

Analytic Toeplitz operators on the Hardy space H^p : a survey

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Abstract

Toeplitz operators on Hardy spaces H^p have been studied extensively during the past 40 years or so. An important special case is that of the operators of multiplication by a bounded analytic function φ : $M_\varphi(f) = \varphi f$ (analytic Toeplitz operators). However, many results about them are either only formulated in the case $p = 2$, or are not so easy to find in an explicit form.

The purpose of this paper is to give a complete overview of the spectral theory of these analytic Toeplitz operators on a general space H^p , $1 \leq p < \infty$. The treatment is kept as elementary as possible, placing a special emphasis on the key role played by certain extremal functions related to the Poisson kernel.

Introduction

Denote by \mathbb{T} the unit circle and by \mathbb{D} the unit disk in the complex plane \mathbb{C} . For $1 \leq p < \infty$, the classical Hardy space H^p is defined as the set of all functions f analytic in \mathbb{D} such that the (increasing) integral means

$$M_p(r; f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}$$

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are bounded for $0 < r < 1$. It is a Banach space with respect to the norm

$$\|f\|_p = \lim_{r \rightarrow 1^-} M_p(r; f).$$

Let H^∞ denote the Banach space of all bounded analytic functions in \mathbb{D} equipped with the usual supremum norm. By a generalization of a theorem of Fatou, whenever $f \in H^p$ and $1 \leq p \leq \infty$, the radial limits $\tilde{f}(\theta) = \lim_{r \rightarrow 1^-} f(re^{i\theta})$ exist for almost every θ in $[0, 2\pi)$. Moreover, the space H^p can be identified with the closed subspace of $L^p(\mathbb{T}, d\theta/(2\pi))$ consisting of the functions with vanishing Fourier coefficients with negative indices, and the H^p norm can also be computed by integrating the boundary values:

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |\tilde{f}(\theta)|^p d\theta \right)^{1/p}. \quad (1)$$

We refer the reader to the standard texts, *e.g.*, [H], [Du], [K], or [G].

A function φ defined on \mathbb{D} is said to be a *pointwise multiplier* of H^p if $\varphi f \in H^p$ for all f in H^p . By the Closed Graph Theorem, we see that this happens if and only if the *multiplication operator* $M_\varphi : H^p \rightarrow H^p$, given by $M_\varphi(f) = \varphi f$, is bounded. The function φ is called the *symbol* of M_φ . The multiplication operators form a special subclass within the larger family of Toeplitz operators.

The *Toeplitz operator* with *symbol* φ , denoted T_φ , is defined for any φ in $L^\infty(\mathbb{T})$ as the multiplication by φ followed by the analytic Szegő projection from $L^p(\mathbb{T})$ onto its closed subspace H^p ; that is, $T_\varphi(f) = P(\varphi f)$. This makes sense even for $p \neq 2$, when there is no orthogonality. Thus, the operators of multiplication by H^∞ functions are precisely the Toeplitz operators with analytic symbols, and are often also called *analytic Toeplitz operators*. The theory of Toeplitz operators with various types of general $L^\infty(\mathbb{T})$ symbols was developed over a long period by a number of important analysts. It would take us way beyond the scope of this article to give an overview of the general theory, or even to list the names of important contributors and their achievements. We only mention a few references in book form (or collections of articles), in the order of appearance: Douglas [Do], Nikol'skiĭ [N1], [N2], Böttcher and Silbermann [BS], Zhu [Z], and Böttcher and Karlovich [BK].

In spite of the vast literature on this important subject, most of the papers are usually only concerned with the case $p = 2$ and rely on the Hilbert space techniques (which do not work for other values of p). Sometimes the study of Toeplitz operators requires a restriction to a certain important class of symbols, which automatically excludes the most general analytic multipliers. Consequently, it does not seem easy to find in the existing literature an elementary exposition on multiplication operators on general H^p spaces. The purpose of this survey is precisely to fill this gap in the (otherwise extremely rich) literature on the Toeplitz operators. We try to present as complete an account of results for $1 \leq p < \infty$ as can fit in a short space, in a unified and systematic way. Though often simple, our results are always conclusive and complete. The exposition includes a characterization of the analytic Toeplitz operators and the computation of their norms (Section 1), essential norms

and approximation numbers (Section 2), as well as a detailed classification of spectral points and a discussion of the Fredholm property and Fredholm index (Section 3).

Although almost all the results presented here are likely to be known at least when $p = 2$ (either they can be found in the texts cited, or are considered “folk knowledge” among the experts), some of them still might be stated here for the first time explicitly in the context of general H^p spaces. The methods employed resemble those used in our recent article [V] on the Bergman space multipliers. We emphasize the role played by the *Poisson integrals* and by the closely related *extremal functions for point evaluations*, which take part in most proofs.

There are two relevant recent papers: [BDL], where the multipliers between weighted H_w^∞ spaces are studied, and [SZ], where a characterization of the multipliers from H^p into H^q when $q \leq p$ is contained as a special case, and the commutants of multiplication operators on H^2 are described when the symbols are inner functions. Analytic Toeplitz operators on the closely related Bergman spaces have also received a lot of attention (*cf.* [At], [Ax], [L], [V]); occasionally it will be useful to compare the results in the two situations.

1 Boundedness, norms, and general facts

We begin by characterizing the bounded analytic Toeplitz operators on H^p in terms of their symbols and computing their norms.

The following sharp estimate is well-known and has various proofs. A typical one relies on the Hardy space factorization methods, reducing the problem to the easy case $p = 2$ (see p. 285 of [N1], Appendix 2). Another proof can be given by using the isometries of H^p generated by the involutive conformal disk automorphisms and reducing the problem to the estimate for the origin. Yet another proof is possible by a typical variational method for the problem of maximizing $|f(\zeta)|$ among all functions f in H^p of norm one and the usual guess-and-check strategy. The proof is left as an exercise.

Lemma 1. *If $1 \leq p < \infty$, $f \in H^p$, and $\zeta \in \mathbb{D}$, then*

$$(1 - |\zeta|^2)^{1/p} |f(\zeta)| \leq \|f\|_p.$$

For arbitrary ζ in \mathbb{D} , the equality holds for

$$f_\zeta(z) = \frac{(1 - |\zeta|^2)^{1/p}}{(1 - \bar{\zeta}z)^{2/p}},$$

(with a suitable analytic branch chosen in the denominator) and $\|f_\zeta\|_p = 1$.

The sharpness of the above estimate will be crucial in some of the proofs that follow.

Proposition 2. *Let $1 \leq p < \infty$. Then the following statements are equivalent:*

(a) $\varphi H^p \subset H^p$.

(b) φ is analytic in \mathbb{D} and the multiplication operator $M_\varphi : H^p \rightarrow H^p$, given by $M_\varphi(f) = \varphi f$, is bounded.

(c) $\varphi \in H^\infty$.

If any (and therefore each) of the above conditions is fulfilled, then the norm of the operator $M_\varphi : H^p \rightarrow H^p$ is $\|M_\varphi\| = \|\varphi\|_\infty$.

Proof. (c) \Rightarrow (a) is trivial.

To prove (a) \Rightarrow (b), observe first because of $\varphi = 1 \cdot \varphi$ we have $\varphi \in H^p$, hence φ is analytic in \mathbb{D} . Next, note that norm convergence in H^p implies the uniform convergence on compact sets in \mathbb{D} (for example, by Lemma 1). Now a standard application of the Closed Graph Theorem shows that M_φ is bounded.

(b) \Rightarrow (c). Suppose M_φ is a bounded operator on H^p . Then for each fixed ζ in \mathbb{D} and for every f in H^p we have

$$(1 - |\zeta|^2)^{1/p} |\varphi(\zeta) f(\zeta)| \leq \|\varphi f\|_p \leq \|M_\varphi\| \cdot \|f\|_p.$$

By Lemma 1, we can choose $f = f_\zeta$ so that $(1 - |\zeta|^2)^{1/p} |f_\zeta(\zeta)| = \|f_\zeta\|_p = 1$. The inequality $|\varphi(\zeta)| \leq \|M_\varphi\|$ then follows for arbitrary ζ in \mathbb{D} .

The norm formula follows from the inequality $\|\varphi\|_\infty \leq \|M_\varphi\|$ obtained above and from the obvious estimate $\|\varphi f\|_p \leq \|\varphi\|_\infty \|f\|_p$. ■

Remarks. (1) Another proof can be given, similar to the one of Proposition 1.7 in [At] for Bergman spaces: show that $\|\varphi^n\|_p \leq \|M_\varphi\|^n$ for all n and then let $n \rightarrow \infty$.

(2) The inequality $\|M_\varphi\| \geq \|\varphi\|_\infty$ holds in more general spaces of analytic functions. See Lemma 11 of [DRS] or Proposition 3 of [BSh]. We included the specific details above, as they will be useful later.

(3) It can be observed from the above proof that, given φ in H^∞ , there exists a sequence $(\zeta_n)_{n=1}^\infty$ in \mathbb{D} such that $\lim_{n \rightarrow \infty} \|\varphi f_{\zeta_n}\|_p = \|\varphi\|_\infty$. Moreover, by invoking the maximum modulus principle, one gets $|\zeta_n| \rightarrow 1$. This is used in the forthcoming Theorem 5.

A sequence $(a_n)_{n=1}^\infty$ in \mathbb{D} is the zero set of some H^p function if and only if it satisfies the *Blaschke condition*: $\sum_{n=1}^\infty (1 - |a_n|) < \infty$. For such a sequence, the corresponding Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$$

(when $a_n = 0$, the term z is to be used instead of the corresponding fraction) defines an H^∞ function whose boundary values have modulus one; this function vanishes only at the points a_n , taking the multiplicities into account. Thus, by (1), the Blaschke product B which corresponds to the zero set of some f in H^p is an isometric zero-divisor: f/B is a zero-free function in H^p such that $\|f/B\|_p = \|f\|_p$.

Corollary 3. Denote by \mathcal{M}_p the algebra of all bounded analytic Toeplitz operators on H^p . The following statements are true.

(a) The mapping $\mathcal{I}_p : H^\infty \rightarrow \mathcal{M}_p$ given by $\mathcal{I}_p(\varphi) = M_\varphi$ is an algebra isomorphism and an isometry.

(b) The unit ball of the Banach algebra \mathcal{M}_p is the closure of the convex hull of the set of all multipliers M_B whose symbols are Blaschke products.

Proof. (a) It is clear that $\mathcal{I}_p : H^\infty \rightarrow \mathcal{M}_p$ is an algebra isomorphism; the equality of the norms follows from Proposition 2.

(b) By a theorem of D. Marshall (*cf.* [K], Chapter VII, Section B, or [G], Chapter 5, Corollary 2.6), the unit ball of H^∞ is the closed convex hull of Blaschke products. ■

The most elementary multiplication operator on H^p is the *shift operator* M_z , that is, the operator of multiplication by z . We now show that the property of commuting with this operator characterizes the pointwise multiplication operators among all bounded operators on H^p . This resembles Theorem 2 from the classical paper [BH] by Brown and Halmos on the Laurent (multiplication) operators on $L^2(\mathbb{T})$.

Theorem 4. A bounded operator T acting on H^p ($1 \leq p < \infty$) is a pointwise multiplication operator if and only if it commutes with the shift operator M_z .

Proof. Any two multiplication operators commute, so the necessity is clear. For the sufficiency, suppose that T is bounded on H^p and commutes with the operator of multiplication by z . We will show that T coincides with the operator M_φ of multiplication by the analytic function $\varphi = T(1)$. Observe that $T(z^n) = z^n \cdot T1$ by assumption, hence $T(p) = p \cdot \varphi$ for every polynomial p .

We claim φ is also bounded. First of all, note that $\|\varphi\|_p = \|T1\|_p \leq \|T\|$. Since the polynomials are dense in H^p , $1 \leq p < \infty$ ([Du], Chapter 3), there exists a sequence of polynomials (q_n) which converges to φ in the H^p norm (and, thus, also uniformly on the compact subsets of \mathbb{D}). Then $Tq_n \rightarrow T\varphi$ in H^p norm, hence also uniformly on compact sets. On the other hand, $Tq_n = \varphi q_n \rightarrow \varphi^2$ uniformly on compact sets. It follows that $T\varphi = \varphi^2$, hence $\varphi^2 \in H^p$, too. Proceed inductively to show that $T\varphi^n = \varphi^{n+1}$ and $\varphi^n \in H^p$, for all n in \mathbb{N} . Thus, we have $\|\varphi^2\|_p \leq \|T\| \cdot \|\varphi\|_p \leq \|T\|^2$ and, by induction, $\|\varphi^n\|_p \leq \|T\|^n$. This tells us that $\|\varphi\|_{np} \leq \|T\|$. Let $n \rightarrow \infty$ to conclude that $\|\varphi\|_\infty \leq \|T\|$. This proves the claim.

Now for each f in H^p there is a sequence of polynomials (p_n) such that $p_n \rightarrow f$ in the H^p norm. Since T is a continuous operator and $\varphi \in H^\infty$, we have $Tf = \lim_{n \rightarrow \infty} T(p_n) = \lim_{n \rightarrow \infty} \varphi p_n = \varphi f$. This completes the proof. ■

2 Compactness

Given a Banach space X , let us agree to denote by $\mathcal{B}(X)$ the set of all bounded linear operators on X , and by $\mathcal{K}(X)$ the set of the compact ones among them. Recall that for an operator $T \in \mathcal{B}(X)$ its *essential norm* is defined as

$$\|T\|_e = \inf\{\|T - K\| : K \in \mathcal{K}(X)\}.$$

This norm measures the non-compactness of T . Since $\mathcal{K}(X)$ is closed in $\mathcal{B}(X)$ in the operator norm topology, it follows that T is compact if and only if $\|T\|_e = 0$. In order to obtain a further measure of non-compactness, the *n-th approximation number* of T is defined as

$$a_n(T) = \inf\{\|T - A_n\| : A_n \in \mathcal{B}(X), \text{rank } A_n \leq n\}.$$

Since the finite-rank operators are compact, it follows that $\|T\|_e \leq a_n(T) \leq \|T\|$ for all positive integers n . For the properties of approximation numbers, the reader may consult [P].

We return to analytic Toeplitz operators $M_\varphi : H^p \rightarrow H^p$ with some precise statements about their compactness, analogous to the corresponding results for Bergman spaces obtained in [V]. The simplest example is, as we said, the shift operator M_z , and the reader may wish to test the results of this section on this example. Several results for the Bergman space, as well as some for H^2 (identified with l^2) can be found in Chapter XI, Section 4 of [C]. Various results on other multiplication operators can be deduced from the theorems in Appendix IV, pp. 299–398 of [N1], as well as from some of the other sources cited.

Theorem 5. *Let $1 \leq p < \infty$, $\varphi \in H^\infty$, and consider the operator $M_\varphi : H^p \rightarrow H^p$. Then we have the following equality of norms and approximation numbers:*

$$\|M_\varphi\|_e = \|M_\varphi\| = \|\varphi\|_\infty = a_n, \text{ for all } n.$$

Thus, M_φ is compact if and only if $\varphi \equiv 0$.

Proof. We already know that $\|M_\varphi\|_e \leq a_n \leq \|M_\varphi\|$, and Proposition 2 tells us that $\|M_\varphi\| = \|\varphi\|_\infty$. Thus, we need only show that $\|M_\varphi\|_e \geq \|\varphi\|_\infty$.

By Remark (3) following Proposition 2, there exists a sequence of points $(\zeta_n)_{n=1}^\infty$ in \mathbb{D} such that

$$\lim_{n \rightarrow \infty} \|\varphi f_{\zeta_n}\|_p = \|\varphi\|_\infty \quad \text{and} \quad \lim_{n \rightarrow \infty} |\zeta_n| = 1.$$

The functions f_{ζ_n} all have norm one and converge pointwise to zero as $n \rightarrow \infty$. Also, H^p is a dual space of a Banach space whenever $1 \leq p < \infty$ (see [K], Chapter VII). Thus, a corollary on p. 272 of [BSh] (see also p. 318 of [Sh]) implies that $f_{\zeta_n} \rightarrow 0$ weakly. For any compact operator K on H^p , the sequence $K f_{\zeta_n} \rightarrow 0$ strongly. This implies

$$\|M_\varphi - K\| \geq \limsup_{n \rightarrow \infty} \|\varphi f_{\zeta_n} - K f_{\zeta_n}\|_p \geq \limsup_{n \rightarrow \infty} (\|\varphi f_{\zeta_n}\|_p - \|K f_{\zeta_n}\|_p) = \|\varphi\|_\infty.$$

Taking the infimum over all compact operators K acting on H^p , we deduce the desired inequality. ■

Corollary 6. *The mapping \mathcal{J} of H^∞ into the Calkin algebra $\mathcal{B}(H^p)/\mathcal{K}(H^p)$, given by $\mathcal{J}(\varphi) = \{M_\varphi + K : K \in \mathcal{K}(H^p)\}$, is an algebra isomorphism and an isometry.*

Proof. The algebra isomorphism part follows from the conclusion on compactness in Theorem 5. The isometry part follows from the formula for essential norm. ■

3 Fredholm and spectral properties

Note that the multiplication operator M_φ on H^p is injective if and only if $\varphi \not\equiv 0$, by the basic properties of analytic functions. Hence, M_φ has no eigenvalues unless $\varphi \equiv \text{const}$. The next question is that of determining the other points in the spectrum. Recall that the *spectrum* of a bounded operator T is the set of all λ in \mathbb{C} for which the operator $T - \lambda I$ is not invertible. It turns out that the spectra of multipliers of Hardy spaces have the same simple description as in Bergman spaces (*cf.* [Ax] or [V]).

A function g in H^p is said to be a *cyclic vector* for H^p if the polynomial multiples of g are dense in H^p (in its norm topology). This is equivalent to saying that the constant function one can be approximated by $p^n g$ in the norm, for some sequence of polynomials $(p_n)_{n=1}^\infty$. This equivalence holds for a generic class of functional Banach spaces (*cf.* Proposition 5 in Shields [Sh]). However, in H^p spaces there is a more direct characterization. Namely, the classical theorem of Beurling which describes the invariant subspaces of H^2 under the operator M_z extends without difficulties to the other values $1 \leq p < \infty$ (as was shown by Srinivasan and Wang; see [K], Chapter 4), and from here one easily sees that the only cyclic vectors in H^p are the outer functions for H^p (see [Du], Chapter 7, or [G], Chapter II, Section 7). Recall that a function F is *outer* for H^p if

$$F(z) = e^{it_0} \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \psi(t) dt \right\}$$

for some real number t_0 and a non-negative function $\psi \in L^p(\mathbb{T})$ such that $\log \psi \in L^1(\mathbb{T})$.

The above results will help us describe the continuous spectrum of M_φ . Recall that a complex number λ in the spectrum of a bounded operator T belongs to the *continuous spectrum* $\sigma_c(T)$ if and only if $T - \lambda I$ is one-to-one and has dense range ([DS], Chapter VII, p. 580). If $T - \lambda I$ is injective, but does not have dense range, then we say that $\lambda \in \sigma_r(T)$, the *residual spectrum* of T .

Theorem 7. *Let $1 \leq p < \infty$.*

(a) *For a non-zero H^∞ symbol φ , the operator M_φ acting on H^p is onto (equivalently, it is invertible) if and only if φ is bounded away from zero in \mathbb{D} .*

(b) *The spectrum of M_φ is $\sigma(M_\varphi) = \overline{\varphi(\mathbb{D})}$.*

(c) *Assume $\varphi \not\equiv \text{const}$. Then the continuous spectrum $\sigma_c(M_\varphi)$ consists of those values of λ for which $\varphi - \lambda$ is an outer H^∞ function, and it is a subset of $\overline{\varphi(\mathbb{D})} \setminus \varphi(\mathbb{D})$. The residual spectrum $\sigma_r(M_\varphi)$ is the set of all λ for which $\varphi - \lambda$ is not outer.*

Proof. (a) By the injectivity of M_φ , the equivalence of the surjectivity and invertibility is immediate by a usual application of the Closed Graph Theorem or the Open Mapping Theorem. Proposition 2 tells us that either of the two properties is equivalent to the requirement that φ be bounded away from zero.

(b) The trivial identity $M_\varphi - \lambda I = M_{\varphi - \lambda}$ implies that the spectrum of M_φ is $\sigma(M_\varphi) = \overline{\varphi(\mathbb{D})}$.

(c) By definition, $\lambda \in \sigma_c(M_\varphi)$ if and only if the set $(\varphi - \lambda)H^p$ is dense in H^p (since the operator $M_\varphi - \lambda I = M_{\varphi - \lambda}$ is injective). Because the polynomials are dense in H^p for $1 \leq p < \infty$, it is easily seen that this is equivalent to saying that the polynomial multiples of $\varphi - \lambda$ are dense in H^p , that is, to $\varphi - \lambda$ being a cyclic vector. By generalized Beurling's theorem, the only cyclic vectors in H^p are the outer functions. The outer functions for H^p differ with the value of p . However, an outer H^p function which belongs to H^∞ is also going to be an outer function for H^∞ (for example, by a theorem on p. 69, Chapter 5 of [H]). By Proposition 4 of Shields' survey article [Sh], in order that $\varphi - \lambda$ be cyclic, it must not vanish in \mathbb{D} , so $\lambda \notin \varphi(\mathbb{D})$, and we are done. ■

The operator T is said to have *closed range* if $T(X)$ is a closed subspace of X . Since M_φ is always injective on H^p (unless $\varphi \equiv 0$), it follows from the Open Mapping Theorem ([DS], Lemma VI.6.1) that a nonzero multiplication operator M_φ has closed range if and only if there exists a positive constant m such that the inequality

$$\|\varphi f\|_p \geq m\|f\|_p \quad (2)$$

holds for all $f \in H^p$; that is, if and only if

$$\inf\{\|\varphi f\|_p : f \in H^p, \|f\|_p = 1\} > 0. \quad (3)$$

The characterization of the multipliers of Hardy spaces with closed range is simpler than the one for Bergman spaces (compare with Luecking [L]), due to the better boundary behavior of Hardy functions. A statement like the one below will still make sense for Bergman spaces in the case of symbols φ which belong to the disk algebra (see [V], for example) or similar special classes.

Theorem 8. *Let $1 \leq p < \infty$, $\varphi \in H^\infty$, and let $\tilde{\varphi}$ denote the boundary function on \mathbb{T} . Then M_φ (acting on any H^p , $1 \leq p < \infty$) has closed range if and only if there exists a constant $m > 0$ such that $|\tilde{\varphi}| \geq m$ almost everywhere on \mathbb{T} . In fact,*

$$\inf\{\|M_\varphi f\|_p : f \in H^p, \|f\|_p = 1\} = \text{ess inf}\{|\tilde{\varphi}(\theta)| : \theta \in [0, 2\pi)\}.$$

Proof. The sufficiency of essential boundedness from below for having the closed range property is obvious from the norm formula (1) and inequality (2).

To prove the necessity, suppose that M_φ has closed range; then (2) holds for some $m > 0$ and all f in H^p . By assumption, the boundary values $\tilde{\varphi} \in L^\infty[0, 2\pi]$.

Choose an increasing sequence $(r_n)_{n=1}^\infty$ of positive numbers which tend to one. Let

$$P_n(t) = \frac{1 - r_n^2}{1 - 2r_n \cos t + r_n^2}, \quad t \in [0, 2\pi],$$

be the corresponding sequence of Poisson kernels, and therefore an approximate identity. Then the convolutions

$$(|\tilde{\varphi}|^p * P_n)(t) = \int_0^{2\pi} |\tilde{\varphi}(\theta)|^p \frac{1 - r_n^2}{1 - 2r_n \cos(t - \theta) + r_n^2} d\theta$$

converge to $|\tilde{\varphi}(t)|^p$ in $L^1[0, 2\pi]$ ([H], Chapter 3), hence a subsequence converges for almost every t in $[0, 2\pi]$. Fix such a t outside the exceptional set of measure zero, and denote the indices of the corresponding subsequence again by n in order not to burden the notation. Choose

$$\zeta_n = r_n e^{it} \in \mathbb{D}, \quad \text{so that} \quad |\zeta_n| = r_n \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Let

$$f_n(z) = \frac{(1 - |\zeta_n|^2)^{1/p}}{(1 - \overline{\zeta_n} z)^{2/p}},$$

as in Lemma 1. The pointwise convergence $(|\tilde{\varphi}|^p * P_n)(t) \rightarrow |\tilde{\varphi}(t)|^p$ and inequality (2) now yield

$$\begin{aligned} |\tilde{\varphi}(t)|^p &= \lim_{n \rightarrow \infty} \int_0^{2\pi} |\tilde{\varphi}(\theta)|^p P_n(t - \theta) \frac{d\theta}{2\pi} = \lim_{n \rightarrow \infty} \int_0^{2\pi} |\tilde{\varphi}(\theta)|^p \frac{1 - |\zeta_n|^2}{|1 - \overline{\zeta_n} e^{i\theta}|^2} \frac{d\theta}{2\pi} \\ &= \lim_{n \rightarrow \infty} \| \varphi f_n \|_p^p \geq m^p \limsup_{n \rightarrow \infty} \| f_n \|_p^p = m^p. \end{aligned}$$

Since this holds for almost every t , the statement about the essential boundedness from below follows. The actual formula for the infimum in (3) is easy to read off from the proof. ■

Like for the multipliers of H^∞ , the answer can also be expressed in terms of the Shilov boundary (see [BDL] for a general result on weighted H^∞ spaces), but the one above is quite descriptive.

Example. Here is an example of an operator M_φ with closed range whose symbol is zero “often” near the unit circle, in the sense that its boundary values $\tilde{\varphi}$ are zero at each rational point of \mathbb{T} (which is still allowed by Theorem 8). Let (q_n) be an enumeration of all rational numbers in $[0, 2\pi)$, and choose a double array of numbers $r_{n,k}$ in $(0, 1)$ so that $\sum_{k=1}^\infty (1 - r_{n,k}) < 1/n^2$ for each fixed n in \mathbb{N} . Then the set $\{r_{n,k} e^{iq_n} : n, k \in \mathbb{N}\}$ satisfies the Blaschke condition, and we can form the Blaschke product B which has precisely these points as zeros. Then, on the one hand, B has radial limits zero at every point e^{iq_n} and, on the other hand, $|\tilde{B}| = 1$ almost everywhere on \mathbb{T} , whence the operator M_B has closed range by Theorem 8.

Recall that λ is in the *approximate point spectrum* of T , denoted by $\sigma_{ap}(T)$, if there is a sequence (f_n) of unit vectors such that $\|(T - \lambda I)f_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Corollary 9. $\sigma_{ap}(M_\varphi) = \text{ess ran}\{\tilde{\varphi}(\mathbb{T})\}$.

Proof. By Proposition 6.4 of Chapter VII of [C], we know that $\lambda \notin \sigma_{ap}(M_\varphi)$ if and only if $M_\varphi - \lambda I = M_{\varphi-\lambda}$ is bounded from below. By our Theorem 8, it follows that $\lambda \in \sigma_{ap}(M_\varphi)$ if and only if $\text{ess inf}\{|\tilde{\varphi}(\theta) - \lambda| : \theta \in [0, 2\pi)\} = 0$. This is equivalent to saying that the measure of the set $\{\theta \in [0, 2\pi) : |\tilde{\varphi}(\theta) - \lambda| < \varepsilon\}$ is positive for all $\varepsilon > 0$. By the very definition of the essential range, this is the same as saying that $\lambda \in \text{ess ran } \tilde{\varphi}(\mathbb{T})$. ■

Among the operators with closed range, an important class is that of the *Fredholm operators*: those for which both the dimension M of the kernel and the codimension N of the range are finite. The difference $M - N$ is called the *Fredholm index*. A complex number λ is said to belong to the *essential spectrum* $\sigma_e(T)$ of a bounded operator T if $T - \lambda I$ is not Fredholm. For the basic theory in the Hilbert space case, we refer the reader to [C], Chapter XI.

Every multiplication operator is injective, hence $M = 0$. It has been known for quite some time that a general Toeplitz operator on H^2 is Fredholm if and only if its symbol is bounded away from zero on a neighborhood of the boundary (cf. [N1] and [Do], for example); the situation is similar for Bergman spaces A^p (see Axler's remarkable paper [Ax]). We now prove a statement of this type for an analytic Toeplitz operator on H^p . The index part resembles Proposition 4.3 of [V], but is more general.

Theorem 10. *Let $1 \leq p < \infty$ and $\varphi \in H^\infty$. Then the multiplication operator M_φ (acting on H^p) is Fredholm if and only if $|\tilde{\varphi}| \geq m$ almost everywhere on \mathbb{T} for some $m > 0$ and φ has finitely many zeros in \mathbb{D} . If φ has n zeros in \mathbb{D} (counting the multiplicities), then the Fredholm index of M_φ is equal to $-n$.*

Proof. If M_φ is Fredholm, then it automatically has closed range, hence $\tilde{\varphi}$ is essentially bounded away from zero on \mathbb{T} by Theorem 8. The other condition stated is also necessary: suppose that φ has infinitely many zeros in \mathbb{D} and denote them by a_n , $n \in \mathbb{N}$. Assume first that they are all simple zeros and denote by B_n the single Blaschke factor corresponding to the point a_n . The functions φ/B_n are easily seen to be linearly independent as follows. Consider some finite collection denoted B_1, B_2, \dots, B_n (for the sake of simplicity) and suppose that $\sum_{k=1}^n c_k \varphi/B_k \equiv 0$. Evaluate the function on the left-hand side at a_j for any fixed j to conclude that $c_j = 0$. The linear independence is easily transferred to the cosets $(\varphi/B_n) + \varphi H^p$ in the quotient set $H^p/\varphi H^p$. Namely, the equality of the cosets

$$\sum_{k=1}^N c_k \varphi/B_k + \varphi H^p = \varphi H^p$$

implies $\sum_{k=1}^N c_k \varphi/B_k = \varphi f$ for some f in H^p . Evaluate both sides again at a_j to infer that $c_j = 0$. In the case of multiple zeros, only minimum modifications are

needed: if a_k is a zero of multiplicity $m > 1$, use the collection of powers B_k, B_k^2, \dots, B_k^m instead of B_k , and everything else is similar.

In order to prove the converse, suppose that $|\tilde{\varphi}| \geq m$ on \mathbb{T} for some $m > 0$ and φ has only finitely many simple zeros $a_k, 1 \leq k \leq N$, in \mathbb{D} . Let B_k be a single Blaschke factor which corresponds to a_k . We claim that an arbitrary function g in H^p can be written as

$$g = \sum_{k=1}^N c_k \varphi / B_k + \varphi f \quad (4)$$

where $f \in H^p$. This will in turn imply that the cosets $\varphi / B_n + \varphi H^p$ (which are linearly independent) generate the quotient set $H^p / \varphi H^p$, and the statement of the theorem follows.

To prove the claim, note that the function $B/\varphi \in H^\infty$, as $|\tilde{B}/\tilde{\varphi}| \leq 1/m$ a.e. on \mathbb{T} . Next, choose $c_k = (gB_k/\varphi)(a_k)$. It is immediate that the H^p function

$$h = g - \sum_{k=1}^N c_k \varphi / B_k$$

vanishes at each a_j . Thus, $f = h/\varphi \in H^p$, too. We have obtained the representation (4) we sought. For multiple zeros, the proof can be modified as before. ■

Corollary 11. *A complex number $\lambda \in \sigma_e(M_\varphi)$ if and only if one of the following happens: either $\tilde{\varphi} - \lambda$ is not essentially bounded away from zero on \mathbb{T} , or φ takes on the value λ infinitely often in \mathbb{D} .*

Example. The Blaschke product B constructed in the example following Theorem 8 induces the multiplication operator M_B on H^p which has closed range but is not Fredholm. Moreover, since every value in the closed disk $\overline{\mathbb{D}}$ is taken on infinitely often by B (by Theorem 6.6 of Chapter II and the discussion on p. 80 of [G]), it follows that $\sigma_e(M_B) = \overline{\mathbb{D}}$. By Corollary 9, the approximate point spectrum is much smaller in this case, as $\sigma_{ap}(M_B) \subset \mathbb{T}$. This is in a sharp contrast with the shift operator M_z , which is Fredholm of index -1 and has the property $\sigma_e(M_z) = \mathbb{T} = \sigma_{ap}(M_z)$.

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