.3}$$

(For the details see [2]).

Moreover, if we multiply the j-th equation of (2.1) by $\rho_1 g''_{jm}(t) + \rho'_1 g'_{jm}(t)$ and we sum with respect to j, we easily obtain, by using (2.3), the estimate

$$\int_0^t \|(\rho_1 v'_m)'\|^2_{L^2} ds \le c_8 \quad \forall t \in [0, T^*]. \tag{2.4}$$

Since the estimates (2.3)-(2.4) are established, then we can pass to the limit in (2.1). The argument for the linear terms is classical, we only have to justify the limit in the non linear term.

Indeed, from the estimates (2.3)-(2.4), it follows that Dv_m is bounded in $C^0(0, T^*, L^2(0,1))$. Moreover, since

$$\begin{aligned}\Big|\|Dv_m(t)\|^2_{L^2} - \|Dv_m(\tau)\|^2_{L^2}\Big| &\le \\ &\le 2|t-\tau|^{\frac{1}{2}} \|Dv_m\|_{C^0(0,T^*;L^2(0,1))} \|Dv'_m(t)\|_{L^2(0,T^*;L^2)},\end{aligned}$$

we deduce from the Ascoli-Arzelá theorem and from (2.3) that we can extract a subsequence $\{v_\nu\}$ such that $\left\{\|Dv_\nu(\cdot)\|^2_{L^2}\right\}$ converges uniformly in $[0, T^*]$ to $\|Dv(\cdot)\|^2_{L^2}$.

Since the function $\tilde{M}(\cdot,\cdot)$ belongs to the class $C^1([0,T^*] \times [0,\infty[)$ and Dv_ν is bounded in $L^\infty(0, T^*, H^1(0,1))$ (maybe passing to the new subsequence), we have for $\nu > j$

$$\tilde{M}(t, \frac{1}{L}\|Dv_\nu(t)\|^2_{L^2})(D^2 v_\nu, w_j) \longrightarrow \tilde{M}(t, \frac{1}{L}\|Dv(t)\|^2_{L^2})(D^2 v, w_j). \tag{2.5}$$

Now if we replace in (2.1) m by ν and we pass to the limit when $\nu \to \infty$, we obtain, taking (2.5) into account and the usual classical arguments for the limit in the linear terms,

$$\begin{aligned}((\rho_{1\varepsilon} v_\varepsilon)')', w_j) + \rho_2(v'_\varepsilon, w_j) + \tfrac{1}{L^2}\tilde{M}(t, \tfrac{1}{L}\|Dv_\varepsilon\|^2_{L^2})(Dv_\varepsilon, Dw_j) + \\ + (aDv_\varepsilon, Dw_j) + (a_1 Dv'_\varepsilon, w_j) \\ + (a_2 Dv_\varepsilon, w_j) = (g, w_j) \quad \text{in } L^2(0, T^*).\end{aligned} \tag{2.6}$$

We observe that the estimates (2.3)-(2.4) obtained above are also independent of ε. Therefore, by the same arguments we can pass to the limit in (2.6) when $\varepsilon \to 0$. Thus, we obtain a function v satisfying equation (1.8) in the sense $L^2(0, T^*; L^2(0,1))$.

Consequently, we have the following result.

THEOREM 2.1. *Let*

$$v_0 \in H^2_0(0,1), \quad v_1 \in H^1_0(0,1), \quad g \in L^2(0,T; H^1_0(0,1)).$$

If the assumptions (1.4)-(1.5) *and* (1.12) *hold, then there exists* $T^* > 0$ *such that the initial boundary value problem* (1.8)-(1.10) *admits in* $[0, T^*]$ *a unique solution* v *in the class*

$$\begin{cases} v \in L^\infty(0, T^*; H^1_0(0,1) \cap H^2(0,1)), \quad v_t \in L^2(0, T^*; H^1(0,1)), \\ \sqrt{\rho_1} v_t \in L^\infty(0, T^*; H^1(0,1)), \quad (\rho_1 v_t)_t \in L^2(0, T^*; L^2(0,1)) \end{cases} \tag{2.7}$$

and v *satisfies the equation* (1.8) *in the sense of* $L^2(0, T^*; L^2(0,1))$.

REMARK 2.1. The interval $[0, T^*]$ mentioned in the theorem 2.1 depends on v_0, v_1. But the argument which we have used above allows us to determine T^* as a function of the positive number R such that the solution remains in $[0, T^*]$ for all initial data v_0, v_1 satisfying

$$\|v_1\|_{H^1} + \|v_0\|_{H^2} \leq R.$$

About the uniqueness of solution it's easy to show by a classical argument that the solution is unique.

Now let v the solution obtained from theorem 2.1, then it is easy to see that

$$u = v \circ \tau$$

(see (1.6) for τ) belongs to the class

$$\begin{cases} u \in L^\infty(0, T^*; H_0^1(I_t) \cap H^2(I_t)), \quad \partial_t u \in L^2(0, T^*; H^1(I_t)), \\ \sqrt{\rho_1} u_t \in L^\infty(0, T^*; H^1(I_t)), \quad (\rho_1 u_t)_t \in L^2(0, T^*; L^2(I_t)). \end{cases} \tag{2.8}$$

and from (1.8) it follows $u = v \circ \tau$ satisfies the equation (1.1) in the sense $L^2(0, T^*; L^2(I_t))$. Let u_1, u_2 be two solutions to (1.1), and v_1, v_2 be the functions obtained through the diffeomorphism τ given by (1.6). Then v_1, v_2 are the solutions to (1.8). By the uniqueness result of theorem 2.1, we have $v_1 = v_2$, so $u_1 = u_2$, so that we have the following result.

THEOREM 2.2. *Let I_t, I_0 the intervals $]\alpha(t), \beta(t)[$ and $]\alpha(0), \beta(0)[$ respectively. If we have*

$$u_0 \in H_0^2(I_0), \quad u_1 \in H_0^1(I_0), \quad f \in L^2(0, T; H_0^1(I_t))$$

and the assumptions (1.4)-(1.5) *and* (1.12) *hold, then there exists $T^* > 0$ such that the equation* (1.1) *with the conditions* (1.2)-(1.3) *admits in $[0, T^*]$ a unique solution u which belongs to the class in* (2.8) *and satisfies the equation* (1.1) *in the sense of $L^2(0, T^*; L^2(I_t))$.*

3 Global Solution

In what follows, besides hypotheses (1.4)-(1.5) and (1.12), we assume also that the following conditions are satisfied

$$\left|\tilde{M}(t, \lambda)\right| \leq S_0, \quad \left|\tilde{M}_\lambda(t, \lambda)\right| \leq S_1 \quad \forall (t, \lambda) \in [0, \infty[\times[0, \Lambda], \tag{3.1}$$

where S_0 and S_1 are a positive constants independent of t and Λ is given in (3.3),

$$|\alpha^{(i)}(t)| \leq \varepsilon_0, \quad |\beta^{(i)}(t)| \leq \varepsilon_0 \quad \forall t > 0 \quad (i = 1, 2, 3), \tag{3.2}$$

where ε_0 is a suitable positive number which will be determined latter (see 3.22) ($\varphi^{(k)}$ denotes the the k-th derivative of the generic real function $\varphi^{(k)}$, the first and the second derivative are denoted by φ' and φ'').

We assume also that

$$\|Dv\|_{L^\infty(0,\infty;L^2)} \le \Lambda \tag{3.3}$$

where Λ is given positive number.

Finally we assume that

$$\rho_2'(t) \le -\delta_1, \quad \forall t \in [0,\infty[\quad (\delta_1 > 0), \tag{3.4}$$

where δ_1 is a positive constant.

In what follows c_i $(i = 1, \dots, 14)$ denote some positive constants which depend only on m_0, δ_0, δ_1, Λ, S_0, S_1, $L_0 = L(0)$ and on norms of ρ_1 in $W^{2,\infty}(0,\infty)$ and ρ_2 in $W^{1,\infty}(0,\infty)$.

A PRIORI ESTIMATES. We consider the scalar product in $L^2(0,1)$ of the equation (1.8) with v_t. We obtain the following equality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathcal{L}_1(t) &+ (\rho_2 - \tfrac{1}{2}\rho_1')\|v_t\|^2_{L^2} + \tfrac{L'}{2L^3}\tilde{M}(t, \tfrac{1}{L}\|Dv\|^2_{L^2})\|Dv\|^2_{L^2} - \\ &- \tfrac{1}{2L}\int_0^{\frac{1}{L}\|Dv\|^2_{L^2}} \tilde{M}_t(t,\lambda)d\lambda + \\ &+ \tfrac{L'}{2L^2}\hat{M}\Big(t, \tfrac{1}{L}\|Dv\|^2_{L^2}\Big) - \tfrac{1}{2}\int_0^1 (Da_1)|v_t|^2 dy = \sum_{i=1}^3 I_i \end{aligned}\tag{3.5}$$

where

$$\mathcal{L}_1(t) = \rho_1\|v_t\|^2_{L^2} + \int_0^1 a|Dv|^2dy + \tfrac{1}{L}\hat{M}\Big(t, \tfrac{1}{L}\|Dv\|^2_{L^2}\Big), \tag{3.6}$$

$$\hat{M}(t,\lambda) = \int_0^\lambda M(t,\sigma)d\sigma$$

and

$$I_1 = -\int_0^1 a_2(Dv)v_t dy, \quad I_2 = -\frac{1}{2}\int_0^1 a_t|Dv|^2, \quad I_3 = \int_0^1 g v_t dy.$$

If we recall the expressions of coefficient a and a_2 (see (1.11)), we have, taking into account the hypothesis (3.2) (see also (1.5))

$$\Big|\sum_{i=1}^3 I_i\Big| \le \tfrac{\delta_0}{2}\|v_t\|^2_{L^2} + c_1\tfrac{\varepsilon_0}{L^2}\|D^2v\|^2_{L^2} + \tfrac{1}{\delta_0}\|g\|^2_{L^2},$$

where we have used also the obvious inequality $\|Dv\|_{L^2} \le \|D^2v\|_{L^2}$.

If we joint this estimate with (3.5) and taking into account (1.4), (1.15) and respectively the last and the second condition of (1.5) and (1.12), we obtain

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_1(t) + \tfrac{\delta_0}{2}\|v_t\|^2_{L^2} \le c_1\tfrac{\varepsilon_0}{L^2}\|D^2v\|^2_{L^2} + \tfrac{1}{\delta_0}\|g\|^2_{L^2}. \tag{3.7}$$

In other words, the scalar product in $L^2(0,1)$ of the equation (1.8) with $-D^2v_t$, gives the equality

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\mathcal{L}_2(t) &+ (\rho_2 - \tfrac{1}{2}\rho_1')\|Dv_t\|^2_{L^2} + \tfrac{\beta'}{2L}|Dv_t(t,1)|^2 - \tfrac{\alpha'}{2L}|Dv_t(t,0)|^2 - \\ &- \tfrac{1}{2L^2}\|D^2v\|^2_{L^2}\tilde{M}_t\Big(t, \tfrac{1}{L}\|Dv\|^2_{L^2}\Big) + \\ &+ \tfrac{L'}{L^3}\tilde{M}(t, \tfrac{1}{L}\|Dv\|^2_{L^2})\|D^2v\|^2_{L^2} = \sum_{i=1}^7 I_i \end{aligned}\tag{3.8}$$

where

$$\mathcal{L}_2(t) = \rho_1 \|Dv_t\|^2_{L^2} + \tfrac{1}{L^2}\tilde{M}\Big(t, \tfrac{1}{L}\|Dv\|^2_{L^2}\Big)\|D^2v\|^2_{L^2} + \\ + 2\int_0^1 (Da + a_1 + a_2)(Dv)(D^2v)dy + \int_0^1 a|D^2v|^2 dy \tag{3.9}$$

and

$$\begin{aligned} I_1 &= \tfrac{1}{2L^3}\tilde{M}_\lambda(t, \tfrac{1}{L}\|Dv\|^2_{L^2})\|D^2v\|^2_{L^2}\tfrac{d}{dt}\|Dv\|^2_{L^2}, \\ I_2 &= -\tfrac{L'}{2L^4}\tilde{M}_\lambda(t, \tfrac{1}{L}\|Dv\|^2_{L^2})\|D^2v\|^2_{L^2}\|Dv\|^2_{L^2}, \\ I_3 &= \int_0^1 (a_{2t} + Da_t)(Dv)(D^2v)dy, \\ I_4 &= -\int_0^1 (a_2 + Da)(Dv_t)(D^2v)dy, \\ I_5 &= \frac{1}{2}\int_0^1 a_t(D^2v)^2 dy, \quad I_6 = \int_0^1 (Dg)(Dv')dy, \\ I_7 &= -\int_0^1 (Da_1)|Dv_t|^2 dy \end{aligned}$$

(here $\tilde{M}_\lambda(t, \lambda_0)$ denotes the partial derivative with respect λ calculated at the point λ_0).

By virtue of the hypotheses (3.1) and (3.3), we have

$$|I_1 + I_2| \le \tfrac{S_1\Lambda}{2L_0}\tfrac{1}{L^2}\|Dv_t\|_{L^2}\|D^2v\|^2_{L^2} + \tfrac{\varepsilon_0 S_1}{L_0^2}\tfrac{1}{L^2}\|Dv\|^2_{L^2}\|D^2v\|^2_{L^2},$$

where $L_0 = L(0) = \beta(0) - \alpha(0) > 0$.

Moreover, by using (3.2) (see also (1.11)), it follows

$$\Big|\sum_{i=3}^{6} I_i\Big| \le c_2\varepsilon_0\tfrac{1}{L^2}\|D^2v\|^2_{L^2} + \tfrac{\delta_0}{2}\|Dv_t\|^2_{L^2} + \tfrac{1}{\delta_0}\|Dg\|^2_{L^2}.$$

Finally, we have by using (3.3) and (1.15)

$$|I_7| \le c_3\varepsilon_0\|Dv_t\|^2_{L^2}.$$

If we add the above estimates with (3.8), we get the inequality

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_2(t) + (\tfrac{\delta_0}{2} - c_3\varepsilon_0)\|Dv_t\|^2_{L^2} \le c_2\tfrac{\varepsilon_0}{L^2}\|D^2v\|^2_{L^2} + \\ c_4\tfrac{1}{L^2}\|Dv_t\|_{L^2}\|D^2v\|^2_{L^2} + c_5\tfrac{1}{L^2}\|Dv\|^2_{L^2}\|D^2v\|^2_{L^2} + \tfrac{1}{\delta_0}\|Dg\|^2_{L^2}. \tag{3.10}$$

Now we consider the scalar product in $L^2(0,1)$ of the equation (1.8) with $-D^2v$. we get

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_3(t) + \tfrac{1}{L^2}\tilde{M}\Big(t, \tfrac{1}{L}\|Dv\|^2_{L^2}\Big)\|D^2v\|^2_{L^2} - \tfrac{1}{2}\rho_2'\|Dv\|^2_{L^2} + \\ + \int_0^1 a|D^2v|^2 dy = \sum_{i=1}^{5} I_i \tag{3.11}$$

where

$$\mathcal{L}_3(t) = 2\int_0^1 \rho_1(Dv_t)(Dv)dy + \rho_2\|Dv\|_{L^2}^2 \tag{3.12}$$

and

$$I_1 = \rho_1\|Dv_t\|_{L^2}^2, \quad I_2 = \int_0^1 a_2(Dv)(D^2v)dy,$$
$$I_3 = -\int_0^1 (Da)(Dv)(D^2v)dy, \quad I_4 = -\int_0^1 Dg(Dv)dy,$$
$$I_5 = \int_0^1 a_1(Dv_t)(D^2v)dy.$$

It's easy to see from the hypotheses (3.2)-(3.3), and the expressions (1.11) of the functions a_1 and a_2 that

$$|I_2 + I_3 + I_4| \leq \tfrac{m_0}{8L^2}\|D^2v\|_{L^2}^2 + c_6\varepsilon_0\|Dv\|_{L^2}^2 + \Lambda\|Dg\|_{L^2},$$
$$|I_5| \leq \tfrac{m_0}{8L^2}\|D^2v\|_{L^2}^2 + c_7\varepsilon_0\|Dv_t\|_{L^2}^2.$$

These estimates, together with (3.11) give, taking account into (3.4),

$$\begin{aligned}\tfrac{d}{dt}\mathcal{L}_3(t) + (\tfrac{\delta_1}{2} - c_6\varepsilon_0)\|Dv\|_{L^2}^2 + \tfrac{m_0}{4L^2}\|D^2v\|_{L^2}^2 \leq \\ \leq (c_7\varepsilon_0 + \|\rho_1\|_{L^\infty})\|Dv_t\|_{L^2}^2 + \Lambda\|Dg\|_{L^2}.\end{aligned} \tag{3.13}$$

Now we can enunciate and prove our main result about the global existence and uniqueness solution.

We have the

THEOREM 3.2. *Let* $(u_0, u_1) \in H_0^2(I_0) \times H_0^1(I_0)$, $f \in L^2(0,\infty; H_0^1(I_t)) \cap L^1(0,\infty; H_0^1(I_t))$. *We assume that conditions* (3.1)-(3.3) *are satisfied and the norms* $\int_0^\infty \|f\|_{H^1(I_t)}^2 dt$, $\int_0^\infty \|f\|_{H^1(I_t)} dt$, $\|u_0\|_{H^2(I_0)}$ *and* $\|u_1\|_{H^1(I_0)}$ *are small enough. Then the equation* (1.1) *admits one and only one solution* u *in the class*

$$\begin{cases} u \in L^\infty(0,\infty; H^2(I_t) \cap H_0^1(I_t)),\ u \in L^2(0,\infty; H^2(I_t)), \\ \sqrt{\rho_1}u_t \in L^\infty(0,\infty; H^1(I_t)),\ u_t \in L^2(0,\infty; H^1(I_t)), \\ (\rho_1 u_t)_t \in L^2(0,\infty; L^2(I_t)). \end{cases} \tag{3.14}$$

and satisfies the equation (1.1) *in the sense of* $L^2(0,\infty; L^2(I_t))$.

Proof. Theorem 3.2 will follow from theorem 2.1 (see also remark 2.1), if we obtain an *a priori* estimate which shows that the norm

$$\rho_1\|v_t\|_{H^1}^2 + \|Dv(t)\|_{L^2}^2 + \tfrac{1}{L^2}\|D^2v(t)\|_{L^2}^2$$

is uniformly bounded for all $t \in [0,\infty[$. In fact, if this norm is bounded for all t by the same constant, then the theorem 2.1 implies that the solution v can be continued to the interval $[t, t+T'[$ and repeating this argument, we can extend v to the whole interval $[0,\infty[$.

To this end, we introduce the functions $\mathcal{L}(t)$, $\mathcal{E}(t)$ and $\mathcal{D}(t)$; by putting

$$\mathcal{L}(t) = \mathcal{L}_1(t) + \kappa\mathcal{L}_2(t) + \mathcal{L}_3(t), \tag{3.15}$$

$$\mathcal{E}(t) = \rho_1\|v_t\|^2_{H^1} + \|Dv(t)\|^2_{L^2} + \tfrac{1}{L^2}\|D^2v(t)\|^2_{L^2}, \tag{3.16}$$

$$\mathcal{D}(t) = \|v_t\|^2_{H^1} + \|Dv(t)\|^2_{L^2} + \tfrac{1}{L^2}\|D^2v(t)\|^2_{L^2} \tag{3.17}$$

(see (3.6) (3.9) and (3.12)) for the expressions of $\mathcal{L}_i(t)$ $(i = 1, 2, 3)$), where κ is a constant which will be determined in the following. In fact we want choose k and ε_0 such that

$$\overline{c}\mathcal{E}(t) \le \mathcal{L}(t) \le \underline{c}\mathcal{E}(t) \tag{3.18}$$

$$\mathcal{L}(t) + k_1\int_0^t \mathcal{D}(s)ds \le \mathcal{L}_0 + k_2\int_0^t \mathcal{D}(s)(\mathcal{L}(s) + \mathcal{L}^{\frac{1}{2}}(s))ds \tag{3.19}$$

where

$$\mathcal{L}_0 = \mathcal{L}(0) + k_3\int_0^\infty (\|g\|^2_{H^1} + \|g\|_{H^1})dt, \tag{3.20}$$

while k_i $(i = 1, 2, 3)$ and $\overline{c}$ are positives constants depending only on m_0, δ_0, δ_1, Λ and ε_0.

We determine the constants κ and the positive number ε_0 in the inequality (3.7), (3.10) and (3.13) by the following relations:

$$\kappa = 1 + \max\Big(c_{10}, \frac{2\|\rho_1\|_{L^\infty}}{\delta_0}\Big), \tag{3.21}$$

$$\varepsilon_0 = \min\Big(1, \frac{\delta_1}{2c_6}, \frac{\delta_0}{2(\kappa c_3 + c_7)}, \frac{m_0}{4(\kappa c_2 + c_1)}, \frac{1}{2\kappa c_9}, \frac{m_0}{c_8}\Big). \tag{3.22}$$

We remark that it is easy to see from (3.2) (see also (1.12) and the last condition of (1.5)which implies that $\rho_2 \ge \delta_0$), that we have

$$\Big|2\int_0^1 (Da + a_1 + a_2)(Dv)(D^2v)dy\Big| \le c_8\frac{\varepsilon_0}{L^2}\|D^2v\|^2_{L^2} + c_9\rho_2\varepsilon_0\|Dv\|^2_{L^2}, \tag{3.23}$$

$$\Big|2\int_0^1 \rho_1(Dv_t)(Dv)dy\Big| \le \frac{1}{2}\rho_2\|Dv\|^2_{L^2} + c_{10}\rho_1\|Dv_t\|^2_{L^2}. \tag{3.24}$$

If we recall the definition of $\mathcal{L}(\cdot)$ (see (3.15) and also (3.6), (3.9), (3.12)), it is easy to see that the inequalities (3.23)-(3.24) imply (3.18) with a constant $\overline{c}$.

In other words, for (3.19), we recall the inequalities (3.7), (3.10) and (3.13). Now if we sum inequalities (3.7), (3.10) multiplied by κ given in (3.21) and (3.13) and we substitute in these inequalities the expression (3.22) of ε_0, we obtain immediately (3.19)-(3.20) with the constants k_i $(i = 1, 2, 3)$.

Suppose now that

$$\mathcal{L}(0) + k_3\int_0^\infty (\|g\|^2_{H^1}dt + \|g\|_{H^1})dt + < \sigma_0, \tag{3.25}$$

where

$$\sigma_0 = \min\Big(\frac{k_1}{k_2} + \frac{1}{2}\Big(1 - \sqrt{1 + k_1k_2^{-1}}\Big), \overline{c}\Lambda\Big).$$

Then, since we have

$$k_2(r+\sqrt{r}) < k_1$$

for $r \in \Big[0, \frac{k_1}{k_2} + \frac{1}{2}\Big(1 - \sqrt{1+\frac{k_1}{k_2}}\Big)\Big[$, we deduce from (3.18) that

$$\mathcal{L}(t) \le \mathcal{L}_0 \quad \forall t > 0. \tag{3.26}$$

In the same way it follows that

$$\|Dv\|_{L^\infty(0,\infty;L^2)} \le \Lambda$$

which means that the hypothesis (3.3) is verified.

Hence, now we can say that (3.25) implies (3.26) without the hypothesis (3.3). Then, in view of (3.18) and (3.16), with (3.26) we obtain the *a priori* estimate

$$\rho_1(t)\|v_t\|^2_{H^1} + \|Dv(t)\|^2_{L^2} + \frac{1}{L^2}\|D^2 v(t)\|^2_{L^2} \le \frac{1}{\overline{c}}\mathcal{L}_0. \tag{3.27}$$

Since $u = v \circ \tau$ (see (1.6) for τ), it is easy to see that

$$\begin{cases} \|Du\|^2_{H^1(I_t)} \le \frac{1}{L_0}\Big(\|Dv\|^2_{L^2(0,1)} + \frac{1}{L^2}\|D^2 v\|^2_{L^2(0,1)}\Big), \\ \|Du_t\|^2_{L^2(I_t)} \le c\Big(\|Dv\|^2_{L^2(0,1)} + \frac{1}{L^2}\|D^2 v\|^2_{L^2(0,1)} + \|Dv_t\|^2_{L^2(0,1)}\Big) \end{cases} \tag{3.28}$$

where c is a constant which depends on ε_0 and $L_0 = L(0) = \beta(0) - \alpha(0) > 0$. From (3.27) it follows that

$$Du \in L^\infty(0,\infty; H^1(I_t)), \quad \sqrt{\rho_1}Du_t \in L^\infty(0,\infty; L^2(I_t)). \tag{3.29}$$

Moreover, since

$$k_1 - k_2(\mathcal{L}(t) + \mathcal{L}^{\frac{1}{2}}(t)) \ge \nu > 0$$

(see (3.25)-(3.26)), we deduce from (3.19) that

$$\mathcal{D}(\cdot) \in L^1(0,\infty).$$

Recalling the expression of $\mathcal{D}(\cdot)$ (see (3.17)), it's easy to see by using the first inequality of (3.28) that

$$Du \in L^2(0,\infty; H^1(I_t), \quad Du_t \in L^2(0,\infty; L^2(I_t)). \tag{3.30}$$

To conclude that our solution belongs to the class (3.14) it remains, taking into account (3.29)-(3.30), to prove that

$$\begin{cases} u \in L^\infty(0,\infty; L^2(I_t), \quad u \in L^2(0,\infty; L^2(I_t)), \\ \sqrt{\rho}_1 u_t \in L^\infty(0,\infty; L^2(I_t)), \quad u_t \in L^2(0,\infty; L^2(I_t)). \end{cases} \tag{3.31}$$

To this end, first we consider the scalar product in $L^2(I_t)$ of the equation (1.1) with u_t, we obtain, taking into account the relation

$$Du(t,x)\beta'(t) + u_t(t,x)|_{x=\beta(t)} = Du(t,x)\alpha'(t) + u_t(t,x)|_{x=\alpha(t)} = 0$$

which is a immediate consequence of the conditions $u(t,\alpha(t)) = u(t,\beta(t)) = 0$, the inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E_1(t) + (\rho_2(t) + \frac{1}{2}\rho'(t))\|u_t\|^2_{L^2(I_s)} - \frac{1}{2}\int_0^{\|Du\|^2_{L^2(I_t)}} M_t(t,\sigma)d\sigma \\ + \frac{1}{2\beta'}\left|u_t(t,\beta(t))\right|^2\left(M(t,\|Du\|^2_{L^2(I_t)}) - \rho_1(t)\beta'^2(t)\right) + \\ + \frac{1}{2\alpha'}\left|u_t(t,\alpha(t))\right|^2\left(\rho_1(t)\alpha'^2(t) - M(t,\|Du\|^2_{L^2(I_t)})\right) = 0,\end{aligned} \tag{3.32}$$

where

$$E_1(t) = \rho_1\|u_t\|^2_{L^2(I_t)} + \hat{M}(t,\|Du\|^2_{L^2(I_t)}).$$

and

$$\hat{M}(t,\lambda) = \int_0^\lambda M(t,\sigma)d\sigma.$$

If we choose the number ε_0 given in (3.22) (see also (3.2)) such that

$$\varepsilon_0 \le \sqrt{\frac{m_0}{\|\rho_1\|_{L^\infty}}}, \tag{3.33}$$

we get

$$\begin{cases} M(t,\|Du\|^2_{L^2(I_t)}) - \rho_1(t)\beta'^2(t) \ge 0, \\ \rho_1(t)\alpha'^2(t) - M(t,\|Du\|^2_{L^2(I_t)} \le 0 \quad \forall t \ge 0. \end{cases}$$

Thus, if we recall the last conditions of (1.4) and (1.5) and the first condition of (1.12), we deduce from (3.32) that

$$\frac{1}{2}\frac{d}{dt}E_1(t) + \delta_0\|u_t\|^2_{L^2(I_t)} \le 0. \tag{3.34}$$

However, if we consider the scalar product of (1.1) with u, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E_2(t) - \rho_1\|u_t\|^2_{L^2(I_t)} - \frac{1}{2}\rho_2'\|u\|^2_{L^2(I_t)} + \\ + M\left(t,\|Du\|^2_{L^2(I_t)}\right)\|Du\|^2_{L^2(I_t)} = 0,\end{aligned} \tag{3.35}$$

where

$$E_2(t) = 2\int_0^1 \rho_1 u u_t dx + \rho_2\|u_t\|^2_{L^2(I_t)}.$$

Since (see the last condition of (1.5) which implies that $\rho_2 \ge \delta_0$)

$$E_2(t) \ge \frac{1}{2}\rho_2\|u\|^2_{L^2(I_t)} - 2\delta_0^{-1}\|\rho_1\|_{L^\infty}\rho_1\|u_t\|^2_{L^2(I_t)},$$

if we choose the positive number δ_2 such that

$$\delta_2 \ge 2\delta_0^{-1}\|\rho_1\|_{L^\infty} \tag{3.36}$$

we obtain, recalling the expressions of E_1 and E_2,

$$\delta_2 E_1(t) + E_2(t) \ge \frac{1}{2}\rho_2\|u\|^2_{L^2(I_t)} + (\delta_2 - 2\delta_0^{-1}\|\rho_1\|_{L^\infty})\rho_1\|u_t\|^2_{L^2(I_t)} \ge 0. \tag{3.37}$$

Now if we multiply (3.32) by δ_2 given in (3.36) and add it with (3.35), we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\delta_2 E_1(t) + E_2(t)\Big) + \delta_2\delta_1\|u\|^2_{L^2(I_t)} + (\delta_2\delta_0 - \|\rho_1\|_{L^\infty})\|u_t\|^2_{L^2(I_t)} \leq 0.$$

From the above inequality (see also (3.37)), it is easy to verify that u satisfies (3.31) and according to (3.29)-(3.30) we deduce that u belongs to the class (3.14).

Thus the proof of theorem (3.1) is completed. ■

4 Asymptotic behaviour

In this section we shall examine the asymptotic behaviour of the solution given by the theorem 2.2 always in the case where

$$\lim_{t\to\infty} L(t) = \lim_{t\to\infty}(\beta(t) - \alpha(t)) = +\infty.$$

We assume that the hypotheses (3.1) and (3.4) hold and

$$|\alpha^{(i)}(t)| \leq \varepsilon_1 \leq \varepsilon_0, \quad |\beta^{(i)}(t)| \leq \varepsilon_1 \leq \varepsilon_0 \quad \forall t > 0 \quad (i = 1, 2, 3) \tag{4.1}$$

where ε_0 is given in (3.22) (see also (3.33)), while ε_1 will be chosen conveniently in the following (see (4.15)).
Moreover, here we assume that the positive number δ_0 given in (1.5) satisfies

$$\delta_0 > \frac{S_1\Lambda}{L_0}, \tag{4.2}$$

where S_1 and Λ are given in (3.1), (3.3), while $L_0 = L(0)$.

Then we have

THEOREM 4.1 Let $u_0 \in H^2(I_0) \cap H^1_0(I_0)$ **and** $u_1 \in H^1_0(I_0)$**.** *We assume that the hypotheses* (3.1), (3.4) *and* (4.1)-(4.2) *hold, then the solution of the equation* (1.1) *with* $f = 0$ *satisfies for all* $t > 0$ *the inequality*

$$\rho_1(t)\|u_t\|^2_{H^1(I_t)} + \|u(t)\|^2_{H^2(I_t)} \leq CL(t)e^{-\gamma t} \tag{4.3}$$

for suitable positive constants C *and* γ.

We remark (see (4.1)) that the function $L' = \beta' - \alpha'$ is bounded and therefore $L(t)e^{-\gamma t} \to 0$ as $t \to \infty$.

Proof. We have already seen that the scalar product in $L^2(0,1)$ of the equation (1.8) with v_t gives us the inequality

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_1(t) + \tfrac{\delta_0}{2}\|v_t\|^2_{L^2} \leq c_1\tfrac{\varepsilon_1}{L^2}\|D^2 v\|^2_{L^2} + \tfrac{1}{\delta_0}\|g\|^2_{L^2}, \tag{4.4}$$

where

$$\mathcal{L}_1(t) = \rho_1\|v_t\|^2_{L^2} + \int_0^1 a|Dv|^2 dy + \tfrac{1}{L}\hat{M}\Big(t, \tfrac{1}{L}\|Dv\|^2_{L^2}\Big). \tag{4.5}$$

In the same way as (3.8), the scalar product in $L^2(0,1)$ of the equation (1.8) with $-D^2v_t$, gives us the inequality

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_2(t) + (\tfrac{\delta_0}{2} - c_3\varepsilon_1)\|Dv_t\|^2_{L^2} \le c_2\tfrac{\varepsilon_1}{L^2}\|D^2v\|^2_{L^2} + \\ + \tfrac{S_1\Lambda}{2L_0}\tfrac{1}{L^2}\|Dv_t\|_{L^2}\|D^2v\|^2_{L^2} + \tfrac{S_1}{2L_0^2}\varepsilon_1\tfrac{1}{L^2}\|Dv\|^2_{L^2}\|D^2v\|^2_{L^2} + \tfrac{1}{\delta_0}\|Dg\|^2_{L^2} \tag{4.6}$$

where

$$\mathcal{L}_2(t) = \rho_1\|Dv_t\|^2_{L^2} + \tfrac{1}{L^2}\tilde{M}\Big(t, \tfrac{1}{L}\|Dv\|^2_{L^2}\Big)\|D^2v\|^2_{L^2} + \\ + 2\int_0^1 (Da + a_1 + a_2)(Dv)(D^2v)dy + \int_0^1 a(D^2v)^2dy. \tag{4.7}$$

Since

$$\|Dv\|^2_{L^2} \le \Lambda, \quad \frac{1}{L^2}\|D^2v\|^2_{L^2} \le \Lambda$$

(see (3.25), (3.27) and (3.20)), we deduce from (4.6) that

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_2(t) + (\delta_3 - c_3\varepsilon_1)\|Dv_t\|^2_{L^2} \le \\ \le \varepsilon_1(c_2 + \tfrac{S_1\Lambda}{2L_0^2})\tfrac{1}{L^2}\|D^2v\|^2_{L^2} + \tfrac{1}{\delta_0}\|Dg\|^2_{L^2} \tag{4.8}$$

where (see (4.2))

$$\delta_3 = \frac{\delta_0}{2} - \frac{S_1\Lambda}{2L_0} > 0.$$

Moreover if we consider the scalar product in $L^2(0,1)$ of the equation (1.8) with $-D^2v$, we obtain by the same argument as (3.11), the inequality

$$\frac{d}{dt}\mathcal{L}_3(t) + (\tfrac{\delta_1}{2} - c_6\varepsilon_0)\|Dv\|^2_{L^2} + \tfrac{m_0}{4L^2}\|D^2v\|^2_{L^2} \le \\ \le (c_7\varepsilon_1 + \|\rho_1\|_{L^\infty})\|Dv_t\|^2_{L^2} + \Lambda\|Dg\|_{L^2} \tag{4.9}$$

where

$$\mathcal{L}_3(t) = 2\int_0^1 \rho_1(Dv_t)(Dv)dy + \rho_2\|Dv\|^2_{L^2}. \tag{4.10}$$

We put

$$\mathcal{L}(t) = \mathcal{L}_1(t) + \kappa_1\mathcal{L}_2(t) + \mathcal{L}_3(t) \tag{4.11}$$

and we recall the expressions (3.16)-(3.17) of $\mathcal{E}(\cdot)$ and $\mathcal{D}(\cdot)$. The positive number κ_1 will be fixed in the following (see (4.14)). In fact we want to choose k_1 and ε_1 in the inequalities (4.4) and (4.8)-(4.9) such that

$$c_{11}\mathcal{E}(t) \le \mathcal{L}(t) \le c_{12}\mathcal{D}(t), \tag{4.12}$$

$$\frac{d}{dt}\mathcal{L}(t) + c_{13}\mathcal{D}(t) \le 0. \tag{4.13}$$

We determine the constant κ_1 and the number ε_1 in the inequality (4.4) and (4.8)-(4.9) by the following relations

$$\kappa_1 = \max\Big(\kappa, \frac{2\|\rho_1\|_{L^\infty}}{\delta_2}\Big), \tag{4.14}$$

$$\varepsilon_1 = \min\Big(\varepsilon_0, \frac{m_0}{2(\kappa(2c_2 + S_1\Lambda L_0^{-2}) + 2c_1)}, \frac{\delta_3}{\kappa c_3 + c_7}\Big) \tag{4.15}$$

(κ and ε_0 see (3.21)-(3.22)).

From the inequality (3.23)-(3.24) and the relations (4.14)-(4.15), we deduce easily the inequality of the left hand side of (4.12), while the other inequality follows immediately from the expressions of $\mathcal{L}_i$ ($i = 1, 2, 3$) (see also the first condition of (3.1)).

Now, if we multiply (4.8) by κ_1 given in (4.14) and add it with the inequalities (4.4) and (4.9) and we substitute in these inequalities the expression (4.15) of ε_1, we obtain immediately (4.13).

The inequalities (4.12)-(4.13) give us

$$\tfrac{d}{dt}\mathcal{L}(t) + \gamma\mathcal{L}(t) \leq 0 \quad \forall t \geq 0$$

where γ is a positive constant which depends only on the numbers m_0 δ_0, δ_1, Λ and S_0. This above inequality implies that

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\gamma t}$$

which, with a help of (4.12) (see also (3.16)-(3.17)), gives us

$$\rho_1\|v_t\|^2_{H^1} + \|v\|^2_{H^1} + \tfrac{1}{L^2}\|D^2 v\|^2_{L^2} \leq c(\|v_1\|^2_{H^1} + \|v_0\|^2_{H^2})e^{-\gamma t}. \tag{4.16}$$

In order to conclude our result, it is sufficient to see that

$$u = v \circ \tau$$

(see (1.6) for τ) satisfies the following estimates

$$\|u_t\|^2_{H^1(I_t)} \leq cL(\|v_t\|^2_{H^1(0,1)} + \|Dv\|^2_{L^2(0,1)} + \frac{1}{L^2}\|D^2 v\|^2_{L^2(0,1)}\Big),$$
$$\|u\|^2_{H^2(I_t)} \leq cL\Big(\|v\|^2_{H^1(0,1)} + \frac{1}{L^2}\|D^2 v\|^2_{L^2(0,1)}\Big)$$

with a positive constant c independent of t, so that the inequality (4.16) implies

$$\rho_1(t)\|u\|^2_{H^1(I_t)} + \|u\|^2_{H^2(I_t)} \leq cL(t)e^{-\gamma t}.$$

Thus the proof of theorem 4.1 is completed. ■

REMARK 2.2. When f decays in an appropriate way (see [16]) we can obtain the same result as in Theorem 4.1.

Acknowledgement - The authors would like to express their gratitude to Professor G. P Menzala for fruitful discussions concerning this paper.

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Jorge Ferreira, Marcelo M. Cavalcanti and Valéria N. Domingos Cavalcanti:
Department of Mathematics (DMA),
State University of Maringá (UEM)
Av. Colombo, n.5790
Campus Universitário -zona 07 CEP 87020-900 - Maringá - PR, Brazil.

R. Benabidallah:
Dipartimento di Matematica, Università di Pisa,
Via Buonarroti, 2, 56127 Pisa, Italy.