The characteristic numbers of cuspidal plane cubics in $\mathbb{P}^3$

Xavier Hernández Josep M. Miret *

Abstract

We obtain the characteristic numbers of the variety of non degenerate cuspidal plane cubics in $\mathbb{P}^3$, namely, the non-zero intersection numbers which arise from considering 10 (possibly repeated) conditions from among the following: P, that the cuspidal cubic go through a point; ν, that the cuspidal cubic intersect a line; and ρ, that the cuspidal cubic be tangent to a plane. In order to reach this goal, we consider a suitable compactification of the variety of non degenerate cuspidal plane cubics in $\mathbb{P}^3$ and we calculate, using several degeneration formulae, some of its non-zero intersection numbers, including all the characteristic ones.

1 Introduction

The *characteristic numbers problem* for families of algebraic curves in $\mathbb{P}^2$ deals with how many prefixed degree plane curves there are through a given number of points and tangent to a given number of lines. Maillard [5] and Zeuthen [11] independently found the characteristic numbers of the family of cuspidal cubics in $\mathbb{P}^2$; their results were refined by Schubert [10] and cross-checked, in different ways, by Sacchiero [9], Aluffi [1] and Kleiman-Speiser [4]. As a matter of fact, in Miret-Xambó [7, 8] not only the characteristic numbers of cuspidal cubics were computed, but also all those relative to the *singular triangle*.

*Both authors are partially supported by the grant BFM2000-1113-C02-02, DGES (Spain)
Received by the editors January 2002.
Communicated by S. Gutt.
1991 *Mathematics Subject Classification :* 14N10, 14C17.
Key words and phrases : cuspidal cubics, characteristic numbers.

The aim of this paper is to study the case of these curves in projective space. More precisely, we justify the accuracy of the characteristic numbers of the family of cuspidal plane cubics in $\mathbb{P}^3$ given by Schubert in [10]. To do so, we first consider the projective bundle X_{cusp} which is a compactification of the 10-dimensional variety U_{cusp} of non-degenerate cuspidal plane cubics in $\mathbb{P}^3$, and we give consideration to the conditions c (that the cusp be on a plane), q (that the cuspidal tangent intersect a line) and μ (that the plane determined by the cuspidal cubic go through a point), as well as to the properly called *characteristic conditions* : ν (that the cuspidal cubic intersect a line), ρ (that the cuspidal cubic be tangent to a plane), and the second order condition P (that the cuspidal cubic go through a point). Thus, the characteristic numbers we are going to deal with express the amount of non-degenerate cuspidal plane cubics that intersect k given lines, go through i given points and touch $10-2i-k$ given planes. By transversality of general translates [3], these numbers coincide with the degree of the 0-cycles $P^i\nu^k\rho^{10-2i-k}$ of X_{cusp}, so they will be denoted simply by $P^i\nu^k\rho^{10-2i-k}$ in what follows.

We justify a *degeneration formula* which express the ρ condition in terms of the μ, ν conditions and the σ degeneration (whose points correspond with figures of a conic with a tangent line) and, consequently, we check the intersection numbers of the form $\mu^i\nu^k\rho^{10-i-k}$ given by Schubert. Then, the computation of the characteristic numbers $P^i\nu^k\rho^{10-2i-k}$ of the family of cuspidal plane cubics in $\mathbb{P}^3$ follows readily from the incidence formula $P = \nu\mu - 3\mu^2$.

Further, we construct – by means of a suitable blow-up – a new compactification $\mathcal{Z}_{cusp}$ where the condition z (that the line joining the cusp with the inflexion intersect a given line) is well defined. Then, we give the expressions of c, q, z and the conditions v (that the inflexion point lies on a given plane), w (that the inflexional tangent intersect a given line) and y (that the point where the cuspidal and the inflexional tangents intersect lies on a given plane) in terms of μ, ν and σ, and, from them, we get all the non-zero intersection numbers of the form $\mu^i\alpha\nu^k\rho^{9-i-k}$ for $\alpha = c, q, z, v, y, w$.

2 The variety X_{cusp} of cuspidal plane cubics

In the sequel, $\mathbb{P}^3$ will denote the projective space associated to a 4-dimensional vector space over an algebraically closed ground field $\mathbf{k}$ of characteristic 0, and the term *variety* will be used to mean a quasi-projective $\mathbf{k}$-variety. Moreover, we will also write ζ to indicate the degree of a 0-cycle ζ, if the underlying variety can be understood from the context.

Let $\mathbb{U}$ denote the rank 3 tautological bundle over the grassmannian variety Γ of planes of $\mathbb{P}^3$. Then, the projective bundle $\mathbb{P}(\mathbb{U})$ is the variety defined by $\mathbb{P}(\mathbb{U}) = \{(\pi, x_c) \in \Gamma \times \mathbb{P}^3 \mid x_c \in \pi\}$. Let $\mathbb{L}$ be the tautological line subbundle of the rank 3 bundle $\mathbb{U}|_{\mathbb{P}(\mathbb{U})}$ and let $\mathbb{Q}$ be the tautological quotient bundle. Thus, if we write $\mathbb{F}$ to denote the projective bundle $\mathbb{P}(\mathbb{Q}^*)$, it is clear that the elements of the *flag variety* $\mathbb{F}$ are triples (π, x_c, u_q) such that u_q is a line contained in the plane π that goes through the point x_c.

We define E_{cusp} as the subbundle of $S^3\mathbb{U}^*|_{\mathbb{F}}$ whose fiber over $(\pi, x_c, u_q) \in \mathbb{F}$ is the linear subspace of cubic forms defined over π that have multiplicity at least 2 at x_c, and for which u_q is a cuspidal tangent at x_c. Let us consider the natural inclusion map $i : \mathcal{O}_{\mathbb{F}}(-1) \to \mathbb{Q}^*|_{\mathbb{F}}$ and the product map $p : \mathbb{Q}^*|_{\mathbb{F}} \to S^3\mathbb{Q}^*|_{\mathbb{F}}$. There exist maps of vector bundles

$$h : \mathbb{U}^*|_{\mathbb{F}} \otimes \mathcal{O}_{\mathbb{F}}(-2) \to S^3\mathbb{U}^*|_{\mathbb{F}} \quad \text{and} \quad j : S^3\mathbb{Q}^*|_{\mathbb{F}} \to S^3\mathbb{U}^*|_{\mathbb{F}},$$

whose images are clearly contained in E_{cusp}. Proposition 2.1 of [6] ensures that the sequence

$$0 \longrightarrow \mathbb{Q}^*|_{\mathbb{F}} \otimes \mathcal{O}_{\mathbb{F}}(-2) \xrightarrow{\alpha} (\mathbb{U}^*|_{\mathbb{F}} \otimes \mathcal{O}_{\mathbb{F}}(-2)) \oplus S^3\mathbb{Q}^*|_{\mathbb{F}} \xrightarrow{\beta} E_{cusp} \longrightarrow 0, \qquad (1)$$

where $\alpha = \binom{i\otimes 1}{-p}$ and $\beta = h + j$, is an exact sequence of vector bundles over $\mathbb{F}$ and, therefore, it determines a resolution of the vector bundle E_{cusp} which allow us to compute all of its Chern (and Segre) classes.

Let X_{cusp} be the projective bundle $\mathbb{P}(E_{cusp})$ over $\mathbb{F}$. Then, X_{cusp} is the 10-dimensional subvariety of $\mathbb{P}(S^3\mathbb{U}^*|_{\mathbb{F}})$ whose points are pairs $(f, (\pi, x_c, u_q))$ such that f is a cuspidal cubic contained in the plane π, that has a cusp at x_c and u_q as the cuspidal tangent at x_c (see figure 1). Given a point $(\pi, x_c, u_q) \in \mathbb{F}$, and taking projective coordinates x_0, x_1, x_2, x_3 so that $\pi = \{x_3 = 0\}$, $x_c = [1, 0, 0, 0]$ and $u_q = \{x_1 = x_3 = 0\}$, we can express the cubic curves f of the fiber of X_{cusp} over (π, x_c, u_q) as follows:

$$a_0x_0x_1^2 + a_1x_1^3 + a_2x_1^2x_2 + a_3x_1x_2^2 + a_4x_2^3 = 0, \quad a_i \in \mathbf{k}, \quad i = 0, \dots, 4. \qquad (2)$$

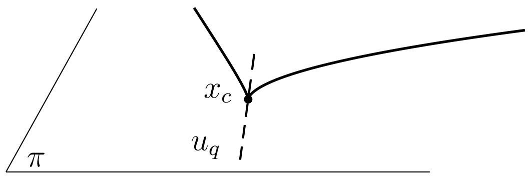

Figure 1: A closed point of X_{cusp} .

We also denote by μ and c the pullbacks to $\mathsf{Pic}(X_{cusp})$ of the hyperplane classes $\mu = c_1(\mathcal{O}_\Gamma(1))$ and $c = c_1(\mathcal{O}_{\mathbb{P}(\mathbb{U})}(1))$, respectively, under the natural projections, so that μ is the class of the hypersurface of X_{cusp} such that π goes through a given point and c coincides with the class of the hypersurface of X_{cusp} such that x_c is on a given plane. The pullback to $\mathsf{Pic}(X_{cusp})$ of the hyperplane class $c_1(\mathcal{O}_{\mathbb{F}}(1))$ coincides with $q - \mu$, where q is the pullback to $\mathsf{Pic}(X_{cusp})$ of the class of the hypersurface of $\mathbb{F}$ such that u_q intersects a given line. In addition, let us denote by ν the class of the hypersurface of X_{cusp} consisting of the pairs $(f, (\pi, x_c, u_q))$ such that f intersect a given line. Then the relation $c_1(\mathcal{O}_{X_{cusp}}(1)) = \nu - 3\mu$ holds in $\mathsf{Pic}(X_{cusp})$ (see [6]). Thus, using the projection formula, we have

$$\int_{X_{cusp}} \mu^i c^j q^h \nu^k = \int_{\mathbb{F}} \mu^i c^j q^h s_{6-i-j-h}(E_{cusp} \otimes \mathcal{O}_\Gamma(-3)), \qquad (3)$$

where $s_t(E_{cusp} \otimes \mathcal{O}_\Gamma(-3))$ denotes the $t-$th Segre class of the vector bundle $E_{cusp} \otimes \mathcal{O}_\Gamma(-3)$, which can be calculated from the resolution (1). This allow us to compute all the intersection numbers of X_{cusp} in the conditions μ, c, q and ν. In particular, we have :

$$\mu^3\nu^7 = 24, \quad \mu^2\nu^8 = 384, \quad \mu\nu^9 = 3216, \quad \nu^{10} = 17760. \tag{4}$$

Let U_{cusp} be the subvariety of X_{cusp} whose points are pairs $(f,(\pi,x_c,u_q))$ such that f is an irreducible cuspidal cubic. In fact, we have that X_{cusp} is a compactification of U_{cusp}, whose boundary $X_{cusp} - U_{cusp}$ consists of two codimension one irreducible components : X_{conic} and X_{trip} (see [6] for a proof), where

- X_{conic} parameterizes pairs $(f,(\pi,x_c,u_q))$ such that f is a cubic consisting of the line u_q and a conic tangent to u_q at x_c (see figure 2). Actually, X_{conic} is a projective subbundle of X_{cusp}, so we can compute the intersection numbers of X_{conic} in the conditions μ, c, q and ν in a very similar way as the one we have used for X_{cusp};

- $X_{trip} = \mathbb{P}(S^3\mathbb{Q}^*|_{\mathbb{F}})$ parameterizes pairs $(f,(\pi,x_c,u_q))$ such that f is a cubic consisting of three lines concurrent at x_c (see figure 2). Notice that X_{trip} is a projective subbundle of X_{cusp} and, in particular, X_{trip} is an irreducible closed set of dimension 9.

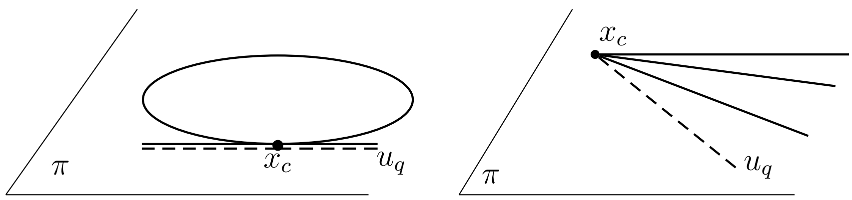

Figure 2: A closed point of X_{conic} and a closed point of X_{trip} .

On the other hand, if we denote the classes in $\mathsf{Pic}(X_{cusp})$ of the X_{conic} and X_{trip} degenerations by σ and τ respectively, we have the next result :

2.1 Proposition. *The group* $\mathsf{Pic}(X_{cusp})$ *is a rank 4 free group generated by* μ, c, q *and* ν. *Furthermore, we have the following relations :*

$$3c = -9\mu + 5\nu - 2\sigma - 3\tau \quad \textit{and} \quad 3q = -6\mu + 4\nu - \sigma - 3\tau.$$

Proof. See theorem 3.1 and proposition 6.1 of [6]. ■

3 Characteristic numbers of X_{cusp}

We denote by ρ the class of the hypersurface of X_{cusp} whose points $(f,(\pi,x_c,u_q))$ satisfy that f is tangent to a given plane. In this section we will consider the tangential structure of the figures of X_{cusp} in order to introduce this condition and compute the characteristic numbers.

Notice that the dual curve f^* of an irreducible cuspidal cubic f is also a cuspidal cubic. Furthermore, the map $f \mapsto f^*$ is a birational map whose indeterminacy locus is the 2-codimensional closed set of X_{cusp} consisting of points $(f, (\pi, x_c, u_q))$ such that f degenerates and contains a double line. If a degenerate cuspidal cubic f is such that $(f, (\pi, x_c, u_q))$ is on the open orbit of X_{conic}, then $f = f' \cdot u_q$ where f' is a non degenerate conic tangent to u_q at x_c, and its dual consists of the dual conic f'^* and the pencil of lines over π with focus x_c. Thus, we may consider the closure $\overline{X}_{conic}$ of the graph of the rational map $X_{conic} \to \mathbb{P}(S^2\mathbb{U}|_{\mathbb{F}})$ that assigns the conic of tangents f'^* to a given cubic $f' \cdot u_q$ where the conic f' has rank ≥ 2. The variety $\overline{X}_{conic}$ is a compactification of X_{conic} where the dual conic is always well defined and, consequently, since we can express ρ in terms of the generators of $\mathsf{Pic}(\overline{X}_{conic})$, we can compute all the non-zero intersection numbers in the conditions μ, ν and ρ. Here we have listed all those numbers:

3.1 Proposition. *The following results hold in* $A^*(X_{cusp})$*:*

$\mu^3\sigma$:	42	87	141	168	141	87	42			
$\mu^2\sigma$:	588	1086	1584	1767	1518	1053	606	294		
$\mu\sigma$:	4296	7068	9222	9393	7626	5136	3003	1587	768	
σ :	20040	28344	31356	26994	18816	11190	6054	3051	1464	696

where the numbers listed to the right of a given $\mu^i\sigma$ *correspond to the intersection numbers* $\mu^i\nu^k\rho^{9-i-k}\sigma$*, for* $k = 9-i, \ldots, 0$.

These numbers coincide with the ones given in the *Tabelle von Zahlen* σ in page 117 of [10]. Now, we are ready to express the condition $\rho \in \mathsf{Pic}(X_{cusp})$ in terms of μ, ν and the degeneration σ; generalizing, in this way, the degeneration formula for ρ in $\mathbb{P}^2$ given by Zeuthen [11] and proved in Kleiman-Speiser [4].

3.2 Proposition. *The following relation holds modulo* $\mathbb{Z}[\tau]$ *in* $\mathsf{Pic}(X_{cusp})$*:*

$$\rho + \nu = 3\mu + 2\sigma.$$

Proof. The degeneration formula of Zeuthen says that $\rho + \nu = 2\sigma$ in $\mathbb{P}^2$ (see [4]). Moreover, we know from proposition 2.1, that there exist a rational number s_0 such that $\rho = s_0\mu - \nu + 2\sigma$ holds modulo $\mathbb{Z}[\tau]$ in $\mathsf{Pic}(X_{cusp})$. From (3) and from the intersection numbers of X_{conic} in the conditions μ, c, q and ν, we have $3\mu^2c\nu^6\rho = 3\mu^2c\nu^6(s_0\mu - \nu + 2\sigma) = 36s_0 + 1500$. From the first formula of proposition 2.1 and from the numbers of proposition 3.1 we get $3\mu^2c\nu^6\rho = \mu^2\nu^6\rho(-9\mu + 5\nu - 2\sigma - 3\tau) = 5\mu^2\nu^7\rho - 2712$. Then, $36s_0 = 5\mu^2\nu^7\rho - 4212$, and by proposition 3.1, $\mu^2\nu^7\rho = \mu^2\nu^7(s_0\mu - \nu + 2\sigma) = 24s_0 + 792$, so $s_0 = 3$. ∎

This proposition implies that the intersection numbers $\mu^i\nu^k\rho^{10-i-k}$ in X_{cusp} can be obtained as $\mu^i\nu^k\rho^{10-i-k} = \mu^i\nu^k\rho^{9-i-k}(3\mu - \nu + 2\sigma)$, because the unique degenerations of the 1-dimensional systems $\mu^i\nu^k\rho^{9-i-k}$ are the ones consisting of a degenerated conic with a tangent line. Thus, from (4) and proposition 3.1, we are now able to compute all the non-zero intersection numbers of the form $\mu^i\nu^k\rho^{10-i-k}$ in X_{cusp}.

3.3 Proposition. *In $A^*(X_{cusp})$ we have:*

μ^3 :	24	60	114	168	168	114	60	24			
μ^2 :	384	864	1488	2022	2016	1524	924	468	192		
μ :	3216	6528	10200	12708	12144	9156	5688	3090	1488	624	
$*$:	17760	31968	44304	49008	43104	30960	18888	10284	5088	2304	960

where the numbers listed to the right of a given μ^i ($*$ for μ^0) correspond to the intersection numbers $\mu^i\nu^k\rho^{10-i-k}$, for $k = 10-i,\dots,0$.

3.4 Proposition. *If, given a point $p \in \mathbb{P}^3$, we denote by P the class of the subvariety of X_{cusp} consisting of pairs $(f,(\pi,x_c,u_q))$ such that $p \in f$, the relation $P = \nu\mu - 3\mu^2$ holds in $A^2(X_{cusp})$.*

Proof. Let Y be the variety defined by the pairs $((f,(\pi,x_c,u_q)),p)$ in $X_{cusp} \times \mathbb{P}^3$ such that $p \in f$. We have projections $Y \xrightarrow{\alpha} X_{cusp}$ and $Y \xrightarrow{\beta} \mathbb{P}(\mathbb{U})$ where $\alpha\left((f,(\pi,x_c,u_q)),p\right) = (f,(\pi,x_c,u_q))$ and $\beta\left((f,(\pi,x_c,u_q)),p\right) = (\pi,p)$. Thus, we get the expected incidence formula by simply applying $\alpha_*\beta^*$ to the relation $\mu^3 - \mu^2 a + \mu a^2 - a^3 = 0$, which holds in $A^*(\mathbb{P}(\mathbb{U}))$. ∎

Finally, from this last formula and from the table of proposition 3.3, we get the characteristic numbers of cuspidal plane cubics in $\mathbb{P}^3$, which ratify the ones given in the *Dritte Tabelle* on page 143 of [10].

3.1 Theorem. *The following results hold in $A^*(X_{cusp})$:*

P^2 :	240	504	804	1014	1008	840	564				
P :	2064	3936	5736	6642	6096	4584	2916	1686	912		
$*$:	17760	31968	44304	49008	43104	30960	18888	10284	5088	2304	960

where the numbers listed to the right of a given P^i ($*$ for P^0) correspond to the characteristic numbers $P^i\nu^k\rho^{10-2i-k}$, for $k = 10-2i,\dots,0$.

4 More fundamental conditions

If we consider a cuspidal cubic of (2), the analytical expressions of the inflexion point x_v, the inflexional tangent u_w, the point x_y where the cuspidal and the inflexional tangent intersect and the line u_z joining the cusp with the inflexion are as follows:

$$\begin{aligned}
x_v &= [27a_1a_4^2 - 9a_2a_3a_4 + 2a_3^3,\ -27a_0a_4^2,\ 9a_0a_3a_4,\ 0],\\
u_w &= \{27a_0a_4^2x_0 + (27a_1a_4^2 - a_3^3)x_1 + (27a_2a_4^2 - 9a_3^2a_4)x_2 = 0, x_3 = 0\},\\
x_y &= [3a_2a_4 - a_3^2,\ 0,\ -3a_0a_4,\ 0],\\
u_z &= \{a_3x_1 + 3a_4x_2 = 0, x_3 = 0\}.
\end{aligned}$$

Notice that there exist cuspidal cubics such that some of these distinguished elements is not well defined. In this section we will construct a new compactification $\mathcal{Z}_{cusp}$ as the blow-up of X_{cusp} along the projective subbundle $\mathbb{P}(\mathbb{U}^*|_{\mathbb{F}} \otimes \mathcal{O}_{\mathbb{F}}(-2))$ (consisting of those cubics with $a_3 = a_4 = 0$), in such a way that it parameterizes the family of cuspidal plane cubics that have the cusp in x_c, for which u_q is a cuspidal tangent at x_c, and where the line u_z is always well defined.

First, we will describe generically the blow-up of a projective bundle $\mathbb{P}(E)$ over a variety X along a projective subbundle $\mathbb{P}(F)$. Let G be the quotient vector bundle E/F and p the canonical projection $p : E \to G$.

4.1 Proposition. *If* $\tilde{p} : \mathbb{P}(E) \to \mathbb{P}(G)$ *is the rational map induced by p, the closure of its graph* $\Gamma_{\tilde{p}} \subseteq \mathbb{P}(E) \times_X \mathbb{P}(G)$ *coincides with* $\mathbb{P}(H)$*, where* H *is the subbundle of* $E|_{\mathbb{P}(G)}$ *whose fiber* $H_{[w]}$ *over* $\mathbb{P}(G)_x$*,* $x \in X$*, consists of those* $v \in E_x$ *such that* $p(v) \in \langle w \rangle$*.*

Proof. Notice that $\mathbb{P}(H)$ is the subvariety of $\mathbb{P}(E) \times_X \mathbb{P}(G)$ consisting of the points $([v], [w])$ such that $p(v) \in \langle w \rangle$. Hence $\mathbb{P}(H)$ contains the graph of $\tilde{p}$. In addition, notice that $\mathbb{P}(H)$ is an irreducible closed set of $\mathbb{P}(E) \times_X \mathbb{P}(G)$ so $\Gamma_{\tilde{p}} \subseteq \mathbb{P}(H)$. On the other hand, $\mathsf{rank}(H) = \mathsf{rank}(F) + 1$. Thus, it turns out that $\mathsf{dim}\mathbb{P}(H) = \mathsf{dim}\mathbb{P}(G) + \mathsf{rank}(F) = \mathsf{dim}\mathbb{P}(E) = \mathsf{dim}(\Gamma_{\tilde{p}})$, and now it is clear that $\Gamma_{\tilde{p}} = \mathbb{P}(H)$.

■

Furthermore, the vector bundle H is just the pullback of $\mathcal{O}_{\mathbb{P}(G)}(-1)$ under the map $E|_{\mathbb{P}(G)} \to G|_{\mathbb{P}(G)}$ induced by the projection $p : E \to G$ whose kernel coincides with $F|_{\mathbb{P}(G)}$. Thus,

$$0 \longrightarrow F|_{\mathbb{P}(G)} \xrightarrow{\iota} H \xrightarrow{\bar{p}} \mathcal{O}_{\mathbb{P}(G)}(-1) \longrightarrow 0,$$

where ι is the inclusion of $F|_{\mathbb{P}(G)}$ in H and $\bar{p}$ derives from p, is an exact sequence of vector bundles. In fact, $\mathbb{P}(F|_{\mathbb{P}(G)})$ is a divisor of $\mathbb{P}(H)$, and as we can assume that X is an affine variety and that the bundles F and E have trivializations such that $\Gamma_{\tilde{p}} = \mathbb{P}(H)$ locally, the following result holds :

4.1 Theorem. *The natural projection* $\mathbb{P}(H) \to \mathbb{P}(E)$ *is isomorphic to the blow-up of* $\mathbb{P}(E)$ *along* $\mathbb{P}(F)$*, and* $\mathbb{P}(F|_{\mathbb{P}(G)})$ *is the exceptional divisor.*

In our case, $E = E_{cusp}$, $F = \mathbb{U}^*_{\mathbb{F}} \otimes \mathcal{O}_{\mathbb{F}}(-2)$ and, from the resolution (1), $G = E/F = S^3\mathbb{Q}^*/(\mathbb{Q}^* \otimes \mathcal{O}_{\mathbb{F}}(-2))$. If we denote by $\mathcal{Z}_{cusp}$ the blow-up of X_{cusp} along the projective subbundle $\mathbb{P}(\mathbb{U}^*|_{\mathbb{F}} \otimes \mathcal{O}_{\mathbb{F}}(-2))$ and by H_{cusp} the corresponding H, we get by the former theorem, setting $\mathbb{G} = \mathbb{P}(G)$, that $\mathcal{Z}_{cusp} = \mathbb{P}(H_{cusp})$ and that the exceptional divisor coincide with $\mathbb{P}(\mathbb{U}^*|_{\mathbb{G}} \otimes \mathcal{O}_{\mathbb{F}}(-2)|_{\mathbb{G}})$ (see figure 3). In fact, the sequence of vector bundles over $\mathbb{G}$

$$0 \to \mathbb{U}^*|_{\mathbb{G}} \otimes \mathcal{O}_{\mathbb{F}}(-2)|_{\mathbb{G}} \to H_{cusp} \to \mathcal{O}_{\mathbb{G}}(-1) \to 0, \tag{5}$$

is exact and allow us to compute all the Chern (and Segre) classes of H_{cusp}.

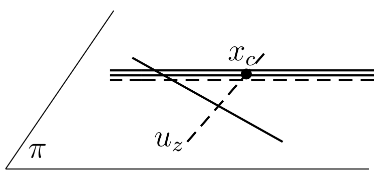

Figure 3: A closed point of the exceptional divisor .

Moreover, we have that the hyperplane class $c_1(\mathcal{O}_{\mathbb{G}}(1))$ coincides with $z+2c-2q-\mu$, where the pullback of z to $\mathsf{Pic}(\mathcal{Z}_{cusp})$ corresponds to the condition of the line u_z intersecting a given line. Then, from the projection formula we have

$$\int_{\mathcal{Z}_{cusp}} \mu^i c^j q^h z^k \nu^t = \int_{\mathbb{G}} \mu^i c^j q^h z^k s_{7-i-j-h-k}(H_{cusp} \otimes \mathcal{O}_\Gamma(-3)), \tag{6}$$

where $s_t(H_{cusp} \otimes \mathcal{O}_\Gamma(-3))$ can be calculated from the resolution (5). Therefore, we can compute all the non-zero intersection numbers of $\mathcal{Z}_{cusp}$ in the conditions μ, c, q, z and ν. For instance, we get the following intersection numbers

$$\mu^2 cz\nu^6 = 284, \quad \mu cqz^2\nu^5 = 192, \quad \mu q^2 z^3 \nu^4 = 34, \quad z^4\nu^6 = 432,$$

which are not listed in [10].

4.2 Proposition. *The group* $\mathsf{Pic}(\mathcal{Z}_{cusp})$ *is a rank 5 free group generated by* μ, c, q, z *and* ν. *Furthermore, we have the following relations:*

$$\begin{aligned} \sigma &= -\mu - c + q + z, \\ \tau &= -\mu + \nu + c - 2q, \\ \eta &= -2\mu + \nu - 2c + 2q - z, \end{aligned}$$

where η *is the class of the exceptional divisor* $\mathbb{P}(\mathbb{U}^*|_{\mathbb{G}} \otimes \mathcal{O}_{\mathbb{F}}(-2)|_{\mathbb{G}})$ *in* $\mathsf{Pic}(\mathcal{Z}_{cusp})$.

Proof. Since σ is the class of $\mathbb{P}(H_{cusp}|_{\mathbb{G}'})$, where $\mathbb{G}'$ is the projective subbundle of $\mathbb{G}$ given by

$$\mathbb{G}' = \mathbb{P}\left(\frac{S^2\mathbb{Q}^* \otimes \mathcal{O}_{\mathbb{F}}(-1)}{\mathbb{Q}^* \otimes \mathcal{O}_{\mathbb{F}}(-2)}\right),$$

the expression of σ coincides with the pullback of the expression of $[\mathbb{G}']$ in $\mathsf{Pic}(\mathbb{G})$. Then, using the formula given in [2], we get

$$\begin{aligned} \sigma &= -\mu + 2c - 2q + z + c_1\left(S^3\mathbb{Q}^*/(S^2\mathbb{Q}^* \otimes \mathcal{O}_{\mathbb{F}}(-1))\right) \\ &= -\mu + 2c - 2q + z + 3(q-c). \end{aligned}$$

Concerning the second relation, we have from [2] that

$$\tau = \nu - 3\mu + c_1\left((\mathbb{U}^*/\mathbb{Q}^*) \otimes \mathcal{O}_{\mathbb{F}}(-2)\right) = -\mu + \nu + c - 2q.$$

Finally, once again by [2], it turns out

$$\begin{aligned} \eta &= \nu - 3\mu + c_1\left(H_{cusp}/(\mathbb{U}^* \otimes \mathcal{O}_{\mathbb{F}}(-2))\right) \\ &= \nu - 3\mu + c_1(\mathcal{O}_{\mathbb{G}}(-1)) = \nu - 2\mu - 2c + 2q - z. \end{aligned}$$

■

Let v, w and y be the classes of $\mathsf{Pic}(\mathcal{Z}_{cusp})$ corresponding, respectively, to the conditions of the inflexion point x_v being on a given plane, the inflexional tangent u_w intersecting a given line and the point x_y lying on a given plane.

4.1 Corollary. *In* $\mathsf{Pic}(\mathcal{Z}_{cusp})$ *the following relations hold modulo* $\mathbb{Z}[\tau] \oplus \mathbb{Z}[\eta]$:

$$\begin{aligned} &3c = -9\mu + 5\nu - 2\sigma, && 3q = -6\mu + 4\nu - \sigma, && 3z = \nu + 2\sigma, \\ &3v = 9\mu - 4\nu + 7\sigma, && 3w = 12\mu - 5\nu + 8\sigma, && 3y = 6\mu - \nu + 4\sigma. \end{aligned}$$

Proof. The first three equalities come from the relations of proposition 4.2. Regarding the other three relations, we will only show the first one, since the proofs of the others are very similar. In order to express v in terms of μ, ν and σ, notice that if we dualize the restriction to μ^3 of $3q = -6\mu + 4\nu - \sigma$, we get $3v\mu^3 = 4\rho\mu^3 - \sigma\mu^3$. Now, taking into account that $\rho = -\nu + 3\mu + 2\sigma$ (from proposition 3.2) we have $3v = s\mu - 4\nu + 7\sigma$, $s \in \mathbb{Z}$. To determine s, notice that the relation $c + v - z = 0$ holds in $A^1(U_{cusp})$ and, using again the relations given in 4.2, we get $s = 9$. ∎

From the relations of corollary 4.1 and taking account of the numbers calculated in propositions 3.1 and 3.3, we have:

$\mu^3 c =$	12	42	96	168	186	132	72				$= \mu^3 w$
$\mu^3 q =$	18	51	105	168	177	123	66				$= \mu^3 v$
$\mu^3 z =$	36	78	132	168	150	96	48				$= \mu^3 y$
$\mu^2 c =$	176	536	1082	1688	1844	1496	956	512			
$\mu^2 q =$	268	670	1228	1771	1846	1453	910	478			
$^*\mu^2 z =$	520	1012	1552	1852	1684	1210	712	352			
$\mu^2 v =$	932	1562	2054	1931	1358	767	362	134			
$\mu^2 w =$	1024	1696	2200	2014	1360	724	316	100			
$^*\mu^2 y =$	704	1280	1844	2018	1688	1124	620	284			
$\mu c =$	1344	3576	6388	8852	9108	7264	4706	2688	1392		
$\mu q =$	2088	4620	7550	9769	9618	7448	4735	2655	1344		
$^*\mu z =$	3936	6888	9548	10498	9132	6476	3898	2088	1008		
$\mu v =$	6888	10380	12382	11039	7650	4348	2195	987	384		
$\mu w =$	7632	11424	13544	11956	8160	4532	2224	954	336		
$^*\mu y =$	5424	8976	11872	12332	10152	6844	3956	2022	912		
$c =$	6592	14800	22336	25560	22864	16672	10380	5836	3040	1504	
$q =$	10568	20120	28220	30930	26912	19238	11790	6515	3320	1592	
$^*z =$	19280	29552	35672	34332	26912	17780	10332	5462	2672	1232	
$v =$	32728	43096	44692	35766	22864	12298	6006	2677	1096	424	
$w =$	36704	48416	50576	41136	26912	14864	7416	3356	1376	512	
$^*y =$	27232	40192	47440	45072	35008	22912	13152	6820	3232	1408	

where the numbers listed to the right (left) of a given $\mu^i\alpha$ — α is one of the conditions c, v, y, w, q or z — correspond with the intersection numbers $\mu^i\alpha\nu^k\rho^{9-i-k}$ $(\mu^i\alpha\nu^{9-i-k}\rho^k)$, for $k = 9-i, \ldots, 0$. It is worth noticing that the numbers in the last table coincide, as are supposed to, with the ones expressed in the *Erste Tabelle* and *Dritte Tabelle* in pages 136 and 143 of [10]. On the other hand, those rows not listed in [10] are marked by *.

References

[1] P. Aluffi. The enumerative geometry of plane cubics II. Nodal and cuspidal cubics. *Math. Ann.*, 289:543–572, 1991.

[2] S. Ilori, A. Ingleton, and A. Lascu. On a formula of D. B. Scott. *Journal London Math. Soc.*, 8:539–544, 1974.

[3] S. Kleiman. Transversality of a general translate. *Compositio Mathematica*, 28:287–297, 1974.

[4] S. Kleiman and R. Speiser. Enumerative geometry of cuspidal plane cubics. In *Vancouver Conference in Algebraic Geometry*, CMS-AMS Conf., pages 227–268, 1984.

[5] S. Maillard. *Recherche des charactéristiques des systèmes élémentaires de courbes planes du troisième ordre*. PhD thesis, Paris, 1871. Publ. by Cusset.

[6] J. Miret. The Picard group of the variety of plane cuspidal cubics of $\mathbb{P}^3$. *Rendiconti del Circolo Matematico di Palermo*, II, Tomo XLIX:61–74, 2000.

[7] J. Miret and S. Xambó. Geometry of complete cuspidal plane curves. Number 1389 in LNM, pages 195–234. Springer, 1989.

[8] J. Miret and S. Xambó. On Schubert's degenerations of cuspidal plane cubics. Number 1436 in LNM, pages 189–214. Springer, 1990.

[9] G. Sacchiero. Numeri caratteristici delle cubiche piane cuspidale. Preprint Univ. di Roma II, 1984.

[10] H. Schubert. *Kalkül der abzählenden Geometrie*. Teubner, 1879. Rep. in 1979 by Springer-Verlag.

[11] H. Zeuthen. Détermination des charactéristiques des systèmes élémentaires des cubiques, I: cubiques douées d'un point cuspidal. *CR. Acad. Sc. Paris*, 74:521–526, 1872.

Departament de Matemàtica,
Universitat de Lleida;
Jaume II, 69 E-25001, Lleida
e-mail: {xavi,miret}@eup.udl.es